Phase Transition of Degeneracy in Minor-Closed Families

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Abstract

Given an infinite family $\mathcal{G}$ of graphs and a monotone property $\mathcal{P}$, an (upper) threshold for $\mathcal{G}$ and $\mathcal{P}$ is a "fastest growing" function $p : \mathbb{N} \to [0, 1]$ such that $\lim_{n\to\infty} \Pr(G_n(p(n)) \in \mathcal{P}) = 1$ for any sequence $(G_n)_{n\in\mathbb{N}}$ over $\mathcal{G}$ with $\lim_{n\to\infty}|V(G_n)| = \infty$, where $G_n(p(n))$ is the random subgraph of G_n such that each edge remains independently with probability $p(n)$.

In this paper we study the upper threshold for the family of H-minor free graphs and for the graph property of being $(r-1)$-degenerate, which is one fundamental graph property that has been shown widely applicable to various problems in graph theory. Even a constant factor approximation for the upper threshold for all pairs (r, H) is expected to be very difficult by its close connection to a major open question in extremal graph theory. We determine asymptotically the thresholds (up to a constant factor) for being $(r-1)$-degenerate for a large class of pairs (r, H), including all graphs H of minimum degree at least r and all graphs H with no vertex-cover of size at most r, and provide lower bounds for the rest of the pairs of (r, H). The results generalize to arbitrary proper minor-closed families and the properties of being r-colorable, being r-choosable, or containing an r-regular subgraph, respectively.

Keywords: Phase transition, random subgraphs, graph minors, degeneracy.

1 Introduction

Studying the properties of random subgraphs of given host graphs is a natural question. Given a host graph G and a real number $0 \leq p \leq 1$, let $G(p)$ be the random subgraph of G where each edge remains independently with probability p. In the case where G is an n-vertex complete graph, this is the well-studied Erdős-Rényi model $\mathbb{G}(n, p)$. Random graph models have broad connections to graph theory and are frequently used to model complex networks in fields such as theoretical computer science, statistical physics, social science, and economics. Besides the well-studied model $\mathbb{G}(n, p)$, rich theories have developed, including the percolation problem, modeling the spread of infectious disease in social network science, and the resilience problem to study the robustness of properties (see e.g. [22, 3, 10, 44].)

A fundamental subject regarding the asymptotic behavior of a random graph model is the study of "threshold phenomena" (or *phase transitions*) for monotone graph properties. A *graph property* $\mathcal{P}$ is a class of graphs such that $\mathcal{P}$ is invariant under graph automorphisms. A graph class $\mathcal{G}$ is *monotone* if every subgraph of a member of $\mathcal{G}$ is in $\mathcal{G}$. We remark that a graph property is also a graph class. So a graph property $\mathcal{P}$ is *monotone* if every subgraph of a member of $\mathcal{P}$ is in $\mathcal{P}$.

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Formally, a function $p^* : \mathbb{N} \to [0,1]$ is a *threshold (probability)* for a monotone graph property $\mathcal{P}$ and an infinite sequence $(G_n)_{n\in\mathbb{N}}$ if the following two conditions hold for any slowly growing function $x(n)$: (1) $G_n(p^*(n)x(n)) \notin \mathcal{P}$ a.a.s.[1]; and (2) $G_n(p^*(n)/x(n)) \in \mathcal{P}$ a.a.s. Thresholds for various graph properties in $\mathbb{G}(n,p)$ were first observed by Erdős and Rényi [15]. These results were further generalized to all monotone set properties and general random set models by Bollobás and Thomason [6] (see also Friedgut and Kalai [18]).

In fact, for any fixed monotone property, the results of Bollobás and Thomason imply the existence of a more general setting of threshold probability for any monotone graph class $\mathcal{G}$, called the *upper threshold*.

Definition 1 (Upper threshold)**.** *Let $\mathcal{P}$ be a monotone graph property and let $\mathcal{G}$ be a monotone graph class. When $\mathcal{G}$ is an infinite family, we say that a function $p^* : \mathbb{N} \to [0,1]$ is an* upper threshold *for $\mathcal{G}$ and $\mathcal{P}$ if the following two conditions hold.*

1. *For every sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with $G_n \in \mathcal{G}$ and $|V(G_n)| = n$, and for any function $q : \mathbb{N} \to [0,1]$ with $p^*(n)/q(n) \to \infty$, the random subgraphs $G_n(q(n))$ are in $\mathcal{P}$ a.a.s.*
2. *There exists a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with $G_n \in \mathcal{G}$ and $|V(G_n)| = n$ such that for any function $q : \mathbb{N} \to [0,1]$ with $q(n)/p^*(n) \to \infty$, the random subgraphs $G_n(q(n))$ are not in $\mathcal{P}$ a.a.s.*

When $\mathcal{G}$ is finite, a function $p^* : \mathbb{N} \to [0,1]$ is an upper threshold *for $\mathcal{G}$ and $\mathcal{P}$ if p^* is $\Theta(1)$.*

In the case when $\mathcal{G}$ consists of the graphs in the sequence $(G_n)_{n\in\mathbb{N}}$ where $|V(G_n)| = n$ for each $n \in \mathbb{N}$, the definition for the upper threshold for $\mathcal{G}$ coincides with the aforementioned definition for a threshold for the sequence $(G_n)_{n\in\mathbb{N}}$.

We denote such a function $p^*(n)$ mentioned in Definition 1 by $p_{\mathcal{G}}^{\mathcal{P}}$. We also abbreviate the upper threshold as threshold for simplicity. Note that for any fixed graph class $\mathcal{G}$ and monotone graph property $\mathcal{P}$, the threshold $p_{\mathcal{G}}^{\mathcal{P}}$ is not unique, as multiplying to it any sufficiently small positive constant factor is again a threshold probability. However, there exists a function f such that every threshold probability is $\Theta(f)$.[2] That is, the order of $p_{\mathcal{G}}^{\mathcal{P}}$ is unique. The aim of this paper is to determine the order of $p_{\mathcal{G}}^{\mathcal{P}}$. [3]

In many natural random structures, it has been observed that phase transitions appear to influence the computational complexity. It has connections to the boundary between easy and hard approximation problems, such as approximate counting problems for the number of independent sets, random SAT problems, vertex cover problem or colorability in random graphs (see [12, 32, 42, 43, 49] for examples).

It is an active line of research to determine the thresholds for various graph properties and the results in this field are too rich to enumerate. Extensive research has been about $\mathbb{G}(n,p)$ (see e.g. [4, 23, 19]), and relatively less is known when the host graphs are other finite graphs. For graph properties which are "global" (such as containing a giant component), the known results tend to depend on special geometric or algebraic features of the host graphs such as being expanders or having spectral conditions. Even with these features, the proofs are already non-trivial (see e.g. [2, 7, 8]).

In this paper, we study the phase transition when $\mathcal{G}$ is a minor-closed family to complement the knowledge in this direction. A graph H is a *minor* of another graph G if H can be obtained

[1] Given a sequence of events $(E_n)_{n\in\mathbb{N}}$ in a probability space, we say E_n happens *asymptotically almost surely* (or *a.a.s.* in short) if $\lim_{n\to\infty} \Pr(E_n) = 1$.

[2] Definition 1 is also called the *crude* threshold in the literature.

[3] Item 1 from Definition 1 has been studied for example in [21]

from a subgraph of G by contracting edges. A family $\mathcal{G}$ of graphs is *minor-closed* if every minor of any member of $\mathcal{G}$ belongs to $\mathcal{G}$. A minor-closed family is *proper* if it does not contain all graphs.

Minor-closed families receive wide attention in graph theory and theoretical computer science. They come up naturally for topological reasons and various kinds of embeddability properties, such as graphs embeddable in a particular surface of bounded genus without edge-crossings, and the graphs embeddable in $\mathbb{R}^3$ such that every cycle forms a non-trivial knot. Minor-closed families are also studied with connection to algorithms and computational complexity, such as [9, 11, 14, 28, 40]. Even though many NP-hard problems in algorithmic graph theory become polynomial time solvable when restricted to minor-closed families, there are still many natural algorithmic problems which are hard even on proper minor-closed families. For example, deciding whether a planar graph is 3-colorable and whether a planar graph is 4-choosable are both NP-hard. Therefore, in any given minor-closed family, graphs are still required to be distinguished. The objective of this paper is to study the phase transitions for some fundamental property of graphs that leads to such a distinction.

The main fundamental property $\mathcal{P}$ studied in this paper is degeneracy, which is known to closely relate to extremal graph theory and understanding other graph properties such as the colorability. A graph G is *r-degenerate* for some nonnegative integer r if every subgraph of G contains a vertex of degree at most r. It can be easily shown that any r-degenerate graph has a proper $(r+1)$-coloring (in fact, is $(r+1)$-choosable) by a very simple greedy algorithm. This simple observation had remained the only known method for decades until the very recent breakthroughs of [34, 37] to provide a general upper bound for Hadwiger's conjecture which is widely considered one of the most difficult questions in graph theory.

In some sense, degeneracy is equivalent with "sparsity." For example, the number of cliques in every r-degenerate graph is at most a linear number of its vertices [50]. On the other hand, a graph is not r-degenerate if and only if it contains a subgraph of minimum degree at least $r+1$. Graphs of large minimum degree are considered dense and contain substructures of certain forms. To name a few, there exist constants c_1, c_2, c_3 such that every graph of minimum degree at least k contains a $K_{c_1 k/\sqrt{\log k}}$ minor [26, 45, 47], a subdivision of $K_{c_2\sqrt{k}}$ [5, 25], and cycles of all even lengths modulo $k - c_3$ [27, 20].

Let r be a positive integer. Let $\mathcal{D}_r$ denote the graph property of being $(r-1)$-degenerate (and equivalently, not containing any subgraph of minimum degree at least r). It is clear that $\mathcal{D}_r$ is a monotone property. Note that $\mathcal{D}_1$ is equivalent with being edgeless which is trivial. For every graph H, let $\mathcal{M}(H)$ be the set of H-minor free graphs. In this paper we mainly consider the following questions on phase transition for being $(r-1)$-degenerate where r is a fixed positive integer.

Question 1.1. *For every graph H and integer $r \geq 2$, what is the threshold probability $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$?*

A more general question is the following.

Question 1.2. *For every integer $r \geq 2$ and every proper minor-closed family $\mathcal{G}$, what is the threshold probability $p^{\mathcal{D}_r}_{\mathcal{G}}$?*

Question 1.1 is a special case of Question 1.2. However, Question 1.1 is already expected to be very difficult for the property $\mathcal{D}_r$. It turns out that understanding the threshold for $(r-1)$-degeneracy is harder than determining the *degeneracy function* d_H for the graph H, which is known to be hard. The function $d_H(n)$ is the minimum d such that any H-minor free graph on n vertices is d-degenerate. Let d^*_H be $\lim_{n\to\infty} d_H(n)$, which is well-defined when H has no isolated vertices.[4]

[4] Note that d_H is a non-decreasing function, as being $(r-1)$-degenerate remains when adding isolated vertices. In addition, a result of Mader [30] implies that $d_H(n)$ has a constant (only depending on H) upper bound for every $n \in \mathbb{N}$. Therefore we can define d^*_H to be $\lim_{n\to\infty} d_H(n)$, which equals $\sup_{n\in\mathbb{N}} d_H(n)$.

A simple observation shows that for any fixed connected graph H, determining whether the answer to Question 1.1 is $\Theta(1)$ for every $r \geq 2$ is equivalent to determining d^*_H. (See Proposition A.1 for the precise description and a proof.)

It is well-known that determining d^*_H for all graphs H is a very challenging problem due to the fact that it is hard to estimate the *extremal function* $f_H(n)$, which is the maximum possible number of edges in an H-minor free graph on n vertices. Mader [30] proved that for every graph H, $\sup_{n\in\mathbb{N}} \frac{f_H(n)}{n}$ exists, and we denote this supremum as f^*_H. It is not hard to see that $f^*_H \leq d^*_H \leq 2f^*_H$. (See Proposition A.2 for a complete proof.) Despite having been extensively studied, even approximating f^*_H within a factor of 2 is not known for general sparse graphs. See for example, [46] for a survey. We remark that a combination of very recent results [33, 38, 39, 48] gives an approximation with a factor $\frac{0.319+\epsilon}{0.319-\epsilon}$ for almost every graph H of average degree at least a function of ϵ (so a density condition for H is still required), where $0 < \epsilon < 1$.

On the other hand, even though Question 1.1 is a special case of Question 1.2, the answer of Question 1.1 provides an approximation for the answer of Question 1.2. By the Graph Minor Theorem [41], for every proper minor closed family $\mathcal{G}$, there exists a finite set $\mathcal{H}$ of graphs such that every graph G in $\mathcal{G}$ does not contain any member of $\mathcal{H}$ as a minor. Hence $\mathcal{G} \subseteq \mathcal{M}(H)$ for every $H \in \mathcal{H}$. So by the definition of the threshold property, $p^{\mathcal{P}}_{\mathcal{G}} = \Omega(p^{\mathcal{P}}_{\mathcal{M}(H)})$ for every $H \in \mathcal{H}$. Therefore the following proposition follows.

Proposition 1.1. *For every proper minor closed family $\mathcal{G}$, there exists a finite set $\mathcal{H}$ of graphs such that $p^{\mathcal{P}}_{\mathcal{G}} = \Omega(\max_{H\in\mathcal{H}} p^{\mathcal{P}}_{\mathcal{M}(H)})$ for every monotone property $\mathcal{P}$.*

One of the main results of this paper, Theorem 1.2 which will be stated soon in Subsection 1.1, answers Question 1.1 for a large family of pairs (r, H) including the ones where either the minimum degree of H is at least r, or there is no vertex-cover of H of size at most r.

As discussed earlier, for any positive integer r, the property $\mathcal{D}_r$ (i.e., being $(r-1)$-degenerate) is closely related to the property of being r-colorable, denoted by χ_r. In fact, it is related to a stronger notion of coloring which is called list-coloring. We say that a graph G is *r-choosable* if for every list-assignment $(L_v : v \in V(G))$ with $|L_v| \geq r$, there exists a function c that maps each vertex $v \in V(G)$ to an element of L_v such that $c(x) \neq c(y)$ for any edge xy of G. Clearly, every r-choosable graph is r-colorable. But the converse is not true. It is known that there exists no integer k such that every bipartite graph is k-choosable.

Let χ^ℓ_r denote the property of being r-choosable. As mentioned earlier, every $(r-1)$-degenerate graph is r-choosable, so $\mathcal{D}_r \subseteq \chi^\ell_r$. Our main result also determines $p^{\chi^\ell_r}_{\mathcal{M}(H)}$ for a large family of pairs (r, H) and implies that $p^{\chi^\ell_r}_{\mathcal{M}(H)}$ and $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ have the same order for those pairs (r, H). The results also generalize to arbitrary proper minor-closed families.

Another property that is closely related to $\mathcal{D}_r$ is the property $\mathcal{R}_r$ of having no r-regular subgraph. We will show bounds for the properties $\mathcal{R}_r$ and χ_r as corollaries in Theorems 1.5 and 1.6, respectively, and a bound on the threshold for planar graphs to be 3-colorable in Corollary 1.7.

The thresholds for these properties are well-studied in $\mathbb{G}(n, p)$ (i.e., when $\mathcal{G}$ consists of complete graphs), all of which are of the form $\Theta(n^{-1})$. It is not hard to guess that $\Theta(n^{-1})$, up to a factor $\log n$, is the correct threshold for being r-degenerate in $\mathbb{G}(n, p)$, by the first moment method.[5] The nature of thresholds are very different when the host graphs are the complete graphs versus H-minor free graphs. This is largely because every H-minor free graph is d-degenerate for some fixed constant d [30] and thus is very sparse, and H-minor free graphs lack symmetry. In general, the

[5] In fact, in $\mathbb{G}(n, p)$ it is much more interesting to determine the sharp thresholds c/n for these properties. The exact constant c is known for r-degeneracy, and has been extensively studied for colorability (see e.g. [36, 24, 31, 1, 29]).

complicated structural nature of H-minor free graphs makes it hard to asymptotically determine the threshold, even for the exponent of n in the threshold. It is expected that the exponent of n in the threshold for the class $\mathcal{M}(H)$ should be significantly larger than -1, the exponent in the threshold for $\mathbb{G}(n,p)$.

1.1 Our Results

Recall that $\mathcal{M}(H)$ is the set of H-minor free graphs and $\mathcal{D}_r$ is the property of being $(r-1)$-degenerate. By the earlier discussion, determining the threshold $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ for all positive integers r is at least as hard as approximating the extremal functions for H-minor free graphs which has been a main open question for many graphs H. If the threshold for all r and H are determined, then these long-standing open questions will be resolved.

In this paper we determine the threshold for $(r-1)$-degeneracy, $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$, for a large class of H. Similar techniques are used to study the threshold for r-choosibility, $p^{\chi^\ell_r}_{\mathcal{M}(H)}$. For all the properties $\mathcal{D}_r, \chi^\ell_r, \chi_r, \mathcal{R}_r$ and all minor closed-families in which the precise thresholds are not determined in this paper, we prove a non-trivial lower bound for the threshold.

1.1.1 Results on being $(r-1)$-degenerate $\mathcal{D}_r$ and being r-choosable χ^ℓ_r

The first theorem determines the threshold for being $(r-1)$-degenerate in the set of H-minor free graphs, denoted by $\mathcal{M}(H)$, for a large class of H. It turns out that the answer is the same for the property χ^ℓ_r of being r-choosable. The threshold is closely related to the minimum size of a *vertex-cover* of H.

Definition 2. *A vertex-cover of a graph G is a subset S of $V(G)$ such that $G - S$ is edgeless. Denote the minimum size of a vertex-cover of a graph H by $\tau(H)$.*

Clearly, $\tau(H) = 0$ if and only if H has no edge. Note that if $\tau(H) = 0$, then no H-minor free graph has more than $|V(H)|$ vertices, so the threshold $p^{\mathcal{P}}_{\mathcal{M}(H)}$ is $\Theta(1)$ for any property $\mathcal{P}$. Hence we are only interested in graphs H with $\tau(H) \geq 1$.

We first introduce simple notations. For any graphs G, H and positive integer t, we define tG to be the disjoint union of t copies of G, and define $G \vee H$ to be the graph that is obtained from a disjoint union of G and H by adding all edges with one end in $V(G)$ and one end in $V(H)$.

The following theorem determines the threshold for $\mathcal{M}(H)$ and for the property of being $(r-1)$-degenerate in many cases including the case $\tau(H) > r$ or the case that H has minimum degree at least r. The same statement also applies to the property of being r-choosable.

Theorem 1.2. *Let $r \geq 2$ be an integer and H a graph (not necessarily connected). Let $\mathcal{P}$ be either of the two properties: $\mathcal{D}_r$ and χ^ℓ_r. In each of the following cases, there exists an integer q_H such that $p^{\mathcal{P}}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$, where q_H is defined as follows.*

1. *If $\tau(H) \geq r+1$, then $q_H = r$.*
2. *If $1 \leq \tau(H) \leq r$ and H is not a subgraph of $K_{\tau(H)-1} \vee tK_{r+2-\tau(H)}$ for any positive integer t, then $q_H = (r+2-\tau(H))r - \binom{r+2-\tau(H)}{2}$.*
3. *If $1 \leq \tau(H) \leq r$, H has minimum degree at least r, and H is not a subgraph of $K_{r-1} \vee tK_2$ for any positive integer t, then $q_H = 2r-1$.*
4. *If $1 \leq \tau(H) \leq r$, H has minimum degree at least r, H is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t, and $H \notin \{K_2, K_3, K_4\}$ then $q_H = 3r-3$.*

Furthermore, if either $H = K_{r+1}$ and $r \le 3$, or H has at most one component on at least two vertices and every component of H is an isolated vertex or a star of maximum degree at most r, then $p^{\mathcal{P}}_{\mathcal{M}(H)} = \Theta(1)$.

Note that Statements 2 and 3 of Theorem 1.2 are consistent since if $\tau(H) \le r$ and $\delta(H) \ge r$, then $\tau(H) = r$. In addition, the constant in $\Theta(n^{-1/q_H})$ may depend on r.

We remark that the graphs H in which the thresholds $p^{\mathcal{P}}_{\mathcal{M}(H)}$ are not determined in Theorem 1.2 belong to the set $\mathcal{H}_r$ of graphs, where

$$\mathcal{H}_r = \{H : 1 \le \tau(H) \le r \text{ and } H \subseteq K_{\tau(H)-1} \vee t^* K_{r+2-\tau(H)} \text{ for some positive integer } t^*\}.$$

Note that Theorem 1.2 also shows[6] that $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ if H is a graph in $\mathcal{H}_r$ with $\tau(H) = 1$. Therefore, the thresholds for being $(r-1)$-degenerate or r-choosable are determined by Theorem 1.2 unless $H \in \mathcal{H}_r$ and $\tau(H) \ge 2$.

We also remark that the number of uncovered cases in $\mathcal{H}_r$ of Theorem 1.2 is not large. Every graph in $\mathcal{H}_r$ has the property that deleting at most $\tau(H)-1$ vertices leads to a graph where every component has at most $r+2-\tau(H) \le r+1$ vertices. Even though every graph W is a subgraph of $K_{\tau(W)} \vee |V(W)|K_1$, which looks close to the definition of the graphs in $\mathcal{H}_r$, there is no control for the maximum degree of the remaining graph if we delete $\tau(W)-1$ vertices.

For those uncovered cases, the next theorem (Theorem 1.3) provides a lower bound of thresholds for the two properties of being $(r-1)$-degenerate and being r-choosable.

To state Theorem 1.3, we need the following definitions.

Definition 3. *Let G be a graph, and let $Z = \{z_1, z_2, ..., z_{|Z|}\}$ be a subset of $V(G)$. For any positive integer k, we define $G \wedge_k Z$ to be the graph obtained from a union of k disjoint copies of G by identifying, for each i with $1 \le i \le |Z|$, the k copies of all z_i into one vertex z_i^*.*

For example, if G is a star and Z consists of the leaves, then $G \wedge_k Z$ is $K_{k,|V(G)|-1}$.

For every nonnegative integer t, we denote the edgeless graph on t vertices by I_t. Note that I_0 is the empty graph that has no vertices and no edges. We remark that $I_t = tK_1$. We use the notation I_t instead of tK_1 for simplicity because the description for t can be complicated.

