INSTITUTE FOR ADVANCED STUDY - SUMMER COLLABORATORS RESEARCH PROJECT REPORT

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1. Introduction

Our project (July 5–August 4, 2017) comprised several parts, all being motivated by Witten's formula (derived via supersymmetric quantum field theory) relating the Donaldson and Seiberg–Witten invariants of a closed, oriented, smooth four-manifold with simple type [33].

2. Instantons, monopoles, and relations between invariants of smooth four-manifolds: July 9–15 and 23–31

2.1. **Overview.** This research project was part of an ongoing collaboration between **Paul Feehan** and **Thomas Leness**. Our goal over the next academic year is to complete our proof of Witten's conjectured formula [33], relating the Donaldson and Seiberg–Witten invariants of a closed four-manifold, and the proof of the superconformal simple type property conjectured by Mariño, Moore, and Peradze, [22, 21]. After our recent advances in [13, 11], the remaining step in each of these two proofs is the completion of our work on the local gluing theorem for SO(3) monopoles, initiated in [9, 12] and discussed in more detail in our monograph [10].

During our visit to IAS, we wrote an outline for a new monograph in which we will construct a 'local gluing map' for SO(3) monopoles; this will incorporate both of our preprints [9, 12] and the results of additional research, based in part on our preprint [4] and monograph [5] concerning Yang–Mills connections.

The *local gluing map* parameterizes a neighborhood of a lower level of the Uhlenbeck compactification of the moduli space of solutions to the SO(3) monopoles. This map is the composition of a *splicing map* (defined by a simple cut-and-paste operation) and a *solution map* (defined by solving the SO(3)-monopole equations). More precisely, the splicing map is defined by using partitions of unity to splice together 'lower energy' solutions to the SO(3)-monopole equations over the four-manifold X with anti-self-dual connections over S^4 by identifying neighborhoods of separated points in X with a neighborhood of the north pole in S^4. Points in image of the splicing map are not solutions to the SO(3)-monopole equations but their failure to be solutions is small in a sense measured by a suitable Sobolev norm. To define the solution map, we adapt [30, 31, 32] and write the SO(3)-monopole equations as a system of quasi-linear elliptic partial differential equations and use the Implicit Function Theorem to find a unique solution to this system. The solution map is then defined by mapping a point in the image of the splicing map to this unique solution.

The construction of the splicing map and the existence of the solution map appear in our manuscript [12]. The remaining properties of the gluing map that we need to verify include: *1*) continuity with respect to bubble-tree limits, *2*) injectivity, and *3*) surjectivity. By *surjectivity*, we mean that the image of the gluing map contains all gauge-equivalence classes of solutions of the SO(3)-monopole equations in a bubble-tree neighborhood of the lower level. In writing an outline for a monograph containing a proof of these properties, we planned how to use the quantitative version of the Implicit Function Theorem to prove the surjectivity properties and

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how to translate our results from [9] on the Uhlenbeck continuity of the gluing map for the anti-self-dual connections to the setting of SO(3) monopoles.

Feehan also noted that a carefully constructed version of our SO(3)-monopole gluing map appears to show that the bubble-tree compactification of the moduli space of SO(3) monopoles admits the structure of real semianalytic set and thus a Whitney stratification. This observation is immediate from the Kuranishi model away from the boundary of the moduli space (see, for example, [3] in the case of the moduli space of anti-self-dual connections). The observation should also translate to bubble-tree compactifications of other moduli spaces (anti-self-dual connections or harmonic maps, for example) whose ends are parameterized by similar gluing maps. We have begun to look for applications of this property.

3. Instantons, monopoles, and relations between Floer homologies of 3-manifolds: July 5–14

3.1. **Overview.** This research project comprised a new collaboration among **Paul Feehan**, **Çağatay Kutluhan**, and **Thomas Leness**. The goal of this collaboration is to establish a relationship between the instanton (Yang–Mills) Floer homology (as defined by Floer [17] for homology three-spheres) and the monopole (Seiberg–Witten) Floer homology (as defined by Kronheimer and Mrowka [19] for closed three-manifolds) and extend the definition of instanton Floer homology to arbitrary closed three-manifolds. The relation between the Euler characteristics of the two Floer homologies for homology three-spheres has been established in [1, 20]. Thus a reasonable first goal would be to generalize that relationship and prove a conjecture due to Kronheimer and Mrowka that the two Floer homologies have the same ranks.

Our plan is to determine the relationship between the instanton and monopole Floer homologies on a three-manifold Y by studying the SO(3)-monopole equations on the cylinder $\mathbb{R}\times Y$. Recall that the instanton and monopole Floer homologies of Y are defined by identifying solutions of the anti-self-dual and Seiberg–Witten equations respectively on $\mathbb{R}\times Y$ with gradient flow lines for the Chern–Simons and Chern–Simons–Dirac functions respectively. The SO(3)-monopole equations, introduced in [29], are higher rank analogues of the Seiberg–Witten equations, corresponding (essentially) to a choice of U(2) vector bundle rather than a U(1) line bundle. As studied by Feehan and Leness in [8, 14, 15, 16, 10], the moduli space of SO(3) monopoles over a closed four-manifold admits an S^1 action whose fixed points are identified with the moduli space of projectively anti-self-dual connections defining the Donaldson invariants and a collection of moduli spaces of Seiberg–Witten monopoles. Over a cylinder, the moduli space of SO(3) monopoles comprises a space of gradient flow lines for a U(2) Chern–Simons–Dirac function admitting an S^1 action whose fixed points are the flow lines appearing in the instanton and monopole Floer homologies.

