We are very grateful to the IAS for the opportunity to work together for two weeks through the Summer Collaborators Program. Our goal was to prove the following conjecture.

Conjecture 1. *There are exactly* **six** *toric symplectic* 4*-manifolds* $X_1, \dots, X_6$ *whose capacity functions* c^{X_i} *admit an* **infinite staircase***. Three of these were previously unknown.*

We have verified the existence of the staircases in a unified way. Our method simply guarantees their existence rather than computing the capacity function exactly.

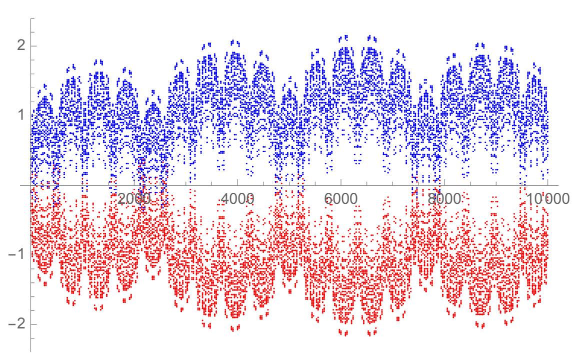
Before arriving at IAS, we had reduced the question of ruling out all other possible infinite staircases to a question in number theory. We spent the bulk of our time together investigating this question in number theory and also discussing it with Peter Sarnak. The time to work together was invaluable and we are also very grateful to Peter for thinking about the question, discussing it with us, and passing it on to some colleagues. The number theoretic question we now consider involves the **fractional part** of a number, defined for a number A to be $\{A\} = A - \lfloor A \rfloor$. To prove Conjecture 1, it is sufficient to prove the following.

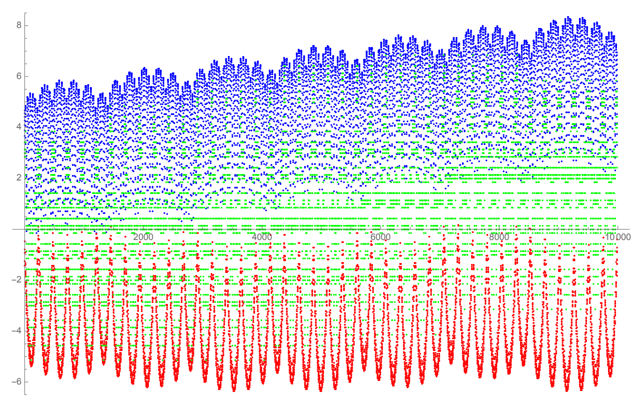
Conjecture 2. *Consider the function*

$$C_\alpha(n) = \sum_{k=1}^{n} \left(\{k\alpha\} - \frac{1}{2} \right).$$

If $\alpha + \frac{1}{\alpha}$ *is rational but not an integer, then* $C_\alpha(n) + C_{1/\alpha}(n)$ *is positive for some integer* n*.*

Remark 3. *In fact, we suspect the conjecture holds for any non-zero real number* α *with* $\alpha + \frac{1}{\alpha} \notin \mathbb{Z}$*. The conjecture is true when* T *is a real variable* [HL]*.*





On the left, when a quadratic surd corresponds to an infinite staircase, the conspiracy forces the blue graph ($C_\alpha(n)$) and red graph ($C_{1/\alpha}(n)$) to be mirror images of one another. In all other cases, we conjecture the sum of the two functions is sometimes positive. This is evident in the example on the right-hand side where the sum is indicated in green and is sometimes positive.

While Conjecture 1 remains elusive, we have explored and continue to explore several methods of attack. If we cannot prove it relatively quickly, we will publish our work to date, including the method to reduce it to Conjecture 2.

There remain a number of interesting combinatorial, topological and analytic questions about embeddings into toric varieties, including the following.

Conjecture 4. *The six infinite staircases* **persist** *for the stablized functions* $c_n^X(a)$.

We had hoped to work on this question at IAS, but already getting up to speed on the number theory relevant to Conjecture 1 took most of our two weeks. We plan to consider this in future work.

REFERENCES

[HL] G.H. Hardy and J.E. Littlewood, "Some problems of Diophantine approximation: The lattice-points of a right-angled triangle." *Proc. London Math. Soc. Ser. 2* **20** *no. 1378*, 15–36.