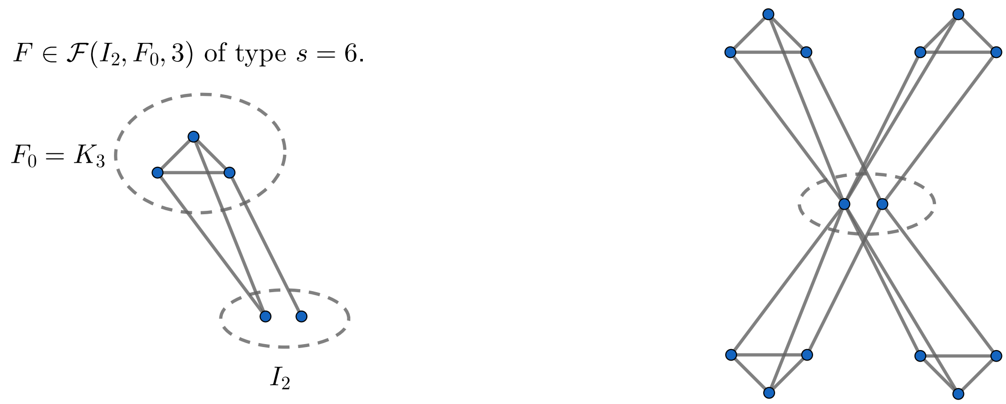
Definition 4. *For graphs G and F_0 and a nonnegative integer r, define $\mathcal{F}(G, F_0, r)$ to be the set consisting of the graphs that can be obtained from a disjoint union of G and F_0 by adding edges between $V(G)$ and $V(F_0)$ such that every vertex in $V(F_0)$ has degree at least r.*

For a graph F in $\mathcal{F}(G, F_0, r)$, the type *of F is the number of edges of F incident with $V(F_0)$, and we call $V(G)$ the* heart *of F.*

Note that every graph in $\mathcal{F}(G, F_0, r)$ has type at least r. Figure (a) in Figure 1 is an example of some $F \in \mathcal{F}(I_2, K_3, 3)$ of type 6.

Definition 5. *For every graph H and positive integer $r \ge 2$, let $s_r(H)$ be the largest integer s with $0 \le s \le \binom{r+1}{2}$ such that for every integer s' with $0 \le s' \le s$, every connected graph F_0 and every graph $F \in \mathcal{F}(I_{\tau(H)-1}, F_0, r)$ of type s', H is a minor of $F \wedge_t I$ for any positive integers t, where I is the heart of F.*

[6] When $\tau(H) = 1$, H is a graph that is a disjoint union of $K_{1,s}$ for some positive integer s and isolated vertices. Since $H \in \mathcal{H}_r$ and $\tau(H) = 1$, H is a subgraph of $t^* K_{r+1}$ for some positive integer t^*, so $s \le r$, and hence every component of H is either an isolated vertex or a star of maximum degree at most r.



(a) F_0 is a triangle. Each vertex of F_0 has degree at least $r = 3$ in F. There are in total $s = 6$ edges incident with vertices in F_0. Thus $F \in \mathcal{F}(I_2, F_0, 3)$ and is of type 6. The vertex-set of I_2 is the heart of F.

(b) $F \wedge_4 Z$, where Z is the heart of F.

Figure 1: An example of a graph $F \in \mathcal{F}(I_2, F_0, 3)$ of type 6 and $F \wedge_4 Z$ for some set Z.

Figure (b) in Figure 1 is an example of $F \wedge_t I_2$ for some $F \in \mathcal{F}(I_2, K_3, 3)$ of type 4 and $t = 4$.

Note that $s_r(H) \geq r-1$, since there exists no connected graph F_0 such that there exists a graph in $\mathcal{F}(I_{\tau(H)-1}, F_0, r)$ of type at most $r-1$. We can now state the theorem which proves the lower bound of the thresholds for the remaining cases of (r, H).

Theorem 1.3. *Let $r \geq 2$ be an integer and $H \in \mathcal{H}_r$. Let $\mathcal{P}$ be either of the two properties $\mathcal{D}_r$ and χ_r^ℓ. If $2 \leq \tau(H) \leq r$, then $p_{\mathcal{M}(H)}^{\mathcal{P}} = \Omega(n^{-1/q_H})$, where $q_H = \max\{\min\{s_r(H)+1, \binom{r+1}{2}\}, (r-\tau(H)+2)r - \binom{r-\tau(H)+2}{2}\}$.*

For any arbitrary proper minor-closed family $\mathcal{G}$, Theorems 1.2 and 1.3 provide a lower bound for $p_{\mathcal{G}}^{\mathcal{D}_r}$ and $p_{\mathcal{G}}^{\chi_r^\ell}$ by Proposition 1.1.

1.1.2 Results on χ_r and $\mathcal{R}_r$

Let χ_r be the property of being r-colorable and $\mathcal{R}_r$ the property of having no r-regular subgraphs. Since every $(r-1)$-degenerate graph is r-colorable, r-choosable, and does not contain any r-regular subgraph, $p_{\mathcal{G}}^{\mathcal{D}_r}$ is a lower bound for the thresholds for the properties χ_r, χ_r^ℓ, and $\mathcal{R}_r$, as stated below.

Proposition 1.4. *For every positive integer r and for every graph class $\mathcal{G}$, the threshold for being $(r-1)$-degenerate is upper bounded by each of the thresholds for the properties of being r-colorable, r-choosable, or having no r-regular subgraphs.*

Recall $\mathcal{H}_r = \{H : 1 \leq \tau(H) \leq r$ and $H \subseteq K_{\tau(H)-1} \vee t^* K_{r+2-\tau(H)}$ for some positive integer $t^*\}$ and we have determined in Theorem 1.2 the threshold for being $(r-1)$-degenerate and being r-choosable for all graphs H unless $H \in \mathcal{H}_r$ and $\tau(H) \geq 2$. The proof of Theorem 1.2 also helps us to determine the thresholds for $\mathcal{R}_r$ and χ_r. Results for thresholds for $\mathcal{R}_r$ and χ_r stated in this paper are easy corollaries of Theorem 1.2. We do not put effort in this paper to further strengthen their upper or lower bounds.

Theorem 1.5. *Let $r \geq 2$ be an integer and H a graph. Then $p^{\mathcal{R}_r}_{\mathcal{M}(H)}$ is $\Theta(n^{-1/q_H})$, where q_H is defined as follows.*

1. *If $\tau(H) \geq r+1$, then $q_H = r$.*
2. *If $1 \leq \tau(H) \leq r$, r is divisible by $r+2-\tau(H)$ and H is not a subgraph of $K_{\tau(H)-1} \vee tK_{r+2-\tau(H)}$ for any positive integers t, then $q_H = (r+2-\tau(H))r - \binom{r+2-\tau(H)}{2}$.*
3. *If $1 \leq \tau(H) \leq r$, r is even, H has minimum degree at least r and H is not a subgraph of $K_{r-1} \vee tK_2$ for any positive integer t, then $q_H = 2r-1$.*

Furthermore, if either $H = K_{r+1}$ and $r \leq 3$, or $H = K_{1,s}$ for some $s \leq r$, then $p^{\mathcal{R}_r}_{\mathcal{M}(H)} = \Theta(1)$.

Theorem 1.6. *Let $r \geq 2$ be an integer and let H be a graph. Then the following hold.*

1. *If $1 \leq \tau(H) \leq 2$ and H is not a subgraph of $K_1 \vee tK_r$ for any positive integer t, then $p^{\chi_r}_{\mathcal{M}(H)} = \Theta(n^{-2/(r(r+1))})$.*
2. *If either $H = K_{r+1}$ and $r \leq 3$, or H has at most one component on more than two vertices and every component of H is an isolated vertex or a star of maximum degree at most r, then $p^{\chi_r}_{\mathcal{M}(H)} = \Theta(1)$.*

Note that we do not obtain the exact value for $p^{\chi_r}_{\mathcal{M}(H)}$ for graphs H with $\tau(H) \geq 3$ except for $H = K_4$. In particular, $p^{\chi_3}_{\mathcal{M}(K_{3,3})}$ is unknown. Note that $\mathcal{M}(K_{3,3})$ contains the set of planar graphs, denoted by $\mathcal{G}_{\text{planar}}$. Since every planar graph on n vertices contains $O(n)$ triangles (by Lemma 4.1) and every triangle-free planar graph is properly 3-colorable by Grőtzsch's theorem, we know $p^{\chi_3}_{\mathcal{G}_{\text{planar}}} = \Omega(n^{-1/3})$. However, we are able to provide the following better estimation for $\mathcal{P}^{\chi_3}_{\mathcal{G}_{\text{planar}}}$ by using Theorem 1.2 and Proposition 1.4.

Corollary 1.7. *The thresholds for the properties of being 2-degenerate and 3-choosable for the set of planar graphs, $\mathcal{G}_{\text{planar}}$, are both $\Theta(n^{-1/5})$. There are positive constants c_1, c_2 such that the threshold for being 3-colorable satisfies $c_1 n^{-1/5} \leq p^{\chi_3}_{\mathcal{G}_{\text{planar}}} \leq c_2 n^{-1/6}$.*

2 Proof Ideas and Algorithmic Implications

2.1 Notations

In this paper, graphs are simple. Let G be a graph and X a subset of $V(G)$. We denote the subgraph of G induced by X by $G[X]$. We define $N_G(X) = \{v \in V(G) - X : v \text{ is adjacent in } G \text{ to some vertex in } X\}$, and define $N_G[X] = N_G(X) \cup X$. For any vertex v, $G - v, N_G(v)$ and $N_G[v]$ are defined to be $G[V(G) - \{v\}]$, $N_G(\{v\})$ and $N_G[\{v\}]$, respectively. The *degree* of a vertex is the number of edges incident with it. The minimum degree of G is denoted by $\delta(G)$. The *length* of a path is the number of its edges. The *distance* of two vertices in G is the minimum length of a path in G connecting these two vertices; the distance is infinity if no such path exists.

For every real number k, we define $[k]$ to be the set $\{x \in \mathbb{Z} : 1 \leq x \leq k\}$. We use $\mathbb{N}$ to denote the set of all positive integers, which does not include 0.

2.2 Proof Ideas and organization of the paper

To determine the order of the threshold probability $p^{\mathcal{P}}_{\mathcal{M}(H)}$ for a graph class $\mathcal{M}(H)$ and a monotone property $\mathcal{P} \in \{\mathcal{D}_r, \mathcal{R}_r, \chi_r, \chi^\ell_r\}$, it suffices to prove that the threshold probability is $O(f)$ and $\Omega(f)$ for some function f. The proof for the upper bound follows from a construction of sequences $(G_n : n \in \mathbb{N})$ of graphs in $\mathcal{M}(H)$ such that $\lim_{n\to\infty} \Pr(G_n(p(n)) \in \mathcal{P}) = 0$ for every function p with $f(n)/p(n) \to 0$. We shall present the construction in Section 3. Roughly speaking, in the construction, our graphs G_n are altered from the complete bipartite graphs such that they have minimum degree at least r in various ways.

The rest of the paper is dedicated to a proof of the lower bound for the threshold probabilities. To prove lower bounds for the thresholds $p^{\mathcal{P}}_{\mathcal{M}(H)}$ for $\mathcal{P} \in \{\mathcal{D}_r, \mathcal{R}_r, \chi_r, \chi^\ell_r\}$, it suffices to prove lower bounds for $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ and then use Proposition 1.4.

We first show a naive approach to prove a lower bound for $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ and then illustrate where the bottleneck is. We then sketch our approach to overcome this difficulty.

If $G(p)$ is $(r-1)$-degenerate, then every subgraph R of G with $\delta(R) \geq r$ has to be destroyed. By destroying we mean some edge of R needs to disappear in $G(p)$. Since $\delta(R) \geq r$, R contains at least $r+1$ vertices, so $|E(R)| \geq r(r+1)/2$. Hence $\Pr(R \subseteq G(p)) = p^{|E(R)|} \leq p^{r(r+1)/2}$. Trivially there are $O(2^{|E(G)|})$ such subgraphs R. And $2^{|E(G)|} = 2^{O(n)}$, where we use the fact that every H-minor free graph on n vertices has $O(n)$ edges. By a trivial union bound (or by linearity of expectation), the probability that some R still remains (or the expected number of R, respectively) in $G(p)$ is at most $p^{r(r+1)/2}2^{O(n)}$. We want the number of R remaining in $G(p)$ to be 0. By letting $p^{r(r+1)/2}2^{O(n)} < 0.01$, we see $p < 2^{-O(n)/(r(r+1))}$. This bound is too weak, even much worse than an easy lower bound[7] $O(n^{-3/2})$.

The strategy mentioned above fails because the union bound above crudely overestimate the number of subgraphs R with $\delta(R) \geq r$. In order to obtain a threshold probability of the form $n^{-1/q}$ by the union bound mentioned above, the union can only afford $\text{poly}(n) \ll 2^{\Theta(n)}$ number of R. This is a common bottleneck in the application of the probabilistic method to graph theory: one needs to find the correct signature of each of the desired objects and then group the objects by the signatures. By showing that the number of signatures is small, the number of groups of the objects is small. One can then apply a union bound to the small number of groups.

The key technical lemma is to find such a signature. That is, we show that for any positive integer r and any proper minor-closed family $\mathcal{G}$, there are a small number of signature sets with desired properties and we can group R by these signatures.

Definition 6. *For any real number c and nonnegative integers q and r, a (c, q, r)-*good signature collection *for a graph G is a collection $\mathcal{C}$ of subsets of $E(G)$ with the following properties.*

1. *Each member of $\mathcal{C}$ has exactly q edges.*
2. $|\mathcal{C}| \leq c|V(G)|$.
3. *For every subgraph of G of minimum degree at least r, its edge-set contains some member in $\mathcal{C}$.*

Condition 3 above implies that all subgraphs of G of minimum degree at least r are destroyed in $G(p)$ as long as all members of $\mathcal{C}$ are destroyed in $G(p)$.

[7]Note that $p(n) = n^{-3/2}$ is an easy lower bound for the three thresholds as this is the bound for a graph on n vertices to be 1-degenerate. For each $t(n)$ with $\lim_{n\to\infty} t(n)/p(n) = 0$, by a union bound, the probability that a given n-vertex graph G has a vertex with degree at least 2 in $G(t(n))$ is at most $n\binom{n}{2}t(n)^2 \leq n^3p(n)^2\left(\frac{t(n)}{p(n)}\right)^2 = \left(\frac{t(n)}{p(n)}\right)^2 \to 0$.

Definition 7. *For a given graph class $\mathcal{G}$ and nonnegative integers q and r, we say $\mathcal{G}$ has (q,r)-*good signature collections *if there is a constant $c = c(\mathcal{G})$ such that for every graph G in $\mathcal{G}$, there is a (c,q,r)-good signature collection for G.*

The following lemma shows that the existence of (q,r)-good signature collections for $\mathcal{G}$ provides a lower bound on the threshold probability in terms of q.

Lemma 2.1. *Let $\mathcal{G}$ be a class of graphs and q,r be positive integers. If $\mathcal{G}$ has (q,r)-good signature collections, then $p_{\mathcal{G}}^{\mathcal{D}_r} = \Omega(n^{-1/q})$.*

Proof. Let $p^* : \mathbb{N} \to [0,1]$ be the function such that $p^*(n) = n^{-1/q}$ for every $n \in \mathbb{N}$. Let $p : \mathbb{N} \to [0,1]$ be a function with $\lim_{n\to\infty} p(n)/p^*(n) = 0$. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of graphs in $\mathcal{G}$ such that $|V(G_n)| = n$ for every $n \in \mathbb{N}$. To show $p_{\mathcal{G}}^{\mathcal{D}_r} = \Omega(n^{-1/q})$, it suffices to show that $\lim_{n\to\infty} \Pr(G_n(p(n)) \in \mathcal{D}_r) = 1$.

For any $n \in \mathbb{N}$, let $\mathcal{C}_n$ be a (c,q,r)-good collection G_n. For each $T \in \mathcal{C}_n$, since $|T| = q$, $\Pr(T \subseteq E(G_n(p))) = p(n)^q$. Since for each subgraph R of G_n with $\delta(R) \geq r$, there exists $T \in \mathcal{C}_n$ with $T \subseteq E(R)$, we know that the probability that $G_n(p)$ contains a subgraph of minimum degree at least r is at most the probability that some member of $\mathcal{C}_n$ is a subset of $E(G_n(p))$ which is at most $|\mathcal{C}_n| p(n)^q$ by a union bound. But as $n \to \infty$,

$$|\mathcal{C}_n| p(n)^q \leq cnp(n)^q = cn(n^{-1/q} \cdot \frac{p(n)}{p^*(n)})^q = c\left(\frac{p(n)}{p^*(n)}\right)^q \to 0.$$

Thus with probability approaching 1 as n approaches infinity, no subgraph of minimum degree at least r is contained in $G_n(p)$. Therefore, $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = 1$. □

We want to emphasize that the value q mentioned in Lemma 2.1 determines the lower bound for $p_{\mathcal{G}}^{\mathcal{D}_r}$. The majority of work of this paper is to identify the largest possible value of q, which turns out to be the value q_H defined in Theorem 1.2, and hence the main theorem Theorem 1.2 is proved. We prove the existence of (q,r)-good signature collections with a large value of q in Lemmas 5.1 and 5.4. We show how these two lemmas imply the main theorems 1.2 and 1.3 in Section 6.

Our proof of the existence of (q,r)-good signature collections with the largest possible value of q is constructive and can be transformed into a quadratic time algorithm (in $|V(G)|$) to construct such a collection. To be more specific, the proof of explicitly finding a (c,q,r)-good signature collection $\mathcal{C}$ is fixed-parameter tractable. The precise statement is as follows.

Proposition 2.2. *For every positive integer r with $r \geq 2$ and every graph H, let q_H be defined as in Theorems 1.2 or 1.3. Then for any positive integer q with $q \leq q_H$, there exist constants $k_{r,H}$ and $c_{H,q}$ and an algorithm such that for any graph $G \in \mathcal{M}(H)$, it finds a $(c_{H,q},q,r)$-good signature collection for G in time at most $k_{r,H}|V(G_n)|$.*

The result above can be generalized to any proper minor-closed family $\mathcal{G}$ by applying the above result to each graph H in the finite set of minimal minor obstructions for $\mathcal{G}$ obtained by the Graph Minor Theorem.

A key sufficient condition for the existence of (q,r)-good signature collections is the following.

Lemma 2.3. *Let H be a graph and let r be a positive integer. If there exist nonnegative real numbers a,t,ζ with $t \leq 2r+1$ such that for every graph $G \in \mathcal{M}(H)$, there exist a subset Z of $V(G)$ with $|Z| \leq \zeta$ and a vertex $z^* \in Z$ such that*

1. *every vertex in Z has degree at most a in G, and*

2. *for every subgraph R of G with $\delta(R) \geq r$ and with $z^* \in V(R)$, $|V(R) \cap Z| \geq t$,*

then G has a $\left(\binom{\zeta}{t}\binom{a}{r}^t, rt - \binom{t}{2}, r\right)$-good signature collection. In other words, $\mathcal{G}$ has $\left(rt - \binom{t}{2}, r\right)$-good signature collections.

Proof. We prove this lemma by induction on the number of vertices in G. The claim trivially holds when $|V(G)| = 1$, as there exists no subgraph of G of minimum degree at least one.

For any set T of t distinct vertices $z_1, ..., z_t$ in Z and every sequence $s = (S_{T,1}, S_{T,2}, ..., S_{T,t})$, where $S_{T,i}$ is a set of r-edges of G incident with z_i for every $i \in [t]$, let $S_s = \bigcup_{j=1}^t S_{T,j}$. Note that $|S_s| \geq rt - \binom{t}{2}$. Let $\mathcal{C}_0$ be the collection of all such possible such sets S_s. Then $|\mathcal{C}_0| \leq \binom{\zeta}{t}\binom{a}{r}^t$ as the number of t-element subsets of Z is at most $\binom{\zeta}{t}$, and each vertex in Z is incident with at most a edges.

The second condition mentioned in the statement of this lemma implies that for every subgraph R of G with $\delta(G) \geq r$ and with $z^* \in V(R)$, the edge-set $E(R)$ contains some member of $\mathcal{C}_0$. Applying the induction hypothesis to $G - z^*$, $G - z^*$ has a $\left(\binom{\zeta}{t}\binom{a}{r}^t, rt - \binom{t}{2}, r\right)$-good signature collection $\mathcal{C}_1$. For every subgraph R of G with $\delta(R) \geq r$ and $z^* \notin V(R)$, R is a subgraph of $G - z^*$ with $\delta(R) \geq r$, so $E(R)$ contains some member of $\mathcal{C}_1$ by the induction hypothesis.

Let $\mathcal{C}_2 = \mathcal{C}_0 \cup \mathcal{C}_1$. Then $\mathcal{C}_2$ has the property that for every subgraph R of G with $\delta(G) \geq r$, $E(R)$ contains some member of $\mathcal{C}_2$. In addition, by the induction hypothesis, $|\mathcal{C}_2| \leq |\mathcal{C}_0| + |\mathcal{C}_1| \leq \binom{\zeta}{t}\binom{a}{r}^t + \binom{\zeta}{t}\binom{a}{r}^t(|V(G)| - 1) = \binom{\zeta}{t}\binom{a}{r}^t|V(G)|$.

Note that $\mathcal{C}_2$ satisfies the conditions of being a $\left(\binom{\zeta}{t}\binom{a}{r}^t, rt - \binom{t}{2}, r\right)$-good signature collection except some member of $\mathcal{C}_2$ possibly has size strictly greater than $rt - \binom{t}{2}$. For each member M of $\mathcal{C}_2$, let $f(M)$ be an arbitrary subset of M of size $rt - \binom{t}{2}$. Note that for every subgraph R of G with $\delta(R) \geq r$, $E(R)$ contains some member M of $\mathcal{C}_2$ and hence contains $f(M)$. Then the collection $\{f(M) : M \in \mathcal{C}_2\}$ is a $\left(\binom{\zeta}{t}\binom{a}{r}^t, rt - \binom{t}{2}, r\right)$-good signature collection for G. □

Note that the exponent of n in $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ is essentially determined by the size q of the members of $\mathcal{C}$ mentioned in Lemma 2.1, and q is determined by the value t mentioned in Lemma 2.3. The majority of work of this paper is to prove the sufficient condition in Lemma 2.3 with the correct value t.

