PROGRESS REPORT: IAS SUMMER COLLABORATORS 2018

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This summer, Jonathan Campbell and I started a new collaboration on scissors congruence. The (very ambitious) goal of the collaboration is to prove Goncharov's conjecture [Gon99, Conjecture 1.8], that

$$H^i\mathcal{P}^\bullet(S^{2n-1}) \cong (gr^\gamma_n K_{2n-i}(\mathbf{C})_{\mathbf{Q}} \otimes \epsilon(n))^+.$$

Here, $\P^\bullet(S^{2n-1})$ is the cochain complex generated by repeatedly applying the Dehn invariant to the scissors congruence group of spheres, and $\epsilon(n)$ is a copy of $\mathbb{Z}$ with $\mathbb{Z}/2$ acting on it by $(-1)^n$. The $\cdot^+$ takes the 1-eigenspace of the $\mathbb{Z}/2$-action. This conjecture is very complicated, and Goncharov came to it by comparing scissors congruence to motives in the (conjectural) abelian category of mixed motives. Our approach instead takes the K-theoretic construction of scissors congruence and compares it to algebraic K-theory directly, using Rognes's filtration on algebraic K-theory [Rog92].

During the three weeks at the IAS we started off the project in the following way:

(1) We proved a direct connection between the K-theoretic construction of scissors congruence of [Zak17] and the homological construction of [Dup01] by proving that they are rationally weakly equivalent.
(2) We began translating the homological approach of [Dup01] and [Cat04, Cat03]. The main advantage of this approach is that a major difficulty of Goncharov's conjecture from the homotopical viewpoint is that the construction of the Dehn invariant (and thus the chain complex we are interested in) has so far proved very difficult. Cathelineau has a construction which is much more algebraic than the classical approach, and we believe that it is much more amenable to a homotopical approach.
(3) We have translated Dupont's proof [Dup01, Chapter 3] of translational scissors congruence to a homotopical approach. This created both a cleaner proof and a stronger result, as a geometric interpretation made some of the algebraic constructions in Dupont's proof much clearer.

References

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[Dup01] Johan L. Dupont, *Scissors congruences, group homology and characteristic classes*, Nankai Tracts in Mathematics, vol. 1, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. MR 1832859

[Gon99] Alexander Goncharov, *Volumes of hyperbolic manifolds and mixed Tate motives*, J. Amer. Math. Soc. **12** (1999), no. 2, 569–618. MR 1649192 (99i:19004)

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[Zak17] I. Zakharevich, *The K-theory of assemblers*, Adv. Math. **304** (2017), 1176–1218. MR 3558230