The chain complexes defining the instanton and monopole Floer homologies of a three-manifold Y are defined by the Chern–Simons and U(1) Chern–Simons–Dirac functions. The critical points of these functions are the *generators* of the complexes and counting the flow lines between the critical points defines the *boundary maps* of the complexes. Because the moduli space of SO(3) monopoles on $Y\times\mathbb{R}$ contains both solutions to the Seiberg–Witten equations and the solutions to the anti-self-dual equations on $Y\times\mathbb{R}$, it is a natural to try to use this moduli space to relate the two Floer homologies. Indeed, the SO(3)-monopole equations on $Y\times\mathbb{R}$ do define a gradient flow-line for a U(2) Chern–Simons–Dirac function and the critical points for this function fall into three categories: *1*) critical points for the U(1) Chern–Simons–Dirac function (that is, Seiberg–Witten monopoles over Y), *2*) critical points for the Chern–Simons function (that is, projectively flat connections over Y), and *3*) 'irreducible' critical points (SO(3) monopoles over Y) that are neither of the preceding two types.

However, for appropriate perturbations, these critical points are isolated and the SO(3)-monopole equations do not give a cobordism between the instanton and Seiberg–Witten critical points in the same way that they do in the four-dimensional theory. To address this complication, Kronheimer and Mrowka have proposed a program using a parameterized moduli space obtained by letting SO(3)-monopoles vary in a homotopically non-trivial loop.

In our discussions at IAS, we laid out a program to develop Kronheimer and Mrowka's idea and analyze the flow lines defined by these parameterized SO(3) monopoles. We identified the following issues that we need to understand before we can make further progress:

(1) Identify appropriate perturbations of the parameterized SO(3)-monopole equations so that the critical points would be non-degenerate.
(2) Identify compactness properties of the parameterized moduli space of SO(3) monopoles on $Y \times \mathbb{R}$. Do the non-compactness properties include both Uhlenbeck bubbling phenomena and solutions 'sliding down the neck'?
(3) Identify when the flow lines given by SO(3)-monopole equations can connect pairs of critical points that lie in different families.
(4) Combine the gluing maps that Feehan and Leness are developing for four-dimensional SO(3) monopoles (as described above) with the method of gluing flow-lines given in [19] for Seiberg–Witten monopoles and the method given in [2, 23] for anti-self-dual connections to glue flow-lines for SO(3) monopoles.
(5) When $b^1(X) = 0$, understanding how the first two families of critical points may fail to be distinct and how to adapt Kronheimer and Mrowka's method in [19] of 'blowing up the reducibles' to the case of SO(3)-monopoles.

While there are certainly more questions to ask, including how the knot Floer homologies are related, our plan was to establish a good understanding of these fundamental issues before proceeding and that understanding was largely achieved.

4. Mathematically rigorous approaches to quantum Yang–Mills theory: July 26–29

4.1. **Overview.** This project was to be a new collaboration between **Paul Feehan** and **Timothy Nguyen**. Unfortunately, Nguyen had to formally withdraw from the project when he did not succeed in obtaining a tenure-track position following his four-year postdoctoral assistant professorship at Michigan State University and three-year postdoctoral fellowship at the Simons Center for Geometry and Physics, Stony Brook. At Tom Spencer's invitation, he still visited IAS for a short period (July 26–29). During that time, Spencer, Nguyen, and Feehan discussed the ideas on mathematically rigorous approaches to quantum field theory that are described below. We reached the conclusion that they are new and well-worth pursuing. Feehan plans to continue his discussions on this topic with Spencer as well as Kevin Costello (Northwestern University), who is now being invited to visit Rutgers during Spring 2018, and his colleagues at Rutgers, Gregory Moore (Physics) and Tadeusz Balaban (Mathematics).

The underlying idea, described in detail in [6], was suggested by Tomasz Mrowka to one of us (Feehan) around the time of preparation of [7]. We seek to construct the quantum Yang–Mills measure — over the configuration space of connections on a principal bundle over a closed Riemannian four-manifold — via a new gauge-invariant finite-dimensional approximation scheme. A standard non-rigorous approach to constructing the quantum Yang–Mills measure involves the use of a lattice regularization which replaces infinite-dimensional integrals over configuration space with finite-dimensional integrals over the principal bundle structure group; see Nguyen [27, 24, 25, 26] for a discussion of the two-dimensional case. In our proposed new approach, we instead work directly in the continuum and use level sets defined by the self-dual component of the curvatures of the connections and spectral projections to construct an exhaustion of the configuration space of connections by a sequence of finite-dimensional subvarieties of increasing dimension. These finite-dimensional subvarieties can be thought of as 'thickened' or 'virtual' moduli spaces, much as encountered in some approaches to the definitions of Donaldson invariants (see Friedman and Morgan [18]) or Gromov-Witten invariants (see Pardon [28] and references cited therein). In [7], it was shown that the Riemannian volume (defined by the L^2 metric) of the moduli space of anti-self-dual connections (which is noncompact due to bubbling phenomenon) is finite. More generally, an attempt to understand the volumes of the finite-dimensional subvarieties — especially when weighted with physical observables and the exponential of the Yang–Mills action — in the limit as the finite-dimensional approximation exhausts the configuration space, provides an interesting geometric-analytic problem that we hope may lead to a mathematically rigorous construction of quantum Yang–Mills theory on four-manifolds.

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