Organization We prove upper bounds for the threshold probabilities in Section 3. We prove the lower bounds in Sections 4, 5, and 6, which is the most involved part of the paper. As we have discussed earlier, the main lemmas are Lemmas 5.1 and 5.4 regarding the existence of good collections, which are proved in Section 5. Lemma 5.1 is simple, but Lemma 5.4 is much more complicated and requires a technical lemma (Lemma 4.4, which is proved in Section 4). We then use Lemmas 5.1 and 5.4 to prove the main theorems in Section 6. Finally we conclude the paper with some remarks in Section 7.

3 Upper bound for the threshold probabilities

Our goal in this section is proving Corollary 3.6 which proves some upper bounds of the thresholds. We will construct sequences of graphs $(G_n : n \in \mathbb{N})$ that are hard to be made $(r-1)$-degenerate by randomly deleting edges. Namely, if p goes to 0 too slow, then $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = 0$. These sequences $(G_n : n \geq 1)$ will be used to establish upper bounds for $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ for different graphs H. The same construction will also be used for proving upper bounds for $p^{\chi^\ell_r}_{\mathcal{M}(H)}$.

A *stable set* in a graph is a subset of pairwise non-adjacent vertices.

Lemma 3.1. *Let Q be a graph and Z a (possibly empty) stable set in Q. Let $q = |E(Q)| \geq 2$. For every $n \in \mathbb{N}$, let $\ell_n = \lfloor \frac{n-|Z|}{|V(Q)|-|Z|} \rfloor$ and let G_n be the graph obtained from $Q \wedge_{\ell_n} Z$ by adding isolated vertices to make G_n have n vertices. Let $p : \mathbb{N} \to [0,1]$ with $\lim_{n\to\infty} n^{-1/q}/p(n) = 0$. Then for every $k \in \mathbb{N}$, $\lim_{n\to\infty} \Pr(Q \wedge_k Z \subseteq G_n(p)) = 1$.*

Proof. Let $\omega(n) = p(n)/n^{-1/q}$ for every $n \in \mathbb{N}$. Note that $\lim_{n\to\infty} 1/\omega(n) = 0$.

Note that for every $n \in \mathbb{N}$, G_n contains ℓ_n edge-disjoint copies of Q, denoted by $A_1, A_2, ..., A_{\ell_n}$. For each $1 \leq i \leq \ell_n$, define a random variable X_i to be 1 if all the edges of A_i remain in the random subgraph $G_n(p)$; let $X_i = 0$ otherwise. Thus $\Pr(X_i = 1) = (p(n))^q$.

Let $X = \sum_{i=1}^{\ell_n} X_i$. Since $E(Q) \neq \emptyset$, $|V(Q)|-|Z| \geq 1$, so $\ell_n \leq n$. By the linearity of expectation,

$$\mathbb{E}[X] = \sum_{i=1}^{\ell_n} \mathbb{E}[X_i] = \ell_n (p(n))^q = \ell_n (n^{-1/q}\omega(n))^q = \frac{\ell_n}{n} \cdot (\omega(n))^q \leq (\omega(n))^q.$$

Since $\omega(n) \to \infty$ and $q \geq 2$, when n is sufficiently large, $(\omega(n))^q \geq \omega(n)k$. This implies $\mathbb{E}[X] \geq \omega(n)k$. Thus, when n is sufficiently large, we have

$$\mathbb{E}[X] - k \geq \mathbb{E}[X] - \mathbb{E}[X]/\omega(n) \geq (\omega(n) - 1)\mathbb{E}[X]/\omega(n). \tag{1}$$

For $1 \leq i < j \leq m$, $\mathbb{E}[X_i X_j] = \Pr(X_i = X_j = 1) = (p(n))^{2q}$. So

$$\mathbb{E}[X^2] = \mathbb{E}[(\sum_{i=1}^{\ell_n} X_i)^2] = \sum_{i=1}^{\ell_n} \mathbb{E}[X_i^2] + 2 \sum_{1\leq i<j\leq \ell_n} \mathbb{E}[X_i X_j] = \ell_n (p(n))^q + \ell_n(\ell_n - 1)(p(n))^{2q}.$$

Note $\Pr(X < k) \leq \Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X] - k)$. By (1) and Chebyshev's inequality, for any sufficiently large n, (and write $p(n)$ as p for conciseness),

$$\begin{aligned}
\Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X] - k) \leq\ & \Pr(|X - \mathbb{E}[X]| \geq (\omega(n) - 1)\mathbb{E}[X]/\omega(n)) \\
\leq\ & \frac{\mathrm{Var}[X]}{((\omega(n) - 1)\mathbb{E}[X]/\omega(n))^2} = \frac{\mathbb{E}[X^2] - \mathbb{E}[X]^2}{((\omega(n) - 1)\mathbb{E}[X]/\omega(n))^2} \\
=\ & \frac{\ell_n p^q + \ell_n(\ell_n - 1)p^{2q} - (\ell_n p^q)^2}{((\omega(n) - 1)\ell_n p^q/\omega(n))^2} \\
=\ & \frac{\ell_n p^q - \ell_n p^{2q}}{((\omega(n) - 1)\ell_n p^q/\omega(n))^2} \\
\leq\ & \frac{\ell_n p^q}{((\omega(n) - 1)\ell_n p^q/\omega(n))^2} \\
=\ & \frac{\omega(n)^2}{(\omega(n) - 1)^2 \ell_n p^q} \\
=\ & \frac{\omega(n)^2}{(\omega(n) - 1)^2 \ell_n \omega(n)^q/n} \to 0
\end{aligned}$$

as $n \to \infty$, since $q \geq 2$ and $\lim_{n\to\infty} \omega(n)^{-1} = 0$. Therefore, $\lim_{n\to\infty} \Pr(X \geq k) = 1$. Note that when $X \geq k$, the union of the copies of Q corresponding to $X_i = 1$ contains $Q \wedge_k Z$ as a subgraph in $G_n(p)$. Hence $\lim_{n\to\infty} \Pr(Q \wedge_k Z \subseteq G_n(p)) = 1$. □

Lemma 3.2. *Let r, r' be integers with $r \geq 2$ and $0 \leq r' \leq r$ and let s be a nonnegative integer. Let F_0 be a connected graph and let $F \in \mathcal{F}(I_{r'}, F_0, r)$ be of type s. Let Z be the heart of F (thus Z is a stable set of size r' in F).*

For every positive integer n, let $\ell_n = \lfloor \frac{n-|Z|}{|V(F_0)|} \rfloor$ and let G_n be an n-vertex graph obtained from $F \wedge_{\ell_n} Z$ by adding isolated vertices to make it have n vertex. Let $p : \mathbb{N} \to [0,1]$ with $\lim_{n\to\infty} n^{-1/s}/p(n) = 0$. Then $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = 0$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \Pr(F \wedge_r Z \subseteq G_n) = 1$. We claim $F \wedge_r Z$ has a subgraph of minimum degree at least r. Every vertex in $V(F)\backslash Z$ has degree at least r in F. So every vertex in $V(F \wedge_r Z)\backslash Z$ has degree at least r in $F \wedge_r Z$. For each vertex in Z, if it has zero degree in F, it has zero degree in $F \wedge_r Z$; if it has degree at least one in F, it has degree at least r in $F \wedge_r Z$ as each of its neighbors has r copies in $F \wedge_r Z$. So some component of $F \wedge_r Z$ has of minimum degree at least r. Therefore, $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = 0$. $\square$

Lemma 3.3. *Let r be a positive integer and w an integer with $0 \leq w \leq r$. Then $I_{r-w} \vee r^{r-w} K_{w+1}$ is not r-choosable.*

Proof. Denote the vertices in $V(I_{r-w})$ by $v_1, v_2, ..., v_{r-w}$. For each i with $1 \leq i \leq r-w$, define a list of r colors $L_{v_i} = \{ri + j : 0 \leq j \leq r-1\}$. Thus $L_{v_i} \cap L_{v_j} = \emptyset$ for $1 \leq i < j \leq r-w$. And for each vertex v in $V(I_{r-w} \vee r^{r-w} K_{w+1}) - \{v_1, v_2, ..., v_{r-w}\}$, we define L_v to be a set of size r that is a union of $\{-1, -2, ..., -w\}$ and a set S_v with $|S_v \cap L_{v_i}| = 1$ for every $1 \leq i \leq r-w$, such that for every distinct vertices $x, y \in V(I_{r-w} \vee r^{r-w} K_{w+1}) - \{v_1, v_2, ..., v_{r-w}\}$, $L_x = L_y$ if and only if x, y are in the same component of $(I_{r-w} \vee r^{r-w} K_{w+1}) - \{v_1, v_2, ..., v_{r-w}\}$. This is possible since there are r^{r-w} components and there are r^{r-w} ways to pick precisely one element from each size-r list L_{v_i} for $1 \leq i \leq r-w$.

Suppose to the contrary that $I_{r-w} \vee r^{r-w} K_{w+1}$ is r-choosable. Then there exists a function f such that $f(v) \in L_v$ for every $v \in I_{r-w} \vee r^{r-w} K_{w+1}$, and $f(x) \neq f(y)$ for every adjacent vertices x, y. By construction, there exists a component C of $(I_{r-w} \vee r^{r-w} K_{w+1}) - \{v_i : 1 \leq i \leq r-w\}$ such that $L_v - \{f(v_i) : 1 \leq i \leq r-w\} = \{-1, -2, ..., -w\}$ for every $v \in V(C)$. Since $|V(C)| = w+1$ and $L_v - \{f(v_i) : 1 \leq i \leq r-w\} = \{-1, -2, ..., -w\}$ for every $v \in V(C)$, there exist two distinct vertices x, y of C such that $f(x) = f(y)$. Since C is isomorphic to K_{w+1}, x is adjacent to y, a contradiction. Therefore, $I_{r-w} \vee r^{r-w} K_{w+1}$ is not r-choosable. $\square$

Lemma 3.4. *Let r be an integer with $r \geq 2$ and let w be an integer with $r \geq w \geq 0$. Let $q = (w+1)r - \binom{w+1}{2}$. For every $n \in \mathbb{N}$ with $n > r$, let G_n be the n-vertex graph obtained from $I_{r-w} \vee \lfloor \frac{n-(r-w)}{w+1} \rfloor K_{w+1}$ by adding isolated vertices to make the number of vertices be n. Let $p : \mathbb{N} \to [0,1]$ be a function with $\lim_{n\to\infty} n^{-1/q}/p(n) = 0$. Then the following hold.*

1. *$\lim_{n\to\infty} \Pr(I_{r-w} \vee r^r K_{w+1} \subseteq G_n(p)) = 1$.*
2. *$\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = \lim_{n\to\infty} \Pr(G_n(p) \in \chi_r^{\ell}) = 0$.*
3. *If $r \neq w$, then $\lim_{n\to\infty} \Pr(G_n(p) \in \chi_{w+1}) = 0$.*
4. *If r is divisible by $w+1$, then $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{R}_r) = 0$.*

Proof. Let $Q = I_{r-w} \vee K_{w+1}$, and let Z be the subset of $V(Q)$ corresponding to $V(I_{r-w})$. For every $n \in \mathbb{N}$ with $n > r$, let $G'_n = Q \wedge_{\lfloor \frac{n-(r-w)}{w+1} \rfloor} Z$. So for every $n \in \mathbb{N}$, G_n is the graph obtained from G'_n by adding isolated vertices to make G_n have n vertices. Note that the number of edges of Q incident with at least one vertex in $V(Q) - Z$ is $(w+1)(r-w) + \binom{w+1}{2} = q$.

Note that for any positive integer k, $I_{r-w} \vee kK_{w+1} = Q \wedge_k Z$. Hence by Lemma 3.1, $\lim_{n\to\infty} \Pr(Q \wedge_{r^r} Z \subseteq G_n(p)) = \lim_{n\to\infty} \Pr(I_{r-w} \vee r^r K_{w+1} \subseteq G_n(p)) = 1$. Since $I_{r-w} \vee r^r K_{w+1}$ has minimum degree at least r, $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = 0$. By Lemma 3.3, $I_{r-w} \vee r^r K_{w+1}$ is not r-choosable, so $\lim_{n\to\infty} \Pr(G_n(p) \in \chi_r^\ell) = 0$.

If $r \neq w$, then $I_{r-w} \vee r^r K_{w+1}$ is not properly $(w+1)$-colorable, so $\lim_{n\to\infty} \Pr(G_n(p) \in \chi_{w+1}) = 0$.

If r is divisible by $w+1$, then $I_{r-w} \vee \frac{r}{w+1} K_{w+1}$ is a r-regular subgraph of $I_{r-w} \vee r^r K_{w+1}$, so $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{R}_r) = 0$. □

To define another sequence of graphs that are hard to be made $(r-1)$-degenerate, we need the following definition.

Definition 8. *Let r be a positive integer with $r \geq 4$. In the graph $I_{r-1} \vee K_3$, let Y be the stable set of size $r-1$ corresponding to $V(I_{r-1})$, and let X be the three vertices in K_3. Let L be a connected graph obtained from $I_{r-1} \vee K_3$ by deleting the edges of a matching of size three between X and Y.*

For every positive integer t, let $L_t = L \wedge_t Y$.

Note that L exists as $r \geq 4$. Also, L_t has $(r-1)+3t$ vertices and $3(r-1)t$ edges.

Lemma 3.5. *Let r be an integer with $r \geq 4$. For every $n \in \mathbb{N}$, let G_n be the n-vertex graph obtained from $L_{\lfloor (n-r+1)/3 \rfloor}$ by adding isolated vertices to make the number of vertices n. Let $p : \mathbb{N} \to [0,1]$ be a function such that $\lim_{n\to\infty} n^{-1/(3r-3)}/p(n) = 0$. Then $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = \lim_{n\to\infty} \Pr(G_n(p) \in \chi_r^\ell) = 0$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \Pr(L_{r^{r-1}} \subseteq G_n(p)) = 1$. Since $L_{r^{r-1}}$ has minimum degree at least r, $\lim_{n\to\infty} \Pr(G_n(p) \in \mathcal{D}_r) = 0$.

Now we show that $L_{r^{r-1}}$ is not r-choosable. Note that it implies that $\lim_{n\to\infty} \Pr(G_n(p) \in \chi_r^\ell) = 0$. We will construct a list of r colors for each vertex v, denoted as L_v. Denote $Y = \{y_1, y_2, ..., y_{r-1}\}$. For each i with $1 \leq i \leq r-1$, define $L_{y_i} = \{ri + j : 0 \leq j \leq r-1\}$. Thus the color lists of y_i and y_j are disjoint for any $i \neq j$. Let $C_1, C_2, ..., C_{r^{r-1}}$ be the r^{r-1} copies of $V(L) - Y$ in $L_{r^{r-1}}$. For each i with $1 \leq i \leq r^{r-1}$, let S_i be a set of size $r-1$ such that $|S_i \cap L_{v_j}| = 1$ for every $1 \leq j \leq r-1$, and $S_k \neq S_{k'}$ for distinct $k, k' \in [r^{r-1}]$. This is possible since there are r^{r-1} ways to pick exactly one element from each of L_{y_i} for $1 \leq i \leq r-1$. For each i with $1 \leq i \leq r^{r-1}$ and each vertex v in C_i, define $L_v = \{-1, -2\} \cup (S_i - L_{y_v})$ where y_v is the vertex in $\{y_1, y_2, ..., y_{r-1}\}$ such that v is not adjacent in $L_{r^{r-1}}$ to y_v. Note that each L_v has size r. Then it is easy to see that $L_{r^{r-1}}$ is not colorable with respect to $(L_v : v \in V(L_{r^{r-1}}))$. □

The following corollary provides an upper bound for $p_{\mathcal{M}(H)}^{\mathcal{D}_r}$.

Corollary 3.6. *Let $r \geq 2$ be an integer and let w be an integer with $r \geq w \geq 0$. Let $\mathcal{G}$ be a monotone class of graphs. Then the following hold.*

1. *If there exists $n_0 \in \mathbb{N}$ such that $\{K_{r,n-r} : n \geq n_0\} \subseteq \mathcal{G}$, then $p_{\mathcal{G}}^{\mathcal{D}_r} = O(n^{-1/r})$, $p_{\mathcal{G}}^{\chi_r^\ell} = O(n^{-1/r})$ and $p_{\mathcal{G}}^{\mathcal{R}_r} = O(n^{-1/r})$.*
2. *If there exists $n_0 \in \mathbb{N}$ such that $\{I_{r-w} \vee tK_{w+1} : t \geq n_0\} \subseteq \mathcal{G}$, then the following hold.*
   - *(a) $p_{\mathcal{G}}^{\mathcal{D}_r} = O(n^{-1/q})$ and $p_{\mathcal{G}}^{\chi_r^\ell} = O(n^{-1/q})$, where $q = (w+1)r - \binom{w+1}{2}$.*
   - *(b) If $r \neq w$, then $p_{\mathcal{G}}^{\chi_{w+1}} = O(n^{-1/q})$, where $q = (w+1)r - \binom{w+1}{2}$.*
   - *(c) If r is divisible by $w+1$, then $p_{\mathcal{G}}^{\mathcal{R}_r} = O(n^{-1/q})$, where $q = (w+1)r - \binom{w+1}{2}$.*

3. *If $r \geq 4$ and there exists $n_0 \in \mathbb{N}$ such that $\{L_t : t \geq n_0\} \subseteq \mathcal{G}$, then $p_{\mathcal{G}}^{\mathcal{D}_r} = O(n^{-1/(3r-3)})$ and $p_{\mathcal{G}}^{\chi_r^\ell} = O(n^{-1/(3r-3)})$.*

4. *Let r, r' be an integers with $r \geq 2$ and $r' \leq r$ and let s be a nonnegative integer. Let F_0 be a connected graph and let $F \in \mathcal{F}(I_{r'}, F_0, r)$ be with type s. Let Z be the heart of F. If there exists $n_0 \in \mathbb{N}$ such that $\{F \wedge_t Z : t \geq n_0\} \subseteq \mathcal{G}$, then $p_{\mathcal{G}}^{\mathcal{D}_r} = O(n^{-1/s})$.*

Proof. Statements 2, 3 and 4 of this corollary immediately follows from Lemmas 3.4, 3.5 and 3.2, respectively. Statement 1 of this corollary following from Statement 2 by taking $w = 0$. □

4 Neighbors of low degree vertices

In this section we prove Lemma 4.4, which is a generalization of the main lemma in the work of Ossona de Mendez, Oum, and Wood [35], where they use the lemma to study defective coloring for a broader class of graphs. We refer interested readers to [35] for details.

We require some notions to formally state Lemma 4.4 and some lemmas to prove it.

The *average degree* of a graph G is $\frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of a graph G is $\max_H \frac{2|E(H)|}{|V(H)|}$, where the maximum is over all subgraphs H of G. The following lemma can be found in [50, Lemma 18] or in the proof of [16, Theorem 1.1].

Lemma 4.1 ([50, Lemma 18],[16, Theorem 1.1])**.** *Let r be a positive integer and let k be a positive real number. If G is a graph of maximum average degree at most k, then G contains at most $\binom{k}{r-1}|V(G)|$ cliques of size r.*

For every nonnegative integer ℓ, we say that a graph G is an *ℓ-subdivision* of a graph H if it can be obtained from H by subdividing each edge of H exactly ℓ times. That is, G can be obtained from H by replacing each edge of H by a path of length $\ell + 1$, where those paths are pairwise internally disjoint. For a set S of nonnegative integers, we say a graph G is an *S-subdivision* of H if for every $e \in E(H)$, there exists $s_e \in S$ such that G can be obtained from H by subdividing each edge e of H exactly s_e times. Thus an ℓ-subdivision is the same as an $\{\ell\}$-subdivision. For every nonnegative integer ℓ, a graph G is a *$(\leq \ell)$-subdivision* of H if it is an $([\ell] \cup \{0\})$-subdivision of H.

The *radius* of a graph G is the minimum k such that there exists a vertex v of G such that every vertex of G has distance from v at most k. Let $\ell \in \mathbb{Z} \cup \{\infty\}$. We say that a graph G contains a graph H as an *ℓ-shallow minor* if H can be obtained from a subgraph G' of G by contracting connected subgraphs of G' of radius at most ℓ. In other words, every branch set of an ℓ-shallow minor is a connected subgraph of radius at most ℓ. Note that G contains H as an ∞-shallow minor if and only if G contains H as a minor; G contains H as a 0-shallow minor if and only if H is a subgraph of G.

The next concept is important in our proof.

Definition 9. *For a graph G, a subset Y of $V(G)$, and an integer r, we say a subgraph H of G is* r-adherent *to Y if $V(H) \cap Y = \emptyset$ and $|N_G(V(H)) \cap Y| \geq r$.*

The proof of the following lemma is inspired by the proof of [35, Lemma 2.2].

Lemma 4.2. *For any $r, t \in \mathbb{N}$ and positive real number k', there exists a real number $\alpha > 0$ such that for every $\ell \in \mathbb{Z} \cup \{\infty\}$, for every graph G, for every $Y \subseteq V(G)$ and every collection $\mathcal{C}$ of disjoint connected subgraphs of $G - Y$ where each is r-adherent to Y and of radius at most ℓ, we have either*

1. *there exists a graph H of average degree greater than k' such that G contains a subgraph isomorphic to a $[2\ell+1]$-subdivision of H,*

2. *G contains $K_r \vee I_t$ as an ℓ-shallow minor, or*

3. *$|\mathcal{C}| \leq \alpha|Y|$.*

Proof. Let $r, t \in \mathbb{N}$ and let k' be a positive real number. Define $\alpha = (t-1)\binom{k'}{r-1} + k'/2$.

Let $\ell \in \mathbb{Z} \cup \{\infty\}$, let G be a graph and $Y \subseteq V(G)$. Let $\mathcal{C}$ be a collection of disjoint connected subgraphs of $G - Y$ where each is r-adherent to Y and of radius at most ℓ.

Assume that for every graph H, if G contains some subgraph isomorphic to a $[2\ell+1]$-subdivision of H, then the average degree of H is at most k'. Assume that G does not contain $K_r \vee I_t$ as an ℓ-shallow minor. We shall show that Statement 3 of this lemma holds.

