EMERGING TOPICS WORKSHOP ON QUANTUM CHAOS AND FRACTAL UNCERTAINTY PRINCIPLE

Description of the topic. Between October 9–13, 2017 the Institute hosted a workshop on Emerging Topics focused on quantum chaos, in particular recent developments featuring fractal uncertainty principle. Quantum chaos studies the behavior of eigenfunctions and waves on manifolds which have chaotic geodesic flows, including the following two questions:

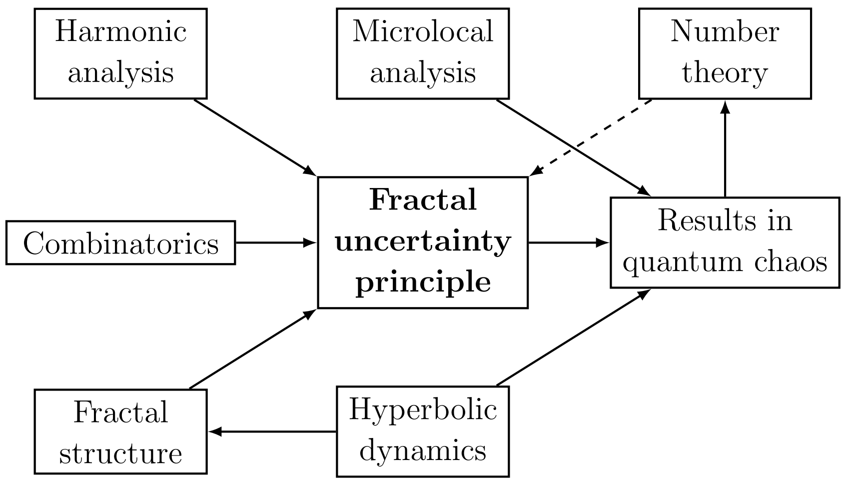
- on a compact manifold, how do eigenfunctions of the Laplacian concentrate at high frequency (in terms of limiting *semiclassical measures*)?
- on a noncompact manifold, do high-frequency solutions to the wave equation decay exponentially fast (i.e. is there an *essential spectral gap*)?

An important tool is the *geometric optics approximation*: at high frequencies and for bounded time solutions to the wave equation propagate along geodesics. It is part of a general theory called *microlocal analysis*. One challenge in quantum chaos is to extend geometric optics approximation to large times and use it together with the chaotic nature of the geodesic flow to obtain results on long time wave dynamics and eigenfunction statistics. For strongly chaotic (hyperbolic) systems, the classical trajectories (geodesics) diverge from each other exponentially fast under a small perturbation and geometric optics approximation is valid for time which is logarithmic in frequency.

In 2015, Dyatlov and Zahl proposed a new approach to the essential spectral gap question, reducing it to showing a *fractal uncertainty principle* (FUP). Roughly speaking, FUP states that no function can be localized near a fractal set in both position and frequency (Fourier) space. Dyatlov–Zahl proved FUP for sets of dimension close to $\frac{1}{2}$; in 2016, Dyatlov and Jin proved FUP for a class of Cantor sets.

At the end of 2016, Bourgain and Dyatlov proved a general FUP, holding for any porous subset of the real line. This result has already seen several important applications to quantum chaos. In particular, together with the work of Dyatlov–Zahl (or a recent much shorter version by Dyatlov and Zworski) it implies that every convex co-compact hyperbolic surface has an essential spectral gap. In 2017, Dyatlov and Jin used the general FUP to show that every semiclassical measure on a compact hyperbolic surface has full support; in particular, eigenfunctions of the Laplacian are bounded below (independently of the eigenvalue) on any nonempty open set. Also in 2017, Bourgain and Dyatlov proved Fourier decay for hyperbolic limit sets, which in particular gave a stronger version of FUP for sets of dimension $\leq \frac{1}{2}$.

The FUP approach goes beyond the time for which geometric optics approximation is valid, and proofs of FUP relied on harmonic analysis, fractal geometry, and combinatorics, bringing new techniques to quantum chaos. There is also hope to use number theory to show stronger versions of FUP in special cases; moreover, some results in quantum chaos (specifically spectral gaps for arithmetic surfaces) have applications to diophantine problems in number theory. The relation of these fields in context of FUP can be described by the following diagram:



The goal of the workshop was to bring together experts from these fields to better understand the different parts of the above diagram and work on several open problems.

Participants. The workshop was organized by Jean Bourgain (IAS) and Semyon Dyatlov (MIT/UC Berkeley/Clay Mathematics Institute). The participants were

- Alexis Drouot (Columbia University),
- Alex Gamburd (The Graduate Center, CUNY),
- Long Jin (Purdue University),
- Alex Kontorovich (Rutgers/IAS),
- Elon Lindenstrauss (The Hebrew University of Jerusalem),
- Alexandr Logunov (IAS),
- Michael Magee (Durham University),
- Frédéric Naud (Université d'Avignon),
- Stéphane Nonnenmacher (Université Paris-Sud),
- Peter Sarnak (IAS),
- Mikhail Sodin (Tel Aviv University),
- Ruixiang Zhang (IAS), and
- Steve Zelditch (Northwestern University).

Organization. The first day of the workshop consisted of two colloquium style lectures: an introduction to quantum chaos (Nonnenmacher) and an overview of applications of FUP (Dyatlov). The next several days had 2–3 lectures each: on FUP and its applications (Dyatlov, Jin, Zhang), Dolgopyat's method (Naud), geometry of limit sets (Magee), long time wave propagation (Nonnenmacher), and the sum-product theorem (Lindenstrauss). These talks were open to the general public and most of them are available as video lectures on the IAS website. There were several informal presentations, including FUP for Cantor sets (Dyatlov), the Beurling–Malliavin theorem (Sodin), and quantum cat map (Nonnenmacher), as well as many open-format discussions.

Outcomes. This workshop was a rare occasion when experts from several different fields could meet and exchange knowledge. For instance, experts on harmonic analysis, combinatorics, and number theory learned about the microlocal techniques used in the passage from FUP to results in quantum chaos. At the same time experts in microlocal analysis and quantum chaos learned about the tools from harmonic analysis and combinatorics that are used in the proof of FUP.

Many open problems were discussed. One for which progress seems most likely is adapting recent results (which were in the setting of hyperbolic surfaces) to more general surfaces with hyperbolic geodesic flows. One of the main issues in this setting is that the stable/unstable foliations are not smooth. It appears that some of the proofs in the paper of Bourgain–Dyatlov can be revisited and combined with hyperbolic parametrix to yield an FUP adapted to such nonsmooth foliations. This would yield statements which only need hyperbolic dynamics of the geodesic flow, not relying on constant curvature.