Workshop on Emerging Topics: Nodal sets of Eigenfunctions

March 20, 2017

Description of the topic

Between February 13 and February 17, 2017, the Institute hosted a workshop on emerging topics focused on understanding the zero sets of Laplacian eigenfunctions and harmonic eigenfunctions, in particular on the recent progress on the Yau and Nadirashvili conjectures. The celebrated Yau conjecture states that if f_λ is a Laplacian eigenfunction on a smooth compact Riemannian manifold without boundary with the eigenvalue λ, then its the $n-1$-dimensional Hausdorff measure grows as $\sqrt{\lambda}$ when $\lambda \to \infty$. In the pioneering works by Donnelly and Fefferman, this conjecture was settled for the manifolds with real-analytic metric. In the smooth case the conjecture remained widely open (except of the lower bound in the two-dimensional case, which is not too difficult). The results available before the Spring of 2016 were quite modest: the best known lower bounds in the higher dimensions did not grow with λ, while the best known upper bound grow with λ over-exponentially. Moreover, as Nadirashvili pointed out about 25 years ago, there is no much hope to proceed toward the Yau conjecture until we will know the answers to related questions about harmonic functions. Probably, the most challenging of them was an innocently looking Nadirashvili's conjecture that for any non-constant harmonic function in $\mathbb{R}^n$ with $n \geqslant 3$, the $n-1$-dimensional Hausdorff measure of its zero set is always infinite.

There were plenty of other questions that we planned to discuss during the workshop. Among them were the behaviour of the number of connected components of the zero set and bounds for the number of critical points, the behaviour of high energy eigenfunctions on compact Riemannian surfaces with metric of negative curvature, in particular, of constant curvature, the properties of the zero sets of the random ensemble of spherical harmonics, the growth of discrete harmonic functions on lattices, to mention a few.

Timelines of the workshop

In the Spring of 2016, Logunov posted two papers in the arXiv in which he achieved an essential progress in the aforementioned problems. He proves the Nadirashvili conjecture on the zero sets of harmonic functions (even its quantitative version), which allowed him to prove the lower bound in the Yau conjecture. In the second paper he found the polynomial upper bound in the Yau conjecture. Both proofs were based on a novel combinatorial-type techniques, introduced in the paper by Logunov and Malinnikova (Spring, 2016) and further developed by Logunov.

In January 2017, Logunov, Malinnikova and Nadirashvili found a version of these techniques which allowed them to almost approach the Yau conjectural upper bound for Laplacian eigenfunctions in bounded Euclidean domains with smooth boundary. They showed that the $n-1$-dimensional Hausdorff measure of the zero set of the Dirichlet eigenfunctions in bounded Euclidean domains with smooth boundary cannot grow faster than $\sqrt{\lambda}\,\log\lambda$ as $\lambda \to \infty$.

Among other very recent results in related areas we mention the first non-trivial lower bound for the variance of the number of connected components of the zero set of random spherical harmonics of large degree obtained by Nazarov and Sodin in Summer 2016, a new, somewhat unexpected and counter-intuitive Liouville-type theorem for discrete harmonic functions on lattices obtained by Buhovski, Logunov, Malinnikova and Sodin in January 2017, and a curious example of a smooth compact Riemannian surface without boundary such that there exists a subsequence of Laplacian eigenfunctions each of which has infinitely many isolated critical points (Buhovski, Logunov, Nazarov, Sodin, January, 2017).

Participants

The workshop was co-organized by Eugenia Malinnikova (NTNU, Trondheim) and Mikhail Sodin (Tel Aviv University). The participants were Lev Buhovski (Tel Aviv University), Charles Fefferman (Princeton University), Hamid Hezari (UC at Irvince), Alexander Logunov (Tel Aviv University), Dan Mangoubi (Hebrew University, Jerusalem), Fedor Nazarov (Kent State University), Peter Sarnak (Princeton University and IAS), John Toth (McGill University, Montreal), Melissa Tacy (Australian National University, Canberra), Steven Zelditch (Northwestern University). Some of the talks were attended by faculty members, postdoctoral fellows and phd students of IAS and Princeton University.

The group photo with most of the participants can be viewed here:
`https://www.math.ias.edu/node/30457`.

Organization

The workshop was organized in the following way. Before that lunch we had two lectures with reports about the recent progress and basic ideas that lead to this progress. These lectures were intended for all interested mathematicians, not only for the workshop participants. The full schedule of lectures with short abstracts is available here: `https://www.math.ias.edu/node/30451`.

The time after lunch was left for discussions and sharing of knowledge. For instance, Melissa Tacy gave a wonderful informal lecture, where she outlined and explained the main ideas and obstacles in the study of Laplacian eigenfunctions on Riemannian surfaces of negative curvature.

As it was expected, Logunov was bombarded by many questions about clarifications of ideas used in his approach to Yau's conjecture.

Outcome

During the workshop (actually, during the Logunov lecture) Nazarov succeeded to improve the upper bound obtained by Logunov, Malinnikova and Nadirashvili to the optimal one.
Theorem: *Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let f_λ be the Dirichlet eigenfunction in Ω with the eigenvalue λ. Then the $n-1$-dimensional Hausdorff measure of the zero set of f_λ does not exceed $C_G\sqrt{\lambda}$, where C_G is a positive constant which depends only on G.*

During Melissa Tacy's talk on small scale equidistribution for random waves Peter Sarnak suggested an alternate way of directly esimating variance. Previous attempts using Levy concentration of measure had not allowed control of variance down the the expected Planck scale. After the talk, a lunch time discussion between Tacy, Sarnak and de Courcy-Ireland determined that these direct methods should yield better results at least in dimension two.

Another outcome was an interesting combinatorial-type conjecture about the local behaviour of the doubling exponent. Any progress in this conjecture will allow to make a progress towards the upper bound in the Yau conjecture.