Let G' be the graph obtained from G by contracting each member of $\mathcal{C}$ into a vertex. Since each member of $\mathcal{C}$ is a subgraph of G of radius at most ℓ, G' is an ℓ-shallow minor of G. In addition, $Y \subseteq V(G')$ since each member of $\mathcal{C}$ is disjoint from Y. Let $Z = V(G') - V(G)$. (That is, Z is the set of the vertices of G' obtained by contracting members of $\mathcal{C}$.) Define G'' to be the graph obtained from $G' - E(G'[Y])$ by repeatedly picking a vertex v in Z that is adjacent in G' to a pair of nonadjacent vertices u, w in $(G' - E(G'[Y]))[Y]$, deleting v, and adding an edge uw, until for any remaining vertex in Z, its neighbors in Y form a clique.

Let $H = G''[Y]$. So some subgraph H' of G' is isomorphic to a 1-subdivision of H. Together with the fact that G contains H' as an ℓ-shallow minor and $V(H) = Y \subseteq V(G) - Z$, we know G contains a subgraph isomorphic to a $[2\ell+1]$-subdivision of H. This implies that for every subgraph L of H, G contains a subgraph isomorphic to a $[2\ell+1]$-subdivision of L. So the average degree of any subgraph of H is at most k' by our assumption. Hence there are at most

$$\binom{k'}{r-1}|V(H)| = \binom{k'}{r-1}|Y| \tag{2}$$

cliques of size r in H by Lemma 4.1.

Since G contains G' as an ℓ-shallow minor, G' does not contain $K_r \vee I_t$ as a subgraph, for otherwise G contains $K_r \vee I_t$ as an ℓ-shallow minor. This implies that for each clique K in $G''[Y]$ of size r, $|\{z \in Z \cap V(G'') : K \subseteq N_{G''}(z)\}| \leq t-1$. In addition, for every $z \in Z \cap V(G'')$, $N_{G''}(z) \cap Y$ is a clique consisting of at least r vertices in H since every member of $\mathcal{C}$ is r-adherent to Y. So a double counting argument applied to (2) implies

$$|Z \cap V(G'')| \leq (t-1)\binom{k'}{r-1}|Y|.$$

Furthermore, by the definition of G'', the vertices in Z but not in $V(G'')$ are the ones being deleted while adding one edge in between two vertices in Y. Thus $|Z - V(G'')| \leq |E(G''[Y])| = |E(H)| \leq k'|Y|/2$. So $|\mathcal{C}| = |Z| \leq (t-1)\binom{k'}{r-1}|Y| + k'|Y|/2 = \alpha|Y|$. □

Let G be a graph. For any vertex x of G and any (possibly negative) real number ℓ, we denote by $N_G^{\leq \ell}[x]$ the set of all the vertices in G whose distance to x is at most ℓ; in particular, $N_G^{\leq 1}[x] = N_G[x]$.

Definition 10. *For a subset Y of $V(G)$, $v \in V(G) - Y$ and integers k and r, we define a (v, Y, k, r)-span (in G) to be a connected subgraph H of $G - Y$ containing v such that $|Y \cap N_G(V(H))| \geq r$, and for every vertex u of H, there exists a path in H from v to u of length at most k.*

Note that if H is a (v, Y, k, r)-span in G, then H is r-adherent to Y, and $V(H) \subseteq N_H^{\leq k}[v] \subseteq N_G^{\leq k}[v]$. A (v, Y, k, r)-span H is *minimal* if no proper subgraph of H is a (v, Y, k, r)-span.

Lemma 4.3. *Let G be a graph, $Y \subseteq V(G)$ and k, r be integers. Then every minimal (v, Y, k, r)-span is a subgraph of a union of at most r paths in $G - Y$ where each path starts from v and is of length at most k. In particular, every minimal (v, Y, k, r)-span contains at most $kr + 1$ vertices.*

Proof. Let H be a minimal (v, Y, k, r)-span. Since H is a (v, Y, k, r)-span, $|N_G(V(H)) \cap Y| \geq r$. So there exists a subset $S = \{v_1, v_2, ..., v_{|S|}\}$ of $V(H)$ with $|S| \leq r$ such that $Y \cap N_G(v_{i+1}) - \bigcup_{j=1}^{i} N_G(v_j) \neq \emptyset$ for each i with $0 \leq i \leq |S| - 1$. Since H is a (v, Y, k, r)-span, for every $u \in S$, there exists a path P_u in $H \subseteq G - Y$ from v to u of length at most k. Hence $\bigcup_{u \in S} P_u$ is a (v, Y, k, r)-span and is a subgraph of H. By the minimality of H, $H = \bigcup_{u \in S} P_u$. Therefore, H is a union of $|S| \leq r$ paths in $G - Y$ where each path starts from v and is of length at most k. Note that $|V(P_u) - \{v\}| \leq k$, so $|V(H)| \leq 1 + rk$. □

Now we are ready to state and prove Lemma 4.4.

Lemma 4.4. *For any $r, t \in \mathbb{N}$, nonnegative integer ℓ, positive real numbers k, k', and nonnegative real number β, there exists an integer d such that for every graph G, either*

1. *the average degree of G is greater than k,*
2. *there exists a graph H of average degree greater than k' such that some subgraph of G is isomorphic to a $[2\ell + 1]$-subdivision of H,*
3. *G contains $K_r \vee I_t$ as an ℓ-shallow minor, or*
4. *there exist $X, Z \subseteq V(G)$ with $Z \subseteq X$ and $|Z| > \beta|V(G) - X|$ such that*
   - (a) *every vertex in X has degree at most d in G,*
   - (b) *for any distinct $z, z' \in Z$, the distance in $G[X]$ between z, z' is at least $\ell + 1$,*
   - (c) *for every $z \in Z$ and $u \in X$ whose distance from z in $G[X]$ is at most ℓ, $|N_G(u) - X| \leq r - 1$, and*
   - (d) *$|N_G(N_{G[X]}^{\leq \ell - 1}[z]) - X| \leq r - 1$ for every $z \in Z$.*

Proof. Let $r, t \in \mathbb{N}$, ℓ be a nonnegative integer, k, k' be positive real numbers, and β be a nonnegative real number. Let α be the one in Lemma 4.2 by taking $r = r$, $t = t$ and $k' = k'$. Let $\gamma = \beta + (\ell r + 1)\alpha$. Define $d = (1 + (1 + \gamma)^{(r+1)^\ell})k$.

Let G be a graph. Assume that the average degree of G is at most k, and assume that there exists no graph H of average degree greater than k' such that some subgraph of G is isomorphic to a $[2\ell + 1]$-subdivision of H. Assume that G does not contain $K_r \vee I_t$ as an ℓ-shallow minor.

For any $Y \subseteq V(G)$ and $v \in V(G) - Y$, we define the *Y-correlation of v* to be the sequence $(a_0, a_1, a_2, ..., a_{\ell-1})$, where $a_i = |Y \cap N_G(N_{G-Y}^{\leq i}[v])|$ for each i with $0 \leq i \leq \ell - 1$. Note that if some entry of the Y-correlation of a vertex v is at least r, then there exists a (v, Y, ℓ, r)-span. Observe that the Y-correlation of v is an empty sequence when $\ell = 0$.

Let X_0 be the set of the vertices of G of degree at most d, and let $Y_0 = V(G) - X_0$. We use the following iterative procedure for each step $i \geq 0$ to define the vertex partition $V(G) = X_i \cup Y_i$. For each $i \geq 0$, we define the following.

- Define $\mathcal{C}_i$ to be a maximal collection of pairwise disjoint subgraphs of $G[X_i]$, where each member of $\mathcal{C}_i$ is a minimal (v, Y_i, ℓ, r)-span for some vertex $v \in X_i$ satisfying that if $\ell \geq 1$, then $|Y_i \cap N_G(N_{G-Y_i}^{\leq \ell-1}[v])| \geq r$.

- $D_i = \bigcup_{H \in \mathcal{C}_i} V(H)$.
- Z_i is a maximal subset of $X_i - D_i$ such that
  - for any two distinct vertices in Z_i, their distance in $G[X_i]$ is at least $\ell + 1$, and
  - for every $z \in Z_i$, $N_{G[X_i - D_i]}^{\leq \ell-1}[z] \cap N_G(D_i) = \emptyset$.
- $X_{i+1} = X_i - (Z_i \cup D_i)$.
- $Y_{i+1} = Y_i \cup Z_i \cup D_i$.

Note that $\{X_i, Y_i\}$ is a partition of $V(G)$ for every $i \geq 0$, where X_i or Y_i is possibly empty.

Claim 4.4.1. *For every nonnegative integer i and $z \in Z_i$,*

- $|N_G(N_{G[X_i]}^{\leq \ell-1}[z]) - X_i| \leq r - 1$, *and*
- *if u is a vertex in X_i such that the distance in $G[X_i]$ from z to u is at most ℓ, then $|N_G(u) - X_i| \leq r - 1$.*

Proof of Claim 4.4.1: We first suppose $N_{G[X_i]}^{\leq \ell-1}[z] \neq N_{G[X_i-D_i]}^{\leq \ell-1}[z]$. So there exists $v' \in N_{G[X_i]}^{\leq \ell-1}[z] - N_{G[X_i-D_i]}^{\leq \ell-1}[z]$. This means that there exists a path in $G[X_i]$ of length at most $\ell - 1$ from z to v' intersecting D_i. Hence there exists $v \in X_i$ such that there exists a path P_v in $G[X_i]$ of length at most $\ell - 1$ from z to v intersecting D_i. We may assume that v is chosen such that $|V(P_v)|$ is as small as possible. Hence P_v is internally disjoint from D_i. Since $z \in Z_i$, $z \notin D_i$. So $v \in D_i$ and P_v contains at least two vertices. Let v'' be the neighbor of v in P_v. Since P_v is internally disjoint from D_i and $z \notin D_i$, it follows that $v'' \in N_{G[X_i-D_i]}^{\leq \ell-1}[z] \cap N_G(v) \subseteq N_{G[X_i-D_i]}^{\leq \ell-1}[z] \cap N_G(D_i)$. However, since $z \in Z_i$, by the definition of Z_i, $N_{G[X_i-D_i]}^{\leq \ell-1}[z] \cap N_G(D_i) = \emptyset$, a contradiction.

Hence

$$N_{G[X_i]}^{\leq \ell-1}[z] = N_{G[X_i-D_i]}^{\leq \ell-1}[z]. \tag{3}$$

So $N_{G[X_i]}^{\leq \ell-1}[z] \cap N_G(D_i) = \emptyset$. Since $z \notin D_i$, $N_{G[X_i]}^{\leq \ell-1}[z] \cap D_i = \emptyset$.

Suppose $|N_G(N_{G[X_i]}^{\leq \ell-1}[z]) - X_i| \geq r$. Since $\{X_i, Y_i\}$ is a partition of $V(G)$, $|N_G(N_{G-Y_i}^{\leq \ell-1}[z]) \cap Y_i| = |N_G(N_{G[X_i]}^{\leq \ell-1}[z]) \cap Y_i| \geq r$. Therefore $G[N_{G-Y_i}^{\leq \ell-1}[z]] = G[N_{G[X_i]}^{\leq \ell-1}[z]]$ is a (z, Y_i, ℓ, r)-span in G; it is disjoint from members in $\mathcal{C}_i$ since $N_{G[X_i]}^{\leq \ell-1}[z] \cap D_i = \emptyset$. Hence there exists a minimal (z, Y_i, ℓ, r)-span H' in G such that $V(H') \subseteq N_{G[X_i]}^{\leq \ell-1}[z]$. But $V(H') \cap D_i \subseteq N_{G[X_i]}^{\leq \ell-1}[z] \cap D_i = \emptyset$, contradicting the maximality of $\mathcal{C}_i$.

Therefore, $|N_G(N_{G[X_i]}^{\leq \ell-1}[z]) - X_i| \leq r - 1$.

Let u be a vertex in X_i such that the distance in $G[X_i]$ from z to u is at most ℓ. Suppose $u \in D_i$. Since $z \notin D_i$, there exists a vertex $u' \in X_i \cap N_G(u)$ such that the distance in $G[X_i]$ from z to u' is at most $\ell - 1$. So $u' \in N_{G[X_i]}^{\leq \ell-1}[z] \cap N_G(u) \subseteq N_{G[X_i]}^{\leq \ell-1}[z] \cap N_G(D_i) = N_{G[X_i-D_i]}^{\leq \ell-1}[z] \cap N_G(D_i)$ by (3). But $z \in Z_i$, so $N_{G[X_i-D_i]}^{\leq \ell-1}[z] \cap N_G(D_i) = \emptyset$, a contradiction.

Hence $u \notin D_i$. If $|N_G(u) - X_i| \geq r$, then the graph consisting of the vertex u is a minimal $(u, Y_i, 0, r)$-span (and hence a minimal (u, Y_i, ℓ, r)-span), so u is contained in D_i by the maximality of $\mathcal{C}_i$, a contradiction. So $|N_G(u) - X_i| \leq r - 1$. $\square$

If there exists a nonnegative integer i^* such that $|Z_{i^*}| > \beta|V(G) - X_{i^*}|$, then by defining $X = X_{i^*}$ and $Z = Z_{i^*}$, we know that $|Z| > \beta|V(G) - X|$, and every vertex in $X \subseteq X_0$ has degree at most d in G; statements 4(b)-4(d) follow from the definition of Z_{i^*} and Claim 4.4.1, so Statement 4 holds.

So we may assume that $|Z_i| \leq \beta|V(G) - X_i| = \beta|Y_i|$ for every nonnegative integer i.

Since $X_{i+1} \subseteq X_i$, we have for any $0 \leq j \leq \ell - 1$,

$$N^{\leq j}_{G[X_{i+1}]}[v] \subseteq N^{\leq j}_{G[X_i]}[v]. \tag{4}$$

Note that if there exists an integer j with $0 \leq j \leq \ell - 1$ such that $N^{\leq j}_{G[X_{i+1}]}[v] = N^{\leq j}_{G[X_i]}[v]$, then it is easy to see that for every $u \in N^{\leq j}_{G[X_{i+1}]}[v]$, the distance between u and v in $G[X_{i+1}]$ is the same as the distance between u and v in $G[X_i]$ by induction on the distance between u and v in $G[X_{i+1}]$, and hence for every integer j' with $0 \leq j' \leq j$, $N^{\leq j'}_{G[X_{i+1}]}[v] = N^{\leq j'}_{G[X_i]}[v]$.

Claim 4.4.2. *Let i be a nonnegative integer, and let v be a vertex in X_{i+1}. Denote the Y_{i+1}-correlation of v by $(a_0, a_1, ..., a_{\ell-1})$, and denote the Y_i-correlation of v by $(b_0, b_1, ..., b_{\ell-1})$. If there exists an integer $k \leq \ell - 1$ such that $N^{\leq k}_{G[X_{i+1}]}[v] \subsetneq N^{\leq k}_{G[X_i]}[v]$, and $N^{\leq j}_{G[X_{i+1}]}[v] = N^{\leq j}_{G[X_i]}[v]$ for every $0 \leq j < k$, then $(a_0, a_1, \ldots, a_{k-1})$ is strictly greater than $(b_0, b_1, \ldots, b_{k-1})$ in the lexicographic order.*

Proof of Claim 4.4.2: Since $N^{\leq k}_{G[X_{i+1}]}[v] \subsetneq N^{\leq k}_{G[X_i]}[v]$, $k \geq 1$.

Since $X_{i+1} \subseteq X_i$, $Y_i \subseteq Y_{i+1}$. By the condition of this claim, $N_G(N^{\leq j}_{G[X_i]}[v]) = N_G(N^{\leq j}_{G[X_{i+1}]}[v])$ for every integer j with $0 \leq j \leq k - 1$. So for every j with $0 \leq j \leq k - 1$, $Y_i \cap N_G(N^{\leq j}_{G[X_i]}[v]) \subseteq Y_{i+1} \cap N_G(N^{\leq j}_{G[X_{i+1}]}[v])$, and hence $a_j \geq b_j$.

Let u be an arbitrary vertex in $N^{\leq k}_{G[X_i]}[v] - N^{\leq k}_{G[X_{i+1}]}[v]$. By the condition of this claim, $N^{\leq k-1}_{G[X_i]}[v] = N^{\leq k-1}_{G[X_{i+1}]}[v]$. So

$$u \in N^{\leq k}_{G[X_i]}[v] = N_{G[X_i]}[N^{\leq k-1}_{G[X_i]}[v]] = N_{G[X_i]}[N^{\leq k-1}_{G[X_{i+1}]}[v]]. \tag{5}$$

Hence the distance between u and v in $G[X_{i+1} \cup \{u\}]$ is at most k. Since $u \notin N^{\leq k}_{G[X_{i+1}]}[v]$, $u \in X_i - X_{i+1} = Y_{i+1} - Y_i$. Together with (5), $u \in Y_{i+1} \cap N_G(N^{\leq k-1}_{G[X_{i+1}]}[v]) - Y_i$.

Since $N_G(N^{\leq k-1}_{G[X_i]}[v]) = N_G(N^{\leq k-1}_{G[X_{i+1}]}[v])$, we have $Y_i \cap N_G(N^{\leq k-1}_{G[X_i]}[v]) \subseteq Y_{i+1} \cap N_G(N^{\leq k-1}_{G[X_{i+1}]}[v])$. Recall that $a_{k-1} = |Y_{i+1} \cap N_G(N^{\leq k-1}_{G[X_{i+1}]}[v])|$ and $b_{k-1} = |Y_i \cap N_G(N^{\leq k-1}_{G[X_i]}[v])|$. Since $u \in Y_{i+1} \cap N_G(N^{\leq k-1}_{G[X_{i+1}]}[v]) - Y_i$, $a_{k-1} > b_{k-1}$. Therefore, $(a_0, a_1, ..., a_{k-1}) > (b_0, b_1, ..., b_{k-1})$. $\square$

Claim 4.4.3. *Let i be a nonnegative integer, and let v be a vertex in X_{i+1}. Denote the Y_{i+1}-correlation of v by $(a_0, a_1, ..., a_{\ell-1})$, and denote the Y_i-correlation of v by $(b_0, b_1, ..., b_{\ell-1})$. If $\ell \geq 1$ and $N^{\leq j}_{G[X_{i+1}]}[v] = N^{\leq j}_{G[X_i]}[v]$ for every integer j with $0 \leq j \leq \ell - 1$, then $(a_0, a_1, \ldots, a_{\ell-1})$ is strictly greater than $(b_0, b_1, \ldots, b_{\ell-1})$ in the lexicographic order.*

Proof of Claim 4.4.3: Since $Y_{i+1} \supseteq Y_i$, and for every integer j with $0 \leq j \leq \ell - 1$, $N^{\leq j}_{G[X_{i+1}]}[v] = N^{\leq j}_{G[X_i]}[v]$, we have $a_j \geq b_j$ for every j with $0 \leq j \leq \ell - 1$.

Since $v \in X_{i+1}$, $v \notin Z_i \cup D_i$ by the definition of X_{i+1}. Assume that there exists $v' \in Y_{i+1} - Y_i = X_i - X_{i+1}$ such that the distance in $G[X_i]$ between v and v' is ℓ' for some $0 \leq \ell' \leq \ell$, then $v' \in \Big(Y_{i+1} \cap N_G(N^{\leq \ell'-1}_{G[X_{i+1}]}[v])\Big) - \Big(Y_i \cap N_G(N^{\leq \ell'-1}_{G[X_i]}[v])\Big)$, so $a_{\ell'-1} > b_{\ell'-1}$. Recall that $a_j \geq b_j$ for every j with $0 \leq j \leq \ell - 1$, so $(a_0, a_1, ..., a_{\ell-1}) > (b_0, b_1, ..., b_{\ell-1})$, and hence the claim follows.

Hence we may assume that there is no $v' \in Y_{i+1} - Y_i = X_i - X_{i+1}$ such that the distance in $G[X_i]$ between v and v' is at most ℓ. Equivalently, $N^{\leq \ell}_{G[X_i]}[v] \cap Y_{i+1} - Y_i = \emptyset$. Since $D_i \cup Z_i = Y_{i+1} - Y_i$, $N^{\leq \ell}_{G[X_i]}[v] \cap (D_i \cup Z_i) = \emptyset$. Therefore any vertex in X_i whose distance to v in $G[X_i]$ is at most ℓ is not in $D_i \cup Z_i$. In other words, for any $j \leq \ell$, we have $N^{\leq j}_{G[X_i]}[v] = N^{\leq j}_{G[X_i-(D_i \cup Z_i)]}[v]$. Thus $N^{\leq \ell-1}_{G[X_{i+1}]}[v] = N^{\leq \ell-1}_{G[X_i-(D_i \cup Z_i)]}[v] = N^{\leq \ell-1}_{G[X_i-D_i]}[v]$. Since $v \in X_{i+1}$, $v \notin Z_i$. Since $N^{\leq \ell}_{G[X_i]}[v] \cap (D_i \cup Z_i) = \emptyset$, by the maximality of Z_i, $N^{\leq \ell-1}_{G[X_i-D_i]}[v] \cap N_G(D_i) \neq \emptyset$. So there exists $x \in N^{\leq \ell-1}_{G[X_i-D_i]}[v] \cap N_G(D_i) = N^{\leq \ell-1}_{G[X_{i+1}]}[v] \cap N_G(D_i)$. Hence there exists $y \in D_i \cap N_G(x)$. Since $y \in N_G(x)$, $y \in N_G(N^{\leq \ell-1}_{G[X_{i+1}]}[v])$. Since $y \in D_i$, $y \in Y_{i+1} - Y_i$. So $(Y_{i+1} - Y_i) \cap N_G(N^{\leq \ell-1}_{G[X_{i+1}]}[v]) \neq \emptyset$. Therefore, $a_{j^*} > b_{j^*}$ for some j^* with $0 \leq j^* \leq \ell - 1$. Recall that $a_j \geq b_j$ for every j with $0 \leq j \leq \ell - 1$. So $(a_0, a_1, ..., a_{\ell-1}) > (b_0, b_1, ..., b_{\ell-1})$. $\square$

Claim 4.4.4. *Let i be a nonnegative integer, and let v be a vertex in X_{i+1}. Denote the Y_{i+1}-correlation of v by $(a_0, a_1, ..., a_{\ell-1})$, and denote the Y_i-correlation of v by $(b_0, b_1, ..., b_{\ell-1})$. If there exists k with $0 \leq k \leq \ell-1$ such that $b_k \geq r$ and $b_j < r$ for every $0 \leq j \leq k-1$, then $(a_0, a_1, ..., a_{k-1})$ is strictly greater than $(b_0, b_1, ..., b_{k-1})$ in the lexicographic order.*

Proof of Claim 4.4.4: If there exists an integer j^* with $0 \leq j^* \leq k$ such that $N^{\leq j^*}_{G[X_{i+1}]}[v] \subsetneq N^{\leq j^*}_{G[X_i]}[v]$, then by Claim 4.4.2, $(a_0, a_1, ..., a_{k-1})$ is strictly greater than $(b_0, b_1, ..., b_{k-1})$. So by (4), we may assume that $N^{\leq j}_{G[X_{i+1}]}[v] = N^{\leq j}_{G[X_i]}[v]$ for every j with $0 \leq j \leq k$. In particular, $N^{\leq k}_{G[X_{i+1}]}[v] = N^{\leq k}_{G[X_i]}[v]$.

