

Non-bipartite k -common graphs*

Daniel Král[†] Jonathan A. Noel[‡] Sergey Norin[§]
 Jan Volec[¶] Fan Wei^{||}

Abstract

A graph H is k -common if the number of monochromatic copies of H in a k -edge-coloring of K_n is asymptotically minimized by a random coloring. For every k , we construct a connected non-bipartite k -common graph. This resolves a problem raised by Jagger, Št'ovíček and Thomason [Combinatorica 16 (1996), 123–141]. We also show that a graph H is k -common for every k if and only if H is Sidorenko and that H is locally k -common for every k if and only if H is locally Sidorenko.

1 Introduction

Ramsey's Theorem states that for every graph H and integer $k \geq 2$, there exists a natural number $R_k(H)$ such that if $N \geq R_k(H)$, then every k -edge-coloring

*The work of the first, second and fourth authors has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 648509). This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains. The first and the fourth were also supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University. The second author was also supported by the Leverhulme Trust Early Career Fellowship ECF-2018-534. The third author was supported by an NSERC Discovery grant. The fifth author was supported by IAS Founders' Circle funding provided by Cynthia and Robert Hillas.

[†]Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic, and Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK. E-mail: dkral@fi.muni.cz.

[‡]Mathematics Institute and DIMAP, University of Warwick, Coventry CV4 7AL, UK. E-mail: j.noel@warwick.ac.uk.

[§]Department of Mathematics and Statistics, McGill University, Montreal, Canada. E-mail: sergey.norin@mcgill.ca.

[¶]Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic. Previous affiliation: Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic. E-mail: jan@ucw.cz.

^{||}School of Mathematics, Institute for Advanced Study, Princeton, US. E-mail: fanwei@ias.edu.

of the complete graph K_N with N vertices contains a monochromatic copy of H . We study the natural quantitative extension of this question, which was first considered by Goodman [15]: *What is the minimum number of monochromatic copies of H in a k -edge-coloring of K_N for large N ?*

A prevailing theme in Ramsey Theory, dating back to an idea of Erdős [7] from the 1940s, is that one of the best ways to avoid monochromatic substructures is by coloring randomly. Therefore, it would be natural to expect the answer to the above question to be the number of monochromatic copies of H in a uniformly random k -edge-coloring of K_N . Following [18], we say that a graph H is *k -common* if the uniformly random k -edge-coloring of K_N asymptotically minimizes the number of monochromatic copies of H . In other words, the number of monochromatic (labelled) copies of H in every k -edge-coloring of K_N is at least

$$(1 - o(1)) \frac{N^{|H|}}{k^{\|H\|-1}}$$

where $|H|$ and $\|H\|$ denote the number of vertices and edges of H , respectively. The most well-studied case is that of 2-common graphs, which are often referred to as *common* graphs; however, we will always say 2-common to avoid any ambiguity.

Only a handful