Since $Y_i \subseteq Y_{i+1}$, for any j with $0 \leq j \leq k-1$, $N_G(N^{\leq j}_{G[X_i]}[v]) \cap Y_i = N_G(N^{\leq j}_{G[X_{i+1}]}[v]) \cap Y_i \subseteq N_G(N^{\leq j}_{G[X_{i+1}]}[v]) \cap Y_{i+1}$. So $a_j \geq b_j$ for every j with $0 \leq j \leq k-1$.

Since $b_k \geq r$, there exists a minimal (v, Y_i, k, r)-span Q in $G[X_i]$ with $V(Q) \subseteq N^{\leq k}_{G[X_i]}[v]$. Since $v \in X_{i+1}$ and $v \in V(Q)$, we have $Q \notin \mathcal{C}_i$. By the maximality of $\mathcal{C}_i$, there exists a member M of $\mathcal{C}_i$ intersecting Q. Together with the fact that $V(M) \subseteq D_i \subseteq Y_{i+1} - Y_i$, we have $V(M) \cap V(Q) \subseteq N^{\leq k}_{G[X_i]}[v] = N^{\leq k}_{G[X_{i+1}]}[v] \subseteq N_G(N^{\leq k-1}_{G[X_{i+1}]}[v])$ and $V(M) \cap V(Q) \subseteq Y_{i+1} - Y_i$. So $\emptyset \neq V(M) \cap V(Q) \subseteq N_G(N^{\leq k-1}_{G[X_{i+1}]}[v]) \cap Y_{i+1} - (N_G(N^{\leq k-1}_{G[X_i]}[v]) \cap Y_i)$. Hence $a_{k-1} > b_{k-1}$. Therefore, $(a_0, a_1, ..., a_{k-1})$ is strictly greater than $(b_0, b_1, ..., b_{k-1})$. $\square$

Claim 4.4.5. $X_0 \subseteq Y_{(r+1)^\ell}$.

Proof of Claim 4.4.5: We first assume that $\ell = 0$. Then for every two distinct vertices in $X_0 - D_0$, their distance in $G[X_0]$ is at least $1 = \ell + 1$. And for every $z \in X_0 - D_0$, $N^{\leq \ell-1}_{G[X_0-D_0]}[z] = N^{\leq -1}_{G[X_0-D_0]}[z] = \emptyset$. So $Z_0 = X_0 - D_0$. Hence $Z_0 \cup D_0 = X_0$. So $X_0 \subseteq Y_1 = Y_{(r+1)^0} = Y_{(r+1)^\ell}$.

Hence we may assume $\ell \geq 1$. Let $v \in X_0$. We shall show that there exists a nonnegative integer i_v such that $v \in Y_{i_v}$ and show $i_v \leq (r+1)^\ell$.

For each nonnegative integer i, if v is in X_i, then let $a^{(i)} = (a^{(i)}_0, a^{(i)}_1, \ldots, a^{(i)}_{\ell-1})$ be the Y_i-correlation of v. By Claims 4.4.2 and 4.4.3 and (4), for every nonnegative integer i, if $v \in X_{i+1}$, then $a^{(i+1)} > a^{(i)}$ in the lexicographic order. So if $v \in X_{i+1}$, then one entry in $a^{(i)}$ will increase its value by at least one. By Claim 4.4.4, if $v \in X_{i+1}$ and there exists j with $0 \leq j \leq \ell - 1$ such that the entry $a^{(i)}_j \geq r$ while $a^{(i)}_{j'} < r$ for all $0 \leq j' < j$, then $a^{(i+1)}_{j'} > a^{(i)}_{j'}$ for some $j' < j$.

Therefore, there exists a nonnegative integer i_v with $i_v \leq r \cdot (r+1)^{\ell-1}$ such that either $v \in Y_{i_v}$ or $a^{(i_v)}_0 \geq r$. Note that if $v \notin Y_{i_v}$, then $a^{(i_v)}_0 \geq r$, so $|N_G(v) \cap Y_{i_v}| \geq r$. So when $v \notin Y_{i_v}$, the graph consists of the vertex v is a $(v, Y_{i_v}, 0, r)$-span, so v is contained in some member of $\mathcal{C}_{i_v}$ by the maximality of $\mathcal{C}_{i_v}$, and hence $v \in Y_{i_v+1} \subseteq Y_{(r+1)^\ell}$. Therefore, $X_0 \subseteq Y_{(r+1)^\ell}$. $\square$

Recall that we assume $|Z_i| \leq \beta |Y_i|$ for every nonnegative integer i. By Lemma 4.2, $|\mathcal{C}_i| \leq \alpha |Y_i|$ for every nonnegative integer i. For every nonnegative integer i, since each member T of $\mathcal{C}_i$ is a minimal (v, Y_i, ℓ, r)-span, it contains at most $\ell r + 1$ vertices by Lemma 4.3. So for every nonnegative integer i,

$$|Y_{i+1} - Y_i| = |Z_i| + \sum_{T \in \mathcal{C}_i} |V(T)|$$
$$\leq |Z_i| + |\mathcal{C}_i| \cdot (\ell r + 1) \leq (\beta + \alpha \cdot (\ell r + 1))|Y_i| = \gamma |Y_i|.$$

Hence $|Y_{i+1}| \leq (1+\gamma)|Y_i|$ for every nonnegative integer i. Therefore, $|Y_i| \leq (1+\gamma)^i |Y_0|$ for every nonnegative i. By Claim 4.4.5, $|X_0| \leq |Y_{(r+1)^\ell}| \leq (1+\gamma)^{(r+1)^\ell}|Y_0|$. Since $V(G) = X_0 \cup Y_0$,

$$|Y_0| \geq \frac{1}{1 + (1+\gamma)^{(r+1)^\ell}} |V(G)|.$$

Therefore, $\sum_{v \in V(G)} \deg_G(v) \geq \sum_{v \in Y_0} \deg_G(v) > d|Y_0| \geq \frac{d}{1+(1+\gamma)^{(r+1)^\ell}} |V(G)| = k|V(G)|$, which implies that the average degree of G is greater than k, a contradiction. This proves the lemma. □

We remark that the main lemma in the work of Ossona de Mendez, Oum, and Wood [35, Lemma 2.2] is implied by the case $(\ell, \beta) = (0, 0)$ of Lemma 4.4 (up to the constant d).

5 Existence of good collections

In this section we prove Lemmas 5.1 and 5.4, which will provide the correct value q for Lemma 2.1.

Lemma 5.1. *For every positive integer r and graph H, there exists a constant $c = c(r, H) > 0$ such that the following holds. For every H-minor free graph G, there exists a collection $\mathcal{C}$ of r-element subsets of $E(G)$ with $|\mathcal{C}| \leq c|V(G)|$ such that for every subgraph R of G of minimum degree at least r, some member of $\mathcal{C}$ is a subset of $E(R)$.*

Proof. Let r be a positive integer and let H be a graph. By [30], there exists a real number k such that every graph of average degree at least k contains H as a minor. Define $c = \binom{k}{r}$.

Let G be an H-minor free graph. Since G has no H-minor, the average degree of G is less than k. So there exists a vertex z^* of G of degree less than k. Let $Z = \{z^*\}$. Then this lemma immediately follows from Lemma 2.3 by taking $a = k$, $t = 1$ and $\xi = 1$. □

The rest of this section dedicates a proof of Lemma 5.4.

Recall that for graphs G and H and a nonnegative integer r, $\mathcal{F}(G, H, r)$ is the set consisting of the graphs that can be obtained from a disjoint union of G and H by adding edges between $V(G)$ and $V(H)$ such that every vertex in $V(H)$ has degree at least r. For a graph W in $\mathcal{F}(G, H, r)$, the *type* of W is the number of edges of W incident with $V(H)$, and the *heart* of W is $V(G)$.

We need the following lemmas.

Lemma 5.2. *For any $r, t, t' \in \mathbb{N}$, $w \in \mathbb{Z}$ with $r \geq w \geq 0$, nonnegative integer s_0 and positive real numbers k, k', there exists an integer d such that for every graph G, either*

1. *the average degree of G is greater than k,*
2. *there exists a graph H of average degree greater than k' such that some subgraph of G is isomorphic to a $[4s_0 + 2w + 5]$-subdivision of H,*
3. *G contains $K_{r-w+1} \vee I_t$ as a $(2s_0 + w + 2)$-shallow minor, or*

4. *there exists $X \subseteq V(G)$ such that*

   (a) *every vertex in X has degree at most d in G,*

   (b) *there exists $v^* \in X$ such that for every subgraph R of G of minimum degree at least r containing v^*, there exists a path in $G[X \cap V(R)]$ of length w starting at v^*, and*

   (c) *either $X = V(G)$, or there exists a nonnegative integer s with $s \leq s_0$ such that either*

      i. *there exists a connected graph F_0 such that G contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s, where I is the heart of F, or*

      ii. *there exists $x^* \in X$ such that for every subgraph R of G of minimum degree at least r containing x^*, there exists a connected subgraph F of $R[X \cap V(R)]$ containing x^* such that the number of edges in R incident with $V(F)$ is at least $s_0 + 1$.*

Proof. Let $r, t, t' \in \mathbb{N}$, $w \in \mathbb{Z}$ with $r \geq w \geq 0$, $s_0 \in \mathbb{Z}$ with $s_0 \geq 0$, and k, k' be positive real numbers. Let $\beta = (s_0+1)^2 \cdot 2^{\binom{s_0+1}{2}} \cdot (r-w+1)(t'-1+r-w)\left(\frac{k'}{2} + \binom{k'}{\lfloor k'/2 \rfloor} \cdot t' \cdot 2^{(r-w)(s_0+1)}\right) t' \cdot 2^{(r-w)(s_0+1)}$. Define d to be the integer mentioned in Lemma 4.4 by taking $(r, t, \ell, k, k', \beta) = (r-w+1, t, 2s_0+w+2, k, k', \beta)$.

Let G be a graph. Suppose that Statements 1-3 of this lemma do not hold. So by Lemma 4.4, there exist $X, Z \subseteq V(G)$ with $Z \subseteq X$ and $|Z| > \beta |V(G) - X|$ such that

(i) every vertex in X has degree at most d in G,

(ii) for any distinct pair of vertices in Z, the distance in $G[X]$ between them is at least $2s_0+w+3$,

(iii) for every $z \in Z$ and $u \in X$ whose distance from z in $G[X]$ is at most $2s_0+w+2$, $|N_G(u)-X| \leq r-w$, and

(iv) $|N_G(N^{\leq 2s_0+w+1}_{G[X]}[z]) - X| \leq r-w$ for every $z \in Z$.

We shall prove that Statement 4 of this lemma holds. Statement 4(a) immediately follows from (i).

We first prove Statement 4(b). Let v^* be any vertex in Z. Suppose to the contrary that there exists a subgraph R of G of minimum degree at least r containing v^* such that the longest path P in $R[X \cap V(R)]$ starting at v^* has length at most $w-1$. For every vertex $v \in V(P)$, $|N_R(v) \cap X - V(P)| \geq |N_R(v)| - |N_R(v) - X| - |N_R(v) \cap V(P)| \geq |N_R(v)| - |N_G(v) - X| - (|V(P)| - 1) \geq r - (r-w) - (w-1) = 1$ where the last inequality follows from (iii) by taking (z, u) in (iii) to be (v^*, v). So P is not a longest path in R starting at v^* since if v is the other end of P, then we can extend P by concatenating a vertex in $N_R(v) \cap X - V(P)$. This leads to a contradiction. Since $R[X \cap V(R)] \subseteq G[X \cap V(R)]$, Statement 4(b) is proved.

Now we prove Statement 4(c). We may assume that $X \neq V(G)$, for otherwise we are done. Assume 4(c)ii does not hold. We shall show 4(c)i holds.

For every $z \in Z$ and every subgraph R of G of minimum degree at least r containing z, define $s_{R,z}$ to be the number of edges of R incident with the vertices in the component of $R[V(R) \cap X]$ containing z. For every $z \in Z$, define $s'_z = \min_R s_{R,z}$, where the minimum is taken over all subgraphs R of G of minimum degree at least r containing z. Note that for every $z \in Z$, $s'_z \geq r$ as the minimum is taken over all subgraphs of minimum degree at least r. If there exists $z \in Z$ such that $s'_z \geq s_0 + 1$, then Statement 4(c)ii holds by taking $x^* = z$.

So we may assume that $s'_z \leq s_0$ for every $z \in Z$. Define s to be an integer with $0 \leq s \leq s_0$ such that $|\{z \in Z : s'_z = s\}|$ is maximum. Let $Z_s = \{z \in Z : s'_z = s\}$. In particular,

$$|Z_s| \geq \frac{1}{s_0+1}|Z| > \frac{\beta}{s_0+1}|V(G) - X|. \tag{6}$$

If there is a vertex $z \in Z_s$ such that for every subgraph R of G of minimum degree at least r containing z with $s_{R,z} = s'_z = s$, the connected component $F_{R,z}$ of $R[V(R) \cap X]$ containing z contains at least $s_0 + 2$ vertices, then for every such R, the number of edges in R incident with $V(F_{R,z})$ is at least $s_0 + 1 \geq s + 1 = s_{R,z} + 1$, a contradiction. So for every $z \in Z_s$, there exists a subgraph R_z of G of minimum degree at least r containing z with $s_{R_z,z} = s$ such that the component F_z of $R_z[V(R_z) \cap X]$ containing z satisfies that

$$|V(F_z)| \leq s_0 + 1. \tag{7}$$

Since there are at most $(s_0+1) \cdot 2^{\binom{s_0+1}{2}}$ non-isomorphic labelled graphs on at most $s_0 + 1$ vertices, there exist a connected (labelled) graph F on at most $s_0 + 1$ vertices and $Z'_s \subseteq Z_s$ with

$$|Z'_s| \geq \frac{|Z_s|}{(s_0+1) \cdot 2^{\binom{s_0+1}{2}}} > \frac{\beta}{(s_0+1)^2 \cdot 2^{\binom{s_0+1}{2}}} |V(G) - X|$$

such that F is isomorphic to each (labelled) F_z for every $z \in Z'_s$, where the we use (6) for the second inequality.

For every $z \in Z'_s$, since F_z is connected and contains at most $s_0 + 1$ vertices,

$$V(F_z) \subseteq N^{\leq s_0}_{G[X]}[z]. \tag{8}$$

By (iv), for every $z \in Z'_s$, $|N_G(N^{\leq s_0}_{G[X]}[z]) - X| \leq r - w$. So there exist an integer p with $0 \leq p \leq r - w$ and a set $Z^*_s \subseteq Z'_s$ with

$$\begin{aligned} |Z^*_s| \geq \frac{|Z'_s|}{r-w+1} &> \frac{\beta}{(s_0+1)^2 \cdot 2^{\binom{s_0+1}{2}} \cdot (r-w+1)} |V(G) - X| \\ &\geq \frac{\beta}{(s_0+1)^2 \cdot 2^{\binom{s_0+1}{2}} \cdot (r-w+1)} \geq t' - 1 + r - w \end{aligned} \tag{9}$$

such that $|N_G(V(F_z)) - X| = p$ for every $z \in Z^*_s$.

A quick remark is that, by (ii), for distinct vertices z_1, z_2 in Z'_s, $N^{\leq s_0}_{G[X]}[z_1]$ and $N^{\leq s_0}_{G[X]}[z_2]$ are disjoint. Together with (8), we have that

$$V(F_z) \cap V(F_{z'}) = \emptyset. \tag{10}$$

We first assume that $p = 0$. Then for every $z \in Z^*_s$, F_z is of minimum degree at least r since R is of minimum degree at least r and $N_G(V(F_z)) \subseteq X$. Since $|Z^*_s| \geq t' + r - w$, the graphs F_z for $z \in Z^*_s$ form at least $r - w + t'$ disjoint copies of F in G. We just showed that F is of minimum degree at least r. Let F' be a disjoint union of F and $r - w$ isolated vertices. Then $F' \in \mathcal{F}(I_{r-w}, F, r)$ and is of type s. Since G contains $r - w + t'$ disjoint copies of F, we know G contains $F' \wedge_{t'} I$ where I is the heart of F', as we can take t' disjoint copies of F and one vertex in each of other $r - w$ copies of F. So Statement 4(c)i holds.

So we may assume that $p \geq 1$. Recall that by the definition of Z^*_s, $|N_G(V(F_z)) - X| = p$ for every $z \in Z^*_s$.

Claim 5.2.1. *If there is a subset $S \subseteq V(G) - X$ such that S equals $N_G(V(F_z)) - X$ for at least $t' \cdot 2^{p(s_0+1)}$ vertices $z \in Z^*_s$, then Statement 4(c) i holds.*

Proof of Claim 5.2.1: For every $z \in Z_s^*$, since $|N_G(V(F_z)) - X| = p$, and each of the copies F_z are isomorphic (as a labelled graph), there are at most $2^{|V(F_z)|p} \leq 2^{(s_0+1)p}$ possibilities for how vertices in F_z are connected in G to the p vertices in the set $N_G(V(F_z)) - X$ by (7) and the fact that there are $|V(F_z)|p$ potential egdes between vertices in F_z and $N_G(V(F_z)) - X$.

Notice that each vertex in F_z has degree at least r in $G[N_G[V(F_z)]]$. By a piegon-hole argument, if S is a subset of $V(G) - X$ such that S equals $N_G(V(F_z)) - X$ for at least $t' \cdot 2^{p(s_0+1)}$ vertices $z \in Z_s^*$, then there are at least t' vertices $z \in Z_s^*$ such that the graphs $G[S \cup V(F_z)] - E[S]$, denoted by F'_z, are isomorphic to a graph F' as a labeled graph. Let F_0 be F_z for one of these t' vertices $z \in Z_s^*$. Then $F' \in \mathcal{F}(I_p, F_0, r)$ and the union of F'_z among these t' vertices in Z_s^* is a subgraph G' of G isomorphic to $F' \wedge_{t'} I$, where I is the stable set corresponding to $V(I_p)$. Let G'' be the union of G' and $r - w - p$ vertices in the remaining $|Z_s^*| - t' \geq r - w$ vertices in Z_s^*. Then G'' is isomorphic to $F'' \wedge_{t'} I''$ for some $F'' \in \mathcal{F}(I_{r-w}, F_0, r)$, where I'' is the union of I and the new $r - w - p$ vertices. Therefore Statement 4(c)i holds. □

Claim 5.2.2. *If Statement 4(c)i does not hold, then*

$$|\{N_G(V(F_z)) - X : z \in Z_s^*\}| \leq \left(\frac{k'}{2} + \binom{k'}{\lfloor k'/2 \rfloor} \cdot t' \cdot 2^{p(s_0+1)}\right) |V(G) - X|.$$

Proof of Claim 5.2.2: By (10), $V(F_z) \cap V(F_{z'}) = \emptyset$ for distinct vertices z_1, z_2 in Z_s^*. Starting from $G[\bigcup_{z \in Z_s^*} V(F_z) \cup (V(G) - X)] - E(G[V(G) - X])$, we obtain a graph H' by repeatedly deleting all the vertices in $V(F_z)$ for some $z \in Z_s^*$ where some pair of distinct vertices y, y' in $N_G(V(F_z)) - X$ are non-adjacent in the current graph, and adding the edge yy'. We continue this process until for every remaining vertex z' in Z_s^*, $N_G(V(F_z)) - X$ is a clique.

Let $H = H'[V(G) - X]$. Since $p \geq 1$ and $V(F_z) \subseteq N_{G[X]}^{\leq s_0}[z]$ by (10) for every $z \in Z_s^*$ (which implies any two vertices in F_z can be connected in F_z by a path of length at most $2s_0$), we know $G[\bigcup_{z \in Z_s^*}((N_G(V(F_z)) - X) \cup V(F_z))]$ contains a $[2s_0 + 1]$-subdivision of H. It implies that G contains a $[2s_0 + 1]$-subdivision of any subgraph of H. Since Statement 2 of this lemma does not hold, the average degree of any subgraph of H is at most k'.

For each vertex $z \in Z_s^*$, either $V(F_z)$ has been deleted thus corresponding to a unique edge in H, or $V(F_z)$ survives in H', in which case $N_G(V(F_z)) - X$ becomes a clique of size $|N_G(V(F_z)) - X| = p$ in H', and thus also a clique of size p in H since $N_G(V(F_z)) - X \subseteq V(G) - X$. There are at most $|E(H)|$ vertices in Z_s^* of the first kind. Since the maximum average degree of H is at most k', $|E(H)| \leq \frac{k'|V(H)|}{2} = \frac{k'}{2}|V(G) - X|$.

For the vertices in Z' of the second kind, note that $N_G(V(F_z)) - X$ is a clique of size p in H. Let c be the number of vertices in Z_s^* of the second kind. By Claim 5.2.1, each $S \subseteq V(G) - X$ is the neighborhood of at most $t' \cdot 2^{p(s_0+1)}$ vertices $z \in Z_s^*$ of the second kind. Since each $z \in Z_s^*$ gives a clique of size p in H and by Lemma 4.1, the number of cliques of size p in H is at most $\binom{k'}{p-1}|V(G) - X| \leq \binom{k'}{\lfloor k'/2 \rfloor}|V(G) - X|$. Combining these two facts, we have $c \leq \binom{k'}{\lfloor k'/2 \rfloor} \cdot t' \cdot 2^{p(s_0+1)}|V(G) - X|$. Therefore,

$$|\{N_G(V(F_z)) - X : z \in Z_s^*\}| \leq |E(H)| + c \leq \left(\frac{k'}{2} + \binom{k'}{\lfloor k'/2 \rfloor} \cdot t' \cdot 2^{p(s_0+1)}\right) |V(G) - X|.$$

□

By Claim 5.2.2, the number of distinct sets of the form $N_G(V(F_z)) - X$ for some $z \in Z_s^*$ is at most $\left(\frac{k'}{2} + \binom{k'}{\lfloor k'/2 \rfloor} \cdot t' \cdot 2^{p(s_0+1)}\right) |V(G) - X|$. However, by (9), $|Z_s^*| > \frac{\beta}{(s_0+1)^2 \cdot 2^{\binom{s_0+1}{2}} \cdot (r-w+1)} \cdot |V(G) - X|$. Therefore there is a subset $S \subseteq V(G) - X$ with $|S| = p \leq r - w$ such that there are at least

$\left(\frac{\beta}{(s_0+1)^2\cdot 2^{\binom{s_0+1}{2}}\cdot(r-w+1)}\right) / \left(\frac{k'}{2}+\binom{k'}{\lfloor k'/2\rfloor}\cdot t'\cdot 2^{p(s_0+1)}\right) \geq t'\cdot 2^{p(s_0+1)}$ vertices z in Z_s^* satisfying $S = N_G(F_z) - X$. Then Statement 4(c)i holds by Claim 5.2.1. This completes the proof. $\square$

Lemma 5.3. *For any $r, t, t' \in \mathbb{N}$, $w \in \mathbb{Z}$ with $r \geq w \geq 0$, nonnegative integer s_0 and positive real numbers k, k', there exist integers c, d such that for every graph G, either*

1. *the average degree of G is greater than k,*
2. *there exists a graph H of average degree greater than k' such that some subgraph of G is isomorphic to a $[4s_0+2w+5]$-subdivision of H,*
3. *G contains $K_{r-w+1} \vee I_t$ as a $(2s_0+w+2)$-shallow minor, or*
4. *there exists a vertex $v^* \in V(G)$ and a collection $\mathcal{C}^*$ of $((w+1)r - \binom{w+1}{2})$-element subsets of $E(G)$ with $|\mathcal{C}^*| \leq c$ such that for every subgraph R of G of minimum degree at least r containing v^*, $E(R)$ contains some member of $\mathcal{C}^*$, and either*
   - (a) *every vertex of G is of degree at most d, and there exists a vertex $x^* \in V(G)$ and a collection $\mathcal{C}$ of $\binom{r+1}{2}$-element subsets of $E(G)$ with $|\mathcal{C}| \leq c$ such that for every subgraph R' of G of minimum degree at least r containing x^*, $E(R')$ contains some member of $\mathcal{C}$, or*
   - (b) *there exists a nonnegative integer s with $s \leq s_0$ such that either*
     - i. *there exists a connected graph F_0 such that G contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s, where I is the heart of F, or*
     - ii. *there exists a vertex $x^* \in V(G)$ and a collection $\mathcal{C}$ of (s_0+1)-element subsets of $E(G)$ with $|\mathcal{C}| \leq c$ such that for every subgraph R' of G of minimum degree at least r containing x^*, $E(R')$ contains some member of $\mathcal{C}$.*

Proof. Let $r, t, t' \in \mathbb{N}$, $w \in \mathbb{Z}$ with $r \geq w \geq 0$, s_0 be a nonnegative integer, and k, k' be positive real numbers. Let d be the number d mentioned in Lemma 5.2 by taking $(r, t, t', w, s_0, k, k') = (r, t, t', w, s_0, k, k')$. Define $c = \binom{d}{r}^{r+1} \cdot (4(r+1)d)^{(r+1)^2} + \binom{d\cdot(s_0+3)d^{s_0+2}}{s_0+1} \cdot 2^{(s_0+3)^2 d^{2s_0+4}}$.

Let G be a graph. Assume that Statements 1-3 of this lemma do not hold. By Lemma 5.2, there exists $X \subseteq V(G)$ such that

- (i) every vertex in X has degree at most d in G,
- (ii) there exists $v^* \in X$ such that for every subgraph R of G of minimum degree at least r containing v^*, there exists a path Q_R in $G[X \cap V(R)]$ of length w starting at v^*, and
- (iii) either $X = V(G)$, or there exists an integer s with $0 \leq s \leq s_0$ such that either
  - (C1) there exists a connected graph F_0 such that G contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s, where I is the heart of F, or
  - (C2) there exists $x^* \in X$ such that for every subgraph R of G of minimum degree at least r containing x^*, there exists a connected subgraph F of $R[X \cap V(R)]$ containing x^* such that the number of edges in R incident with $V(F)$ is at least $s_0 + 1$.

We shall show Statement 4 of this lemma holds.

For every $v \in X$, let $\mathcal{C}_v$ be the collection of all r-element subsets of $E(G)$ such that each of the r edges is incident with v. Since every vertex in X has degree at most d in G, $|\mathcal{C}_v| \leq \binom{d}{r}$ for every $v \in X$.

For every subgraph Q in $G[X]$, let

$$\mathcal{C}_Q = \{ \bigcup_{v \in V(Q)} T_v : (T_v \in \mathcal{C}_v : v \in V(Q))\}.$$

In other words, each member of $\mathcal{C}_Q$ is a union of $|V(Q)|$ sets where each of them consists of r edges incident with a vertex of Q and no two distinct sets are corresponding to the same vertex of Q. For every subgraph Q in $G[X]$, since $|\mathcal{C}_Q| \leq \prod_{v \in V(Q)} |\mathcal{C}_v|$, we have

$$|\mathcal{C}_Q| \leq \binom{d}{r}^{|V(Q)|}. \tag{11}$$

Claim 5.3.1. *Let $u \in X$ and q be a nonnegative integer. If $\mathcal{C}$ is the set consisting of all the members of $\mathcal{C}_Q$ for all connected subgraphs Q in $G[X]$ containing u satisfying that $|V(Q)| = q$ and every vertex $v \in V(Q)$ has degree at least r in G, then every member of $\mathcal{C}$ has size at least $qr - \binom{q}{2}$, and $|\mathcal{C}| \leq \binom{d}{r}^q \cdot (4qd)^{q^2}$.*

Proof of Claim 5.3.1: Since every vertex of Q has degree at least r in G, every member of $\mathcal{C}$ has size at least $qr - \binom{q}{2}$. Since every vertex in X has degree at most d in G, for every q' with $0 \leq q' \leq q$, there are at most $d^{q'} \leq d^q$ paths in $G[X]$ of length q' starting at u. So $|N_G^{\leq q}[u]| \leq qd^q + 1$. Since every connected subgraph Q in $G[X]$ containing u with $|V(Q)| = q$ satisfies $V(Q) \subseteq N_G^{\leq q}[u]$, there are at most $\binom{|N_G^{\leq q}[u]|}{q} \cdot 2^{\binom{|V(Q)|}{2}} \leq (4qd)^{q^2}$ connected subgraphs Q in $G[X]$ containing u with $|V(Q)| = q$. So together with (11), $|\mathcal{C}| \leq \binom{d}{r}^q \cdot |\{Q : Q$ is a connected subgraph in $G[X]$ containing u with $|V(Q)| = q\}| \leq \binom{d}{r}^q \cdot (4qd)^{q^2}$. □

Define $\mathcal{C}_0$ to be the union of $\mathcal{C}_{Q_R}$ over all subgraphs R of G of minimum degree at least r containing v^*, where v^* and Q_R are defined in (ii). By Claim 5.3.1, every member of $\mathcal{C}_0$ has size at least $(w+1)r - \binom{w+1}{2}$ and $|\mathcal{C}_0| \leq \binom{d}{r}^{w+1} \cdot (4(w+1)d)^{(w+1)^2} \leq c$. By (ii), for every subgraph R' of G of minimum degree at least r containing v^*, $E(R')$ contains some member of $\mathcal{C}_0$.

For every $S \in \mathcal{C}_0$, since $|S| \geq (w+1)r - \binom{w+1}{2}$, there exists a subset $f^*(S)$ of S of size $(w+1)r - \binom{w+1}{2}$. Let $\mathcal{C}^* = \{f^*(S) : S \in \mathcal{C}_0\}$. So every member of $\mathcal{C}^*$ has size $(w+1)r - \binom{w+1}{2}$, $|\mathcal{C}^*| \leq |\mathcal{C}_0| \leq c$, and for every subgraph of G of minimum degree at least r containing v^*, its edge-set contains some member of $\mathcal{C}^*$.

Therefore, to prove Statement 4 of this lemma, it suffices to prove Statements 4(a) or 4(b) holds.

We first assume that $X = V(G)$. Then every vertex of G is of degree at most d by (i). If every vertex of G has degree less than r, then there exists no subgraph of G of minimum degree at least r, so Statement 4(a) holds by choosing $\mathcal{C} = \emptyset$ and choosing x^* to be any vertex of G. Hence we may assume that there exists a vertex v of G of degree at least r. For every subgraph R of G of minimum degree at least r containing v, there exists a star T_R on $r+1$ vertices centered at v contained in R. Note that every vertex in such T_R has degree at least r in G since R has minimum degree at least r. Define $\mathcal{C}_1$ to be the union of $\mathcal{C}_{T_R}$ over all subgraphs R of G of minimum degree at least r containing v. By Claim 5.3.1, every member of $\mathcal{C}_1$ has size at least $(r+1)r - \binom{r+1}{2} = \binom{r+1}{2}$ and $|\mathcal{C}_1| \leq \binom{d}{r}^{r+1} \cdot (4(r+1)d)^{(r+1)^2} \leq c$. For every subgraph R' of G of minimum degree at least

r containing v, since $V(T_{R'}) \subseteq V(R')$, $E(R')$ contains some member of $\mathcal{C}_1$. Hence Statement 4(a) holds and we are done.

So we may assume that $X \neq V(G)$. Hence by (iii), there exists a nonnegative integer s with $s \leq s_0$ such that either (C1) or (C2) holds. We may also assume that Statement 4(b)(i) does not hold, for otherwise we are done. In particular, (C2) holds by (iii).

Let $\mathcal{C} = \{E(Q) : Q$ is a subgraph of G obtained from a connected subgraph Q' of $G[X]$ by adding edges of G incident with $V(Q')$ such that $x^* \in V(Q')$ and $|E(Q)| = s_0 + 1\}$. Note that for every connected subgraph Q' of $G[X]$ with $x^* \in V(Q)$ and $|E(Q')| \leq s_0 + 1$, $V(Q') \subseteq N^{\leq s_0+2}_{G[X]}[x^*]$ by the connectedness. Since every vertex in X has degree at most d in G, $|V(Q')| \leq |N^{\leq s_0+2}_{G[X]}[x^*]| \leq (s_0+3)d^{s_0+2}$. Thus the number of such connected graphs Q' is at most $2^{|N^{\leq s_0+2}_{G[X]}[x^*]|} \cdot 2^{\binom{|N^{\leq s_0+2}_{G[X]}[x^*]|}{2}} \leq 2^{(|N^{\leq s_0+2}_{G[X]}[x^*]|)^2} \leq 2^{(s_0+3)^2 d^{2s_0+4}}$. So the number of subgraphs Q of G obtained from such a connected subgraphs Q' of $G[X]$ by adding edges of G incident with $V(Q')$ such that $x^* \in V(Q)$ and $|E(Q)| = s_0 + 1$ is at most $\binom{d|V(Q')|}{s_0+1}$ multiplying by the number of Q', which is at most $\binom{d\cdot(s_0+3)d^{s_0+2}}{s_0+1} \cdot 2^{(s_0+3)^2 d^{2s_0+4}} \leq c$. Hence $|\mathcal{C}| \leq c$. In addition, every member of $\mathcal{C}$ has size $s_0 + 1$.

By (C2), for every subgraph R of G of minimum degree at least r containing x^*, there exists a connected subgraph F'_R of $R[X \cap V(R)]$ containing x^* whose number of edges in R incident to vertices in F'_R is at least $s_0 + 1$, so there exists a connected subgraph F_R of R obtained from a connected subgraph F''_R of F'_R containing x^* by adding edges of R incident with $V(F''_R)$ such that $|E(F_R)| = s_0 + 1$. Note that $E(F_R) \in \mathcal{C}$. Hence $E(R)$ contains $E(F_R) \in \mathcal{C}$. Therefore Statement 4(b)ii holds. □

Now we are ready to prove Lemma 5.4.

Lemma 5.4. *For any $r, t, t' \in \mathbb{N}$, integer w with $r \geq w \geq 0$, and nonnegative integer s_0, there exists an integer c such that for every graph G, either*

1. *G contains $K_{r-w+1} \vee I_t$ as a minor, or*
2. *there exists a collection $\mathcal{C}$ of $\left((w+1)r - \binom{w+1}{2}\right)$-element subsets of $E(G)$ with $|\mathcal{C}| \leq c|V(G)|$ such that for every subgraph R of G with minimum degree at least r, $E(R)$ contains some member of $\mathcal{C}$, and there exists a nonnegative integer s with $s \leq s_0$ such that either*
   - *(a) there exists a connected graph F_0 such that G contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s, where I is the heart of F, or*
   - *(b) there exists a collection $\mathcal{C}$ of $\min\{s_0 + 1, \binom{r+1}{2}\}$-element subsets of $E(G)$ with $|\mathcal{C}| \leq c|V(G)|$ such that for every subgraph R of G with minimum degree at least r, $E(R)$ contains some member of $\mathcal{C}$.*

Proof. Let $r, t, t' \in \mathbb{N}$ and w be an integer with $r \geq w \geq 0$. Let s_0 be a nonnegative integer. Let k be a real number such that every graph with average degree at least k contains $K_{r-w+1} \vee I_t$ as a minor. Note that such a number k exists since we can take k to be any value larger than the supreme of maximum average degree in all $K_{r-w+1+t}$-minor free graphs, and the supreme exists by [45]. Define c and d to be the numbers c and d mentioned in Lemma 5.3 by taking $(r, t, t', w, s_0, k, k') = (r, t, t', w, s_0, k, k)$.

Let G be a graph. We shall prove this lemma by induction on $|V(G)|$. This lemma holds when $|V(G)| = 1$ since there exists no subgraph of G of minimum degree at least r and hence Statement 2 holds. Now we assume that this lemma holds for all graphs with fewer vertices than G.

We may assume that G does not contain $K_{r-w+1} \vee I_t$ as a minor, for otherwise we are done. Since G does not contain $K_{r-w+1} \vee I_t$ as a minor, every subgraph of G has average degree less than k, and G does not contain $K_{r-w+1} \vee I_t$ as a $(2s_0+w+2)$-shallow minor. Similarly, there does not exist a graph H of average degree greater than k such that some subgraph H' of G is a $([4s_0+2w+5])$-subdivision of H, for otherwise H' (and hence G) contains a subdivision of a subgraph of H that contains $K_{r-w+1} \vee I_t$ as a minor, a contradiction.

Hence, applying Lemma 5.3 by taking $(r,t,t',w,s_0,k,k') = (r,t,t',w,s_0,k,k)$, there exists x^* and a collection $\mathcal{C}_{x^*}$ of q-element subsets of $E(G)$ with $|\mathcal{C}_{x^*}| \leq c$ such that for every subgraph R of G of minimum degree at least r containing x^*, $E(R)$ contains some member of $\mathcal{C}_{x^*}$, where q is defined as follows:

- if every vertex of G is of degree at most d, then $q = \binom{r+1}{2}$;
- otherwise, if there exists a connected graph F_0 such that G contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s for some integer s with $0 \leq s \leq s_0$, where I is the heart of F, then $q = (w+1)r - \binom{w+1}{2}$;
- otherwise, $q = \max\{(w+1)r - \binom{w+1}{2}, \min\{s_0+1, \binom{r+1}{2}\}\}$.

Since w is an integer with $0 \leq w \leq r$, $(w+1)r - \binom{w+1}{2} \leq \binom{r+1}{2}$.

Let $G' = G - x^*$. Note that G' does not contain $K_{r-w+1} \vee I_t$ as a minor. So by the induction hypothesis, there exists a collection $\mathcal{C}'$ of q'-element subsets of $E(G')$ with $|\mathcal{C}'| \leq c|V(G')| = c(|V(G)|-1)$ such that for every subgraph R of G' of minimum degree at least r, $E(R)$ contains some member of $\mathcal{C}'$, where

- if there exists a connected graph F_0 such that G' contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s for some integer s with $0 \leq s \leq s_0$, where I is the heart of F, then $q' = ((w+1)r - \binom{w+1}{2})$, and
- otherwise, $q' = \max\{((w+1)r - \binom{w+1}{2}), \min\{s_0+1, \binom{r+1}{2}\}\}$.

Note that if there exists a connected graph F_0 such that G' contains $F \wedge_{t'} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s, where I is the heart of F, then does G. So $q' \leq q$. Hence for every $S \in \mathcal{C}_{x^*}$, there exists a subset $f(S)$ of S of size q' such that $|\{f(S) : S \in \mathcal{C}_{x^*}\}| \leq c$, and for every subgraph R of G of minimum degree at least r containing x^*, $E(R)$ contains some member of $\{f(S) : S \in \mathcal{C}_{x^*}\}$.

Define $\mathcal{C} = \{f(S) : S \in \mathcal{C}_{x^*}\} \cup \mathcal{C}'$. So $\mathcal{C}$ is a collection of q'-element subsets of $E(G)$ with size at most $|\mathcal{C}_{x^*}| + |\mathcal{C}'| \leq c|V(G)|$.

Let R be a subgraph of G of minimum degree at least r. If R contains x^*, then $E(R)$ contains some member of $\{f(S) : S \in \mathcal{C}_{x^*}\} \subseteq \mathcal{C}$. If R does not contain x^*, then R is a subgraph of G' of minimum degree at least r, so $E(R)$ contains some member of $\mathcal{C}' \subseteq \mathcal{C}$. Therefore, Statement 2 holds for G. This proves this lemma. □

6 Proof of Main Theorems

We prove Theorems 1.2, 1.3, 1.5 and 1.6 in this section. We first prove the following lemmas.

Lemma 6.1. *Let r be a positive integer. Let H be a graph that is not a subgraph of $K_r \vee I_t$ for any positive integer t. Then $\{K_{r,s} : s \geq r\} \subseteq \mathcal{M}(H)$.*

Proof. For every integer s with $s \geq r$, every minor of $K_{r,s}$ is a subgraph of $K_r \vee I_s$. Hence, if there is an integer s such that $K_{r,s}$ contains H as a minor, then H is a subgraph of $K_r \vee I_s$, a contradiction. Hence $K_{r,s}$ does not contain H as a minor for every $s \geq r$. $\square$

Lemma 6.2. *Let $r \geq 2$ be an integer. Let H be a graph. Then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/q_H})$, where q_H is defined as follows.*

1. *If H is not a subgraph of $K_r \vee I_t$ for any positive integer t, then $q_H = r$.*
2. *Otherwise let w be the largest integer with $1 \leq w \leq r$ such that H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t.*
   (a) *If H is not a subgraph of $K_{r-w} \vee tK_{w+1}$ for any positive integer t, then $q_H = (w+1)r - \binom{w+1}{2}$.*
   (b) *Otherwise, $q_H = \max\{\min\{s+1, \binom{r+1}{2}\}, (w+1)r - \binom{w+1}{2}\}$, where s is the largest integer with $0 \leq s \leq \binom{r+1}{2}$ such that for every integer s' with $0 \leq s' \leq s$, every connected graph F_0 and every graph $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s', H is a minor of $F \wedge_t I$ for some positive integer t, where I is the heart of F.*

Furthermore, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$ in Statements 1 and 2(a).

Proof. We first assume that H is not a subgraph of $K_r \vee I_t$ for any positive integers t. By Lemma 5.1, there exists a real number c (only depending on r and H) such that for every H-minor free graph G, there exists and a collection $\mathcal{C}$ of r-element subsets of $E(G)$ with $|\mathcal{C}| \leq c|V(G)|$ such that for every subgraph of G of minimum degree at least r, its edge-set contains some member of $\mathcal{C}$. So the threshold $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/r})$ by Lemma 2.1. In addition, by Lemma 6.1, $\mathcal{M}(H)$ contains $\{K_{r,s} : s \geq r\}$. So $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = O(n^{-1/r})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = O(n^{-1/r})$ by the Corollary 3.6. Thus $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/r})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/r})$ by Proposition 1.4. This proves Statement 1.

Now we may assume that H is a subgraph of $K_r \vee I_t$ for some positive integer t. So there exists the largest integer w with $1 \leq w \leq r$ such that H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t. Hence there exists an integer $t_H \geq r$ such that H is a subgraph of $K_{r-w+1} \vee I_{t_H}$. Since $K_{r-w+1, t_H + \binom{r-w+1}{2}}$ contains $K_{r-w+1} \vee I_{t_H}$ as a minor, every H-minor free graph does not contain $K_{r-w+1, t_H + \binom{r-w+1}{2}}$ as a minor and hence does not contain $K_{r-w+1} \vee I_{t_H + \binom{r-w+1}{2}}$ as a minor.

Let c_1 be the number c mentioned in Lemma 5.4 by taking $(r, t, t', w, s_0) = (r, t_H + \binom{r-w+1}{2}, 1, w, \binom{r+1}{2})$. Since every H-minor free graph does not contain $K_{r-w+1} \vee I_{t_H + \binom{r-w+1}{2}}$ as a minor, Lemma 5.4 implies that for every H-minor free graph G, there exists a collection $\mathcal{C}_{G,1}$ of $((w+1)r - \binom{w+1}{2})$-element subsets of $E(G)$ with $|\mathcal{C}_{G,1}| \leq c_1|V(G)|$ such that for every subgraph R of G of minimum degree at least r, $E(R)$ contains some member of $\mathcal{C}_{G,1}$. So $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/((w+1)r - \binom{w+1}{2})})$ by Lemma 2.1. Hence by Proposition 1.4, $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Omega(n^{-1/((w+1)r - \binom{w+1}{2})})$ by Lemma 2.1.

Now we assume that H is not a subgraph of $K_{r-w} \vee tK_{w+1}$ for any positive integer t. Note that for every positive integer s with $s \geq r-w$, every minor of $I_{r-w} \vee sK_{w+1}$ is a subgraph of $K_{r-w} \vee sK_{w+1}$. So for every positive integer t with $t \geq r-w$, $I_{r-w} \vee tK_{w+1}$ does not contain H as a minor. That is, $\{I_{r-w} \vee sK_{w+1} : s \geq r-w\} \subseteq \mathcal{M}(H)$. By Corollary 3.6, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = O(n^{-1/q_H})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = O(n^{-1/q_H})$. This proves Statement 2(a).

Hence we may assume that H is a subgraph of $K_{r-w} \vee tK_{w+1}$ for some positive integer t. Note that it implies that H is a subgraph of $K_{r-w} \vee tK_{w+1}$ for any positive integers t.

We say that a triple (a, F_0, F) is a *standard triple* if a is a nonnegative integer, F_0 is a connected graph, and F is a member of $\mathcal{F}(I_{r-w}, F_0, r)$ of type a. Let s be the largest integer with $0 \leq s \leq \binom{r+1}{2}$ such that for every integer s' with $0 \leq s' \leq s$ and for every standard triple (s', F_0, F), H is a minor of $F \wedge_t I$ for some positive integer t, where I is the heart of F. The number s is well-defined (i.e., $s \geq 0$) since there is no graph F in $\mathcal{F}(I_{r-w}, F_0, r)$ of type 0.

This definition implies that for every integer s' with $0 \leq s' \leq s$ and standard triple (s', F_0, F), there exists an integer $t_{s',F_0,F}$ such that H is a minor of $F \wedge_t I$ for every integer t with $t \geq t_{s',F_0,F}$, where I is the heart of F. In addition, for every integer s' with $0 \leq s' \leq s$ and standard triple (s', F_0, F), since F_0 is connected, we know $|V(F_0)| \leq |E(F_0)| + 1 \leq s' + 1 \leq \binom{r+1}{2} + 1$. So there are only finitely many different standard triple (s', F_0, F) with $0 \leq s' \leq s$. We define t^* to be the maximum $t_{s',F_0,F}$ among all integers s' with $0 \leq s' \leq s$ and standard triples (s', F_0, F). So H is a minor of $F \wedge_{t^*} I$, where I is the heart of F.

Applying Lemma 5.4 by taking $(r, t, t', w, s_0) = (r, t_H + \binom{r-w+1}{2}, t^*, w, s)$, there exists a number c_2 such that for every $K_{r-w+1} \vee I_{t_H+\binom{r-w+1}{2}}$-minor free graph G, there exists an integer s_G with $0 \leq s_G \leq s$ such that either

(i) there exists a connected graph F_0 such that G contains $F \wedge_{t^*} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s_G, where I is the heart of F, or

(ii) there exists a collection $\mathcal{C}$ of $\min\{s+1, \binom{r+1}{2}\}$-element subsets of $E(G)$ with $|\mathcal{C}| \leq c_2|V(G)|$ such that for every subgraph R of G with minimum degree at least r, $E(R)$ contains some member of $\mathcal{C}$.

Let G be an H-minor free graph. Suppose that (i) holds for G. Then there exists a connected graph F_0 such that G contains $F \wedge_{t^*} I$ as a subgraph for some $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type $s_G \leq s$, where I is the heart of F. By the definition of t^*, H is a minor of $F \wedge_{t^*} I$, so G contains H as a minor, contradiction. Hence (ii) holds for G. Therefore, there exists a collection $\mathcal{C}_{G,2}$ of $\min\{s+1, \binom{r+1}{2}\}$-element subsets of $E(G)$ with $|\mathcal{C}_{G,2}| \leq c_2|V(G)|$ such that for every subgraph R of G of minimum degree at least r, $E(R)$ contains some member of $\mathcal{C}_{G,2}$. Hence $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/\min\{s+1,\binom{r+1}{2}\}})$ by Lemma 2.1 and Statement 2(b) holds. This proves the lemma. □

Lemma 6.3. *Let r be a positive integer with $r \geq 2$. Let H be a graph of minimum degree at least r such that H is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t. Let t^* be the minimum such that H is a subgraph of $K_{r-1} \vee t^*K_2$. Then either H is not a minor of $K_{r-2} \vee tK_3$ for any positive integer t, or $2t^* = 3q - 1$ for some positive integer q.*

Proof. We may assume that there exists an H-minor α in $K_{r-2} \vee tK_3$ for some positive integer t, for otherwise we are done. Let Y be the vertex-set $V(K_{r-2})$ in $K_{r-2} \vee tK_3$.

Since $\delta(H) \geq r$ and H is a subgraph of $K_{r-1} \vee t^*K_2$, $H = (K_{r-1} \vee t^*K_2) - S$, where S is a set of edges of $K_{r-1} \vee t^*K_2$ in $E(K_{r-1})$. Hence $I_{r-1} \vee t^*K_2 \subseteq H \subseteq K_{r-1} \vee t^*K_2$. We call each vertex of H in $V(K_{r-1})$ an *inner vertex*, and call each vertex of H in $V(t^*K_2)$ an *outer vertex*.

Claim 6.3.1. *Let A_1 be a branch set of α disjoint from Y. Let X be the vertex-set of the component of $(K_{r-2} \vee tK_3) - Y$ intersecting A_1. Then the following hold.*

1. *A_1 consists of one vertex.*
2. *X is a union of three branch sets of α.*
3. *Every vertex in Y belongs to a branch set, and different vertices of Y belong to different branch sets of α.*

4. *either A_1 is a branch set corresponding to an inner vertex, or $t^* = 1$.*

Proof of Claim 6.3.1 Since $\delta(H) \geq r$, A_1 is adjacent in $K_{r-2} \vee tK_3$ to at least r other branch sets of α. Since A_1 is disjoint from Y, $|A_1| = 1$. Hence every vertex in $Y \cup (X - A_1)$ belongs to a branch set, and different vertices in $Y \cup (X - A_1)$ belong to different branch sets. So Statements 1-3 hold.

Assume that A_1 is a branch set corresponding to an outer vertex. Since every outer vertex is adjacent to all inner vertices, each branch set corresponding to an inner vertex either intersects Y or is contained in X. Since there are $r-1$ inner vertices and $|Y| = r-2$, there exists an inner vertex whose branch set is contained in X, so every branch set corresponding to an outer vertex intersects $Y \cup X$. Hence there are at most $|X \cup Y| - 2t^* = r+1-2t^*$ branch sets corresponding to inner vertices adjacent to A_1. So $r+1-2t^* \geq r-1$. That is, $t^* = 1$. So Statement 4 holds. □

Since $|Y| = r-2$ and $|V(H)| = r-1+2t^* > r-2$, there exists a vertex v of H such that the branch vertex corresponding to v in α is disjoint from Y. Hence there exist a positive integer q and components $C_1, C_2, ..., C_q$ of $(K_{r-2} \vee tK_3) - Y$ such that those C_i are the components of $(K_{r-2} \vee tK_3) - Y$ containing some branch sets disjoint from Y. We may assume that $t^* \neq 1$, for otherwise $2t^* = 3-1$ and we are done. So by Claim 6.4.1, for each $i \in [q]$, $V(C_i)$ is the union of three branch sets of α corresponding to inner vertices. So the number of inner vertices whose branch sets are disjoint from Y is $3q$.

Since each outer vertex is adjacent to all inner vertices, each branch set corresponding to an outer vertex intersects Y and hence contains exactly one vertex in Y (by Claim 6.4.1). Hence by Claim 6.4.1, there are exactly $|Y| - 2t^* = r-2-2t^*$ branch sets corresponding to inner vertices intersecting Y.

Therefore, the number of inner vertices is $3q + r - 2 - 2t^*$. In addition, the number of inner vertices is $|V(I_{r-1})| = r-1$. Hence $2t^* = 3q-1$. This proves the lemma. □

Similar to Lemma 6.3, we can also obtain the following lemma.

Lemma 6.4. *Let r be a positive integer with $r \geq 4$. Let H be a graph of minimum degree at least r such that H is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t. Let t^* be the minimum such that H is a subgraph of $K_{r-1} \vee t^*K_2$. Then either H is not a minor of L_t (defined in Definition 8) for any positive integer t, or $2t^* = 3q$ for some positive integer q.*

Proof. Let us recall the definition of L_t. Let Y be the stable set of size $r-1$ in $I_{r-1} \vee K_3$ corresponding to $V(I_{r-1})$, and let $X = V(I_{r-1} \vee K_3) - Y$. Let L be a connected graph obtained from $I_{r-1} \vee K_3$ by deleting the edges of a matching of size three between X and Y. Denote $Y = \{y_1, y_2, ..., y_{r-1}\}$. For every positive integer t, L_t is the graph obtained from a union of disjoint t copies of L by for each i with $1 \leq i \leq r-1$, identifying the y_i in each copy of L into a new vertex y_i^*.

We may assume that there exists an H-minor α in L_t, for otherwise we are done. Since $\delta(H) \geq r$ and H is a subgraph of $K_{r-1} \vee t^*K_2$, $I_{r-1} \vee t^*K_2 \subseteq H \subseteq K_{r-1} \vee t^*K_2$. We call each vertex of H in $V(K_{r-1})$ an *inner vertex*, and call each vertex of H in $V(t^*K_2)$ an *outer vertex*.

Claim 6.4.1. *Let A_1 be a branch set of α disjoint from Y. Let Z be the vertex-set of the component of $L_t - Y$ intersecting A_1. Then the following hold.*

- *A_1 consists of one vertex.*
- *Z is a union of three branch sets of α.*

- *Every vertex in Y belongs to a branch set, and different vertices of Y belong to different branch sets of α.*

- *A_1 is a branch set corresponding to an inner vertex.*

Proof of Claim 6.4.1: Since $\delta(H) \geq r$, A_1 is adjacent in L_t to at least r other branch sets of α. So $1 \leq |A_1| \leq 2$.

Suppose $|A_1| = 2$. Then $|Y \cup (Z - A_1)| = r$. So each vertex in $Y \cup (Z - A_1)$ is contained in a branch set of α, and different vertices in $Y \cup (Z - A_1)$ are contained in different branch sets. So some branch set of α consists of the single vertex u in $Z - A_1$. Since u is nonadjacent in L_t to some vertex in Y, the branch set consisting of u is adjacent to at most $(|Y| - 1) + 1 = r - 1$ branch sets of α, contradicting $\delta(H) \geq r$.

So $|A_1| = 1$ and Statement 1 holds. Let x_1 be the vertex in A_1. By symmetry, we may assume that y_1 is the vertex in Y nonadjacent to x_1 in L_t. Since $A_1 \cap Y = \emptyset$ and $\delta(H) \geq r$, each vertex in $(Y - \{y_1\}) \cup (Z - A_1)$ is contained in a branch set of α, and different vertices in $(Y - \{y_1\}) \cup (Z - A_1)$ are contained in different branch sets of α. This implies that there exist two different branch sets A_2, A_3 of α other than A_1 such that $A_2 \cap Z \neq \emptyset \neq A_3 \cap Z$, and one of A_2, A_3 is disjoint from Y. By symmetry, we may assume that A_2 is disjoint from Y. So $|A_2| = 1$. Let x_2 be the vertex in A_2. By symmetry, we may assume that y_2 is the vertex in Y nonadjacent to x_2 in L_t. Since $\delta(H) \geq r$, each vertex in $(Y - \{y_2\}) \cup (Z - A_2)$ is contained in a branch set of α, and different vertices in $(Y - \{y_2\}) \cup (Z - A_2)$ are contained in different branch sets of α. This implies that $y_1 \not\in A_3$. So A_3 consists of one vertex, say x_3, in Z. Hence Z is a union of three branch sets A_1, A_2, A_3 of α, where each of A_i consists of one vertex. So Statement 2 holds.

By symmetry, let y_3 be the vertex in Y nonadjacent to x_3 in L_t. Since $\delta(H) \geq r$, each vertex in $(Y - \{y_3\}) \cup (Z - A_3)$ is contained in a branch set of α, and different vertices in $(Y - \{y_3\}) \cup (Z - A_3)$ are contained in different branch sets of α. So y_1 and y_2 are contained in different branch sets. Hence each vertex of Y is contained in a branch set of α other than A_1, A_2, A_3, and different vertices of Y are contained in different branch sets of α. This proves Statement 3.

Suppose that A_1 is the branch set of α corresponding to an outer vertex v_1 of H. Let v_1' be the outer vertex of H adjacent to v_1 in H. Since the neighbors of v_1 are v_1' and the $r - 1$ inner vertices, y_1 is contained in the branch set of α corresponding to an outer vertex other than v_1'. Suppose some of A_2, A_3, say A_2, is the branch set of α corresponding to an outer vertex v_2 of H. Then y_2 is contained in the branch set of α corresponding to an outer vertex. So there are at most $(|Y| - 2) + (|Z| - 2) \leq r - 2$ branch sets corresponding to an inner vertex intersecting $(Y - \{y_1\}) \cup Z$. Since there are $r - 1$ inner vertices, A_1 is nonadjacent to some branch vertex corresponding to an inner vertex, a contradiction. So each of A_2, A_3 is the branch set corresponding to an inner vertex. Hence every branch set corresponding to an outer vertex other than v_1 intersects Y. So there are at most $|Y| - (2t^* - 1) \leq r - 2t^*$ branch sets corresponding to an inner vertex intersecting Y. Since A_1 is adjacent to $r - 1$ branch sets corresponding to inner vertices, $r - 2t^* + 2 = r - 2t^* + (|Z| - 1) \geq r - 1$, we know $t^* = 1$. So v_1 and v_1' are the only outer vertices. But y_1 is contained in the branch set of α corresponding to an outer vertex other than v_1', a contradiction. This proves Statement 4 of the claim. □

Since $|Y| = r - 1$ and $|V(H)| = r - 1 + 2t^* > r - 1$, there exists a vertex v of H such that the branch set corresponding to v in α is disjoint from Y. Hence there exist a positive integer q and components $C_1, C_2, ..., C_q$ of $L_t - Y$ such that those C_i are the components of $L_t - Y$ containing some branch sets disjoint from Y. By Claim 6.4.1, for each $i \in [q]$, $V(C_i)$ is the union of three branch sets of α corresponding to inner vertices. So the number of inner vertices whose branch sets are disjoint from Y is $3q$.

Since each outer vertex is adjacent to all inner vertices, each branch set corresponding to an outer vertex intersects Y and hence contains exactly one vertex in Y (by Claim 6.4.1). Hence by Claim 6.4.1, there are exactly $|Y| - 2t^* = r - 1 - 2t^*$ branch sets corresponding to an inner vertex intersecting Y.

Therefore, the number of inner vertices is $3q + r - 1 - 2t^*$. In addition, the number of inner vertices is $|V(I_{r-1})| = r - 1$. Hence $2t^* = 3q$. This proves the lemma. □

Lemma 6.5. *Let r be a positive integer with $r \geq 2$. Let H be a graph of minimum degree at least r such that H is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t. Then either*

1. *H is not a minor of $K_{r-2} \vee tK_3$ for any positive integer t, or*
2. *$r \geq 4$ and H is not a minor of L_t for any positive integer t, or*
3. *$r \in \{2, 3\}$ and $H = K_{r+1}$.*

Proof. When $r \geq 4$, Statements 1 or 2 hold by Lemmas 6.3 and 6.4. So we may assume that $r \in \{2, 3\}$. We may assume that H is a minor of $K_{r-2} \vee tK_3$ for some positive integer t, for otherwise we are done. Note that for any positive integer t, every minor of $K_{r-2} \vee tK_3$ is a subgraph of $K_{r-2} \vee tK_3$. So H is a subgraph of $K_{r-2} \vee tK_3$ for some positive integer t.

When $r = 2$, H is a subgraph of $K_{r-1} \vee tK_2 = K_1 \vee tK_2$ and a subgraph of tK_3 for some positive integer t, so $H = K_3 = K_{r+1}$ since $\delta(H) \geq 2$.

So we may assume $r = 3$. Hence H is a subgraph of $K_2 \vee tK_2$ and a subgraph of $K_1 \vee tK_3$ for some positive integer t. Since $\delta(H) \geq 3$ and H is a subgraph of $K_2 \vee tK_2$ for some positive integer t, there exists a positive integer t^* such that $H = K_2 \vee t^*K_2$ or $H = I_2 \vee t^*K_2$. In particular, H is 2-connected. Since H is a subgraph of $K_1 \vee tK_3$ for some positive integer t and $\delta(H) \geq 3$, either $H = K_4$ or H has a cut-vertex. So $H = K_4$. This proves the lemma. □

Lemma 6.6. *Let r be a positive integer with $r \geq 2$. Let H be a graph of minimum degree at least r. Then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$, where q_H is defined as follows.*

1. *If H is not a subgraph of $K_r \vee I_t$ for any positive integer t, then $q_H = r$.*
2. *If H is a subgraph of $K_r \vee I_t$ for some positive integer t, and H is not a subgraph of $K_{r-1} \vee tK_2$ for any positive integer t, then $q_H = 2r - 1$.*
3. *If H is a subgraph of $K_r \vee I_t$ and is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t, and $H \neq K_{r+1}$, then $q_H = s + 1$, where s is the largest integer with $0 \leq s \leq \binom{r+1}{2}$ such that for every integer s' with $0 \leq s' \leq s$, every connected graph F_0 and every graph $F \in \mathcal{F}(I_{r-1}, F_0, r)$ of type s', H is a minor of $F \wedge_t I$ for some positive integer t, where I is the heart of F. Furthermore, $2r - 1 \leq s + 1 \leq \binom{r+1}{2}$.*
4. *If $H = K_{r+1}$ and $r \leq 3$, then $q_H = \infty$; if $H = K_{r+1}$ and $r \geq 4$, then $q_H = 3r - 3$.*

Moreover, $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$ for Statements 1, 2 and 4.

Proof. Statement 1 immediately follows from Statement 1 of Lemma 6.2.

So we may assume that H is a subgraph of $K_r \vee I_t$ for some positive integer t. Since H has minimum degree at least r, H is not a subgraph of $K_{r-1} \vee I_t$ for any positive integer t. So 1 equals the largest integer w with $1 \leq w \leq r$ such that H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t. Let $w = 1$.

If H is not a subgraph of $K_{r-1} \vee tK_2 = K_{r-w} \vee tK_{w+1}$ for any positive integer t, then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$, where $q_H = 2r-1$ by Statement 2(a) in Lemma 6.2. So Statement 2 of this lemma holds.

Hence we may assume that H is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t.

Now we assume that $H \neq K_{r+1}$ and prove Statement 3 of this lemma. By Lemma 6.2, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/q_H})$, where $q_H = \max\{\min\{s+1, \binom{r+1}{2}\}, 2r-1\}$ and s is the largest integer with $0 \leq s \leq \binom{r+1}{2}$ such that for every integer s' with $0 \leq s' \leq s$, every connected graph F_0 and every graph $F \in \mathcal{F}(I_{r-1}, F_0, r)$ of type s', H is a minor of $F \wedge_t I$ for some positive integer t, where I is the heart of F. For every positive integer t, define F_t to be the graph that is the disjoint union of I_{r-1} and t copies of K_{r+1}. Clearly, for every positive integer t, $F_t = F \wedge_t I$ for some $F \in \mathcal{F}(I_{r-1}, K_{r+1}, r)$ of type $\binom{r+1}{2}$. Suppose that H is a minor of F_t for some positive integer t. Since the minimum degree of H is at least r, H is a disjoint union of copies of K_{r+1}. On the other hand, since H is a subgraph of $K_r \vee I_t$ for some positive integer t, one can delete at most r vertices to make H edgeless. Therefore H is one copy of K_{r+1}. That is, $H = K_{r+1}$, a contradiction. So H is not a minor of F_t for some positive integer t. In particular, $s \leq \binom{r+1}{2} - 1$. Hence, by the maximality of s, there exists a connected graph F_0^* and a graph $F^* \in \mathcal{F}(I_{r-1}, F_0^*, r)$ of type $s+1 \leq \binom{r+1}{2}$ such that H is not a minor of $F^* \wedge_t I$ for any positive integer t, where I is the heart of F^*. Therefore, $\{F^* \wedge_t I : t \in \mathbb{N}\} \subseteq \mathcal{M}(H)$. By Statement 4 of Corollary 3.6, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = O(n^{-1/(s+1)})$. In addition, since $F^* \in \mathcal{F}(I_{r-1}, F_0^*, r)$, $|V(F_0^*)| \geq 2$. Note that for any two vertices in F_0^*, there are at least $r + (r-1) = 2r-1$ edges of F^* incident with them. So $s+1 \geq 2r-1$. Hence $\max\{\min\{s+1, \binom{r+1}{2}\}, 2r-1\} = s+1$ and $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/(s+1)})$ and hence $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(s+1)})$. This proves Statement 3.

Now we assume that $H = K_{r+1}$ and prove Statement 4.

So H is a subgraph of $K_r \vee I_t$ and $K_{r-1} \vee tK_2$ for some positive integer t. Recall that $w = 1$. Note that for every nonnegative integer s', connected graph F_1 and graph $F' \in \mathcal{F}(I_{r-w}, F_1, r)$ of type s', if $|V(F_1)| \geq 3$, then $s' \geq 3r-3$ since for any $S \subseteq V(F_1)$ with $|S| = 3$, there are at least $3r - \binom{3}{2} = 3r-3$ edges of F' incident with S. So if F_1 is a connected graph and F' is a member of $\mathcal{F}(I_{r-w}, F_1, r)$ of type at most $3r-4$, then $|V(F_1)| \leq 2$, so $|V(F_1)| = 2$ since $w = 1$, and hence $F' = I_{r-1} \vee K_2$. Hence for every nonnegative integer s' with $0 \leq s' \leq 3r-4$, connected graph F_1 and graph $F' \in \mathcal{F}(I_{r-w}, F_1, r)$ of type s', H is a minor of $F' \wedge_t I$ for some positive integer t, where I is the heart of F'. Therefore, by Statement 2(b) in Lemma 6.2, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Omega(n^{-1/q})$, where $q \geq \max\{\min\{3r-4+1, \binom{r+1}{2}\}, 2r-1\} = \max\{3r-3, 2r-1\} = 3r-3$, since $r \geq 2$.

If $r = 2$, then every H-minor free graph is a forest and does not contain any subgraph of minimum degree at least two, thus G itself (which is also $G(p)$ where p is the constant function $p = 1$) is already 1-degenerate, so $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$. If $r = 3$, then $H = K_4$, and by [13], every K_4-minor free graph contains a vertex of degree at most two, so no subgraph of any H-minor free graph has minimum degree at least $r = 3$, and hence $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$. Recall that $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(1)$ when $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ by Proposition 1.4.

Hence we may assume that $r \geq 4$. Since $K_{r+1} = K_{r-1} \vee K_2$, L_t is K_{r+1}-minor free by Lemma 6.4. Hence $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = O(n^{-1/(3r-3)})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = O(n^{-1/(3r-3)})$ by Statement 3 of Corollary 3.6. This completes the proof. □

Lemma 6.7. *Let $r \geq 2$ be an integer. Let H be a graph with $\delta(H) \geq r$. If $H \neq K_{r+1}$ and H is a subgraph of $K_{r-1} \vee tK_2$ and a subgraph of $K_r \vee I_t$ for some positive integer t, then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$.*

Proof. Let t^* be the minimum positive integer such that H is a subgraph of $K_{r-1} \vee t^* K_2$. Since

$\delta(H) \geq r$, H can be obtained from $K_{r-1} \vee t^* K_2$ by deleting a set S of edges contained in K_{r-1}.

Let s be the largest integer with $0 \leq s \leq \binom{r+1}{2}$ such that for every integer s' with $0 \leq s' \leq s$, every connected graph F_0 and every graph $F \in \mathcal{F}(I_{r-1}, F_0, r)$ of type s', H is a minor of $F \wedge_t I$ for some positive integer t, where I is the heart of F. We shall prove that $s = 3r-4$.

Suppose to the contrary that $s \leq 3r-5$. Since $r \geq 2$, $3r-5 \leq \binom{r+1}{2} - 1$. So by the maximality of s, there exist an integer s' with $0 \leq s' \leq 3r-5+1$, a connected graph F_0 and a graph $F \in \mathcal{F}(I_{r-1}, F_0, r)$ of type s' such that H is not a minor of $F \wedge_t I$ for any positive integer t, where I is the heart of F. If $|V(F_0)| \geq 3$, then for any $Z \subseteq V(F_0)$ with $|Z| = 3$, there exist at least $|Z|r - \binom{|Z|}{2} = 3r-3 > s'$ edges of F incident with $Z \subseteq V(F_0)$, a contradiction. So $|V(F_0)| \leq 2$. Hence $F_0 = K_1$ or K_2. Since the heart of F has size $r-1$, $F_0 = K_2$. So $F = I_{r-1} \vee K_2$. Since H is a subgraph of $K_{r-1} \vee tK_2$ which is a minor of $F \wedge_{t'} I$ where I is the heart of F for sufficiently large t', we have H is a minor of $F \wedge_{t'} I$ for some sufficiently large positive integer t', where I is the heart of F. This is a contradiction.

So $s \geq 3r-4$. By Lemma 6.5, either H is not a minor of $K_{r-2} \vee tK_3$ for any positive integer t, or $r \geq 4$ and H is not a minor of L_t of any positive integer t. For every positive integer t, let L'_t be the graph obtained from $I_{r-2} \vee tK_3$ by adding an isolated vertex. Since H has no isolated vertex, if H is not a minor of $K_{r-2} \vee tK_3$ for any positive integer t, then H is not a minor of L'_t for any positive integer t. Hence either $r \geq 4$ and H is not a minor of L_t for any positive integer t, or H is not a minor of L'_t for any positive integer t.

Note that for every positive integer t, $L_t = F \wedge_t I$ for some $F \in \mathcal{F}(I_{r-1}, K_3, r)$ of type $3r-3$, where I is the heart of F, and $L'_t = F' \wedge_t I'$ for some $F' \in \mathcal{F}(I_{r-1}, K_3, r)$ of type $3r-3$, where I' is the heart of F'. So $s \leq 3r-4$.

Therefore, $s = 3r-4$. By Statement 3 of Lemma 6.6, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(s+1)}) = \Theta(n^{-1/(3r-3)})$. Hence $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Omega(n^{-1/(3r-3)})$. Recall that either H is not a minor of $K_{r-2} \vee tK_3$ for any positive integer t, or $r \geq 4$ and H is not a minor of L_t of any positive integer t. So $p^{\chi^\ell_r}_{\mathcal{M}(H)} = O(n^{-1/(3r-3)})$ by Statements 2(a) and 3 of Corollary 3.6. Therefore $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$. □

Now we are ready to prove Theorems 1.2, 1.3, 1.5 and 1.6. We first show a connection between vertex-cover and subgraphs of $K_s \vee I_t$ for some integers s, t.

Lemma 6.8. *Let r, w, t be nonnegative integers such that $r \geq 1$ and $r \geq w \geq 0$. Then the following two statements are equivalent:*

1. *H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t but not a subgraph of $K_{r-w} \vee I_t$ for any positive integer t;*
2. *$\tau(H) = r - w + 1$.*

Proof. Let s be a nonnegative integer. Note that if a graph H is a subgraph of $K_s \vee I_k$ for some integer k, then $\tau(H) \leq s$. On the other hand, if $\tau(H) \leq s$, then H is a subgraph of $K_s \vee I_k$ for any sufficiently large integer k by embedding the vertices in a minimum vertex-cover into K_s and the rest of the $|V(H)| - \tau(H)$ vertices to I_k. Therefore H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t is equivalent with $\tau(H) \leq r-w+1$. And H is not a subgraph of $K_{r-w} \vee I_t$ for any positive integer t is equivalent with $\tau(H) > r - w$. □

Proof of Theorem 1.3: Since $2 \leq \tau(H) \leq r$, there exists w with $r-1 \geq w \geq 1$ such that $\tau(H) = r-w+1$. By Lemma 6.8, w is the largest integer with $r-1 \geq w \geq 1$ such that H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t. Note that $w = r - \tau(H) + 1$. Since $H \in \mathcal{H}_r$, H is a subgraph of $K_{r-w} \vee t^* K_{w+1}$ for some positive integer t^*. By Statement 2(b) of Lemma 6.2,

$p^{\mathcal{P}}_{\mathcal{M}(H)} = \Omega(n^{-1/q_H})$, where $q_H = \max\{\min\{s+1, \binom{r+1}{2}\}, (w+1)r - \binom{w+1}{2}\}$, where s is the largest integer with $0 \leq s \leq \binom{r+1}{2}$ such that for every integer s' with $0 \leq s' \leq s$, every connected graph F_0 and every graph $F \in \mathcal{F}(I_{r-w}, F_0, r)$ of type s', H is a minor of $F \wedge_t I$ for some positive integer t, where I is the heart of F. So this theorem follows from the fact $w = r - \tau(H) + 1$. $\square$

Proof of Theorems 1.2: If $\tau(H) \geq r+1$, then H is not a subgraph of $K_r \vee I_t$ for any positive integer t by Lemma 6.8, so $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/r})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/r})$ by Statement 1 of Lemma 6.2. So Statement 1 of Theorem 1.2 holds.

Now we assume that $1 \leq \tau(H) \leq r$ and H is not a subgraph of $K_{\tau(H)-1} \vee tK_{r+2-\tau(H)}$ for any positive integer t. Since $1 \leq \tau(H) \leq r$, there exists w with $r \geq w \geq 1$ such that $\tau(H) = r-w+1$. So H is a subgraph of $K_{r-w+1} \vee I_t$ for some positive integer t but is not a subgraph of $K_{r-w} \vee I_t$ for any positive integer t by Lemma 6.8. Since H is not a subgraph of $K_{\tau(H)-1} \vee tK_{r+2-\tau(H)} = K_{r-w} \vee tK_{w+1}$ for any positive integer t, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/q_H})$, where $q_H = (w+1)r - \binom{w+1}{2}$, by Statement 2(a) of Lemma 6.2. Hence Statement 2 of Theorem 1.2 holds.

Now we assume $\tau(H) \leq r$ and $\delta(H) \geq r$. Then H is a subgraph of $K_r \vee I_t$ for some positive integer t. If H is not a subgraph of $K_{r-1} \vee tK_2$ for any positive integer t, then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(2r-1)})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(2r-1)})$ by Statement 2 of Lemma 6.6. Hence Statement 3 of Theorem 1.2 holds.

Now we assume $\tau(H) \leq r$, $\delta(H) \geq r$, and H is a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t. So H is a subgraph of $K_r \vee I_t$ and a subgraph of $K_{r-1} \vee tK_2$ for some positive integer t. If $H \neq K_{r+1}$, then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$ by Lemma 6.7. If $H = K_{r+1}$ and $r \geq 4$, then $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(n^{-1/(3r-3)})$ by Statement 4 of Lemma 6.6. Hence Statement 4 of Theorem 1.2 holds.

Furthermore, if $\tau(H) = 0$, then H is edgeless, so every graph on more than $|V(H)|$ vertices contains H as a minor, and hence $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$. If H consists of $K_{1,s}$ and isolated vertices for some s with $1 \leq s \leq r$, then every H-minor free graph on more than $|V(H)|$ vertices has maximum degree at most $s-1 \leq r-1$ and hence is $(r-1)$-degenerate, so $\mathcal{P}^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$. If $H = K_{r+1}$ for $r \leq 3$, then $\mathcal{P}^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ by Statement 4 of Lemma 6.6. Recall that $p^{\chi^\ell_r}_{\mathcal{M}(H)} = \Theta(1)$ whenever $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ by Proposition 1.4. This proves Theorem 1.2. $\square$

Proof of Theorem 1.5: Statement 1 holds by Statement 1 of Corollary 3.6, Lemma 6.1, Proposition 1.4 and Statement 1 in Theorem 1.2.

Now we can assume $1 \leq \tau(H) \leq r$. So there exists an integer w with $1 \leq w \leq r$ such that $\tau(H) = r - w + 1$.

We first prove Statement 2. So r is divisible by $w+1$ and H is not a subgraph of $K_{r-w} \vee tK_{w+1}$ for any positive integers t. Since every minor of $I_{r-w} \vee tK_{w+1}$ is a subgraph of $K_{r-w} \vee tK_{w+1}$, $\{I_{r-w} \vee sK_{w+1} : s \geq s_0\} \subseteq \mathcal{M}(H)$ for some sufficiently large s_0. Hence Statement 2 of this theorem follows from Statement 2 of Corollary 3.6, Statement 2 of Theorem 1.2 and Proposition 1.4.

Now we prove Statement 3. Note that for any positive integer t, every minor of $I_{r-1} \vee tK_2$ is a subgraph of $K_{r-1} \vee tK_2$. Hence $\{I_{r-1} \vee sK_2 : s \in \mathbb{N}\} \subseteq \mathcal{M}(H)$. And $K_{r+1} = K_{r-1} \vee K_2$, so $H \neq K_{r+1}$. Hence Statement 3 of this theorem follows from Statement 2(c) of Corollary 3.6 by taking $w = 1$, Statement 3 of Theorem 1.2 and Proposition 1.4.

If either $H = K_{r+1}$ and $r \leq 3$, or $H = K_{1,s}$ for some $s \leq r$, then every graph in $\mathcal{M}(H)$ is $(r-1)$-degenerate and hence $p^{\mathcal{R}_r}_{\mathcal{M}(H)} = p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$. $\square$

Proof of Theorem 1.6: We first prove Statement 1. If $\tau(H) = 1$, then H is a disjoint union of a star and isolated vertices, so H is a subgraph of $K_1 \vee tK_r$ for some positive integer t, a contradiction. So $\tau(H) = 2$. Hence Statement 2 in Theorem 1.2 and Proposition 1.4 implies that $p^{\chi_r}_{\mathcal{M}(H)} = \Omega(n^{-2/(r(r+1))})$. Since every minor of $K_1 \vee tK_r$ is a subgraph of $K_1 \vee tK_r$, we know $\{I_1 \vee sK_r : s \in \mathbb{N}\} \subseteq \mathcal{M}(H)$. By Statement 2(b) of Corollary 3.6 by taking $w = r-1$, we know $p^{\chi_r}_{\mathcal{M}(H)} = O(n^{-2/(r(r+1))})$. This proves Statement 1.

Statement 2 follows from the last sentence of Theorem 1.2 and Proposition 1.4. □

Proof of Corollary 1.7 By Theorem 1.2, $p^{\mathcal{D}_3}_{\mathcal{M}(K_{3,3})} = \Theta(n^{-1/5})$. Since the set of planar graphs is a subset of $\mathcal{M}(K_{3,3})$, Proposition 1.1 implies that the threshold for $\mathcal{G}_{\text{planar}}$ and $\mathcal{D}_3$ is $\Omega(n^{-1/5})$. By Proposition 1.4, the thresholds for $\mathcal{G}_{\text{planar}}$ and for the properties $\mathcal{D}_3$, χ_3 and χ^ℓ_3 are $\Omega(n^{-1/5})$. On the other hand, let I_{r-w} be the edgeless graph on $r-w$ vertices. Then $I_{r-w} \vee tK_{w+1}$ is planar for every positive integer t, when $r = 3$ and $w = 1$. Hence by Corollary 3.6 which is proved later as a corollary of the main theorem, the thresholds for being 2-degenerate and 3-choosable are both $O(n^{-1/5})$. Therefore, the thresholds for planar graphs for being 2-degenerate and 3-choosable are both $\Theta(n^{-1/5})$. Finally, the threshold for being 3-colorable is $O(n^{-1/6})$ by considering a disjoint union of copies of K_4 and none of the copies of K_4 can have all the six edges remaining in the random subgraph. □

7 Concluding Remarks and Comments

In this paper, we initiate a systematic study of threshold probabilities for monotone properties in the random model $G(p)$ where G belongs to a given proper minor-closed family $\mathcal{G}$. In particular, we study four properties (1) $\mathcal{D}_r$: being $(r-1)$-degenerate, (2) χ^ℓ_r: being r-choosable, (3) $\mathcal{R}_r$: non-existence of r-regular subgraphs, and (4) χ_r: being r-colorable.

In general, not much is known in the literature for the threshold probability $p^{\mathcal{P}}_{\mathcal{G}}$ when graphs in $\mathcal{G}$ are sparse. To the best of our knowledge, this is the first paper considering this problem when $\mathcal{G}$ is a minor closed family which is one of the most natural classes of sparse graphs.

We provide lower bounds for $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)}$ for all pairs (r, H) in which $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ and $p^{\chi^\ell_r}_{\mathcal{M}(H)}$ are not determined in this paper. The lower bounds for $p^{\mathcal{D}_r}_{\mathcal{M}(H)}$ offer immediate lower bounds for $p^{\mathcal{R}_r}_{\mathcal{M}(H)}$ and $p^{\chi_r}_{\mathcal{M}(H)}$. We do not try to strengthen those lower bounds for $p^{\mathcal{R}_r}_{\mathcal{M}(H)}$ and $p^{\chi_r}_{\mathcal{M}(H)}$ in this paper and leave the following question for future research.

Question 7.1. *For any integer $r \geq 2$ and graph H, what are $p^{\mathcal{R}_r}_{\mathcal{M}(H)}$ and $p^{\chi_r}_{\mathcal{M}(H)}$? And more generally, what are $p^{\mathcal{R}_r}_{\mathcal{G}}$ and $p^{\chi_r}_{\mathcal{G}}$ for any given proper minor-closed family?*

In this paper, the threshold we studied is also called the *crude* threshold. A sharp threshold is an alternation of Definition 1. Let $\mathcal{P}$ be a monotone property and $\mathcal{G}$ a family of graphs. A function $p^* : \mathbb{N} \to [0, 1]$ is an *(upper) sharp threshold* for $\mathcal{G}$ and $\mathcal{P}$ if the following hold.

1. for every sequence $(G_i)_{i\in\mathbb{N}}$ of graphs with $G_i \in \mathcal{G}$ and $|V(G_i)| \to \infty$ and any $\epsilon > 0$, the random subgraphs $G_i((1-\epsilon)p(n_i))$ are in $\mathcal{P}$ a.a.s. where $n_i = |V(G_i)|$;

2. there is some sequence $(G_i)_{i\in\mathbb{N}}$ of graphs with $G_i \in \mathcal{G}$ and $|V(G_i)| \to \infty$ such that for any $\epsilon > 0$, the random subgraphs $G_i((1+\epsilon)p(n_i))$ are not in $\mathcal{P}$ a.a.s. where $n_i = |V(G_i)|$.

In [17], Friedgut provides a necessary and sufficient condition to check whether there is a sharp threshold for a general class of random models. However it is not an easy task to apply to our model. The next natural question is:

Question 7.2. *What are the sharp thresholds for properties $\mathcal{D}_r, \chi_r^\ell, \chi_r, \mathcal{R}_r$ for minor-closed families?*

It is also interesting to study other global properties, where some natural algorithms are NP-hard even on some proper minor-closed families, such as the set of planar graphs.

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A Appendix

Proposition A.1. *For every integer r with $r \geq 2$ and every connected graph H, $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ if and only if $r-1 \geq d^*_H$.*

Proof. Since $d_H(n)$ is non-decreasing in n, we know $r-1 \geq d^*_H$ if and only if $r-1 \geq d_H(n)$ for every $n \in \mathbb{N}$. Hence it suffices to prove that $\mathcal{P}^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ if and only if $r-1 \geq d_H(n)$ for every $n \in \mathbb{N}$.

If $r-1 \geq d_H(n)$ for every $n \in \mathbb{N}$, then every graph $G \in \mathcal{M}(H)$ on sufficiently many vertices is already $(r-1)$-degenerate and thus the threshold probability is $\Theta(1)$.

Now we show that $p^{\mathcal{D}_r}{}_{\mathcal{G}} = \Theta(1)$ implies that $r-1 \geq d_H(n)$ for every $n \in \mathbb{N}$. For every graph $G \in \mathcal{M}(H)$, let $p(G)$ be the supremum of all p such that the random subgraph $G(p)$ is $(r-1)$-degenerate with probability at least 0.9. Note that such $p(G)$ exists since degeneracy is a monotone property. For every $n \in \mathbb{N}$, let $p(n)$ be the minimum of $p(G)$ among all graphs $G \in \mathcal{M}(H)$ on n vertices. Note that there are only finite number of graphs on n vertices. Since adding isolated vertices to any $G \in \mathcal{M}(H)$ results in a $G' \in \mathcal{M}(H)$ on more vertices, and $p(G) = p(G')$, the function p is non-increasing. Hence $\lim_{n\to\infty} p(n)$ exists.

Let $p^* = \lim_{n\to\infty} p(n)$. We claim that $p^* = 1$ or $p^* = 0$. Suppose to the contrary that $0 < p^* < 1$. Let p' be any real number with $0 < p' < p^*$. Let $G \in \mathcal{M}(H)$ be a graph such that $p(G) < 1$, and let a be the probability that $G(p')$ is $(r-1)$-degenerate. Thus $0.9 \leq a$. Since $p(G) < 1$, G is not $(r-1)$-degenerate, $a \leq 1 - {p'}^{e(G)}$. In particular, $0 < a < 1$. For every $k \in \mathbb{N}$, let G_k be a union of k disjoint copies of G. Thus when $k \geq \lceil \log_a(1/2) \rceil$, the probability that at least one copy of $G_k(p')$ is not $(r-1)$-degenerate is $1 - a^k \geq 1 - a^{\lceil \log_a(1/2) \rceil} \geq 0.5 > 0.1$. So $p(G_k) \leq p'$ for every $k \geq \lceil \log_a(1/2) \rceil$. That is, $p(|V(G_k)|) \leq p'$ for every $k \geq \lceil \log_a(1/2) \rceil$. Hence $(p(n_k) : k \geq \lceil \log_a(1/2) \rceil)$ is a subsequence of $(p(n) : n \in \mathbb{N})$, where $n_k = |V(G_k)|$, such that $p(n_k) \leq p'$ for every $k \geq \lceil \log_a(1/2) \rceil$. Therefore, $p^* \leq p'$, a contradiction.

Suppose $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$ and $p^* = 0$. Let $q(n) = \max\{p(n) + \frac{1}{n}, 1\}$ for every $n \in \mathbb{N}$. Since $q(n) > p(n)$ for every $n \in \mathbb{N}$, there exist $G_1, G_2, ...$ such that $|V(G_n)| = n$ and $\Pr(G_n(q) \in \mathcal{D}_r) < 0.9$ for every $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \frac{q(n)}{1} \leq p^* = 0$, but $\lim_{n\to\infty} \Pr(G_n(q) \in \mathcal{D}_r) \neq 1$, contradicting $p^{\mathcal{D}_r}_{\mathcal{M}(H)} = \Theta(1)$.

Therefore, if $p^{\mathcal{D}_r}{}_{\mathcal{G}} = \Theta(1)$, then $p^* = 1$. Thus $p(n) = 1$ for all $n \in \mathbb{N}$ since $p(n)$ is non-increasing in n. Suppose that there exists $G \in \mathcal{M}(H)$ such that G is not $(r-1)$-degenerate. Let $w = (0.2)^{1/n^2}$. Then $\Pr(G(w) \in \mathcal{D}_r) = 1 - \Pr(G(w) \notin \mathcal{D}_r) \leq 1 - \Pr(G(w) = G) = 1 - w^{|E(G)|} \leq 1 - w^{n^2} < 0.9$. So $p(G) \leq w$ and hence $p(|V(G)|) < 1$, a contradiction.

Therefore every graph in $\mathcal{M}(H)$ is $(r-1)$-degenerate, which is equivalent to $r-1 \geq d_H(n)$ for every $n \in \mathbb{N}$. □

Proposition A.2. *Let H be a graph. Then $f^*_H \leq d^*_H \leq 2f^*_H$.*

Proof. By the definition of d_H, every H-minor free graph G is $d_H(|V(G)|)$-degenerate, so G contains a vertex of degree at most $d_H(|V(G)|)$. Hence every H-minor free graph on n vertices contains at most $\sum_{i=1}^{n} d_H(i)$ edges by induction. Since d_H is non-decreasing, every H-minor free graph on n vertices contains at most $\sum_{i=1}^{n} d_H(i) \leq d_H(n)n$ edges. That is, $f_H(n) \leq d_H(n)n$ for every $n \in \mathbb{N}$. Hence $f_H^* = \sup_{n\in\mathbb{N}} \frac{f_H(n)}{n} \leq \sup_{n\in\mathbb{N}} d_H(n) = d^*$.

By the definition of f_H^*, $|E(G)|/|V(H)| \leq f_H^*$ for every H-minor free graph G. So every H-minor free graph G contains a vertex of degree at most $2|E(G)|/|V(G)| \leq 2f_H^*$. Hence every H-minor free graph is $2f_H^*$-degenerate. That is, $d_H(n) \leq 2f_H^*$ for every $n \in \mathbb{N}$. Therefore, $d^* = \sup_{n\in\mathbb{N}} d_H(n) \leq 2f_H^*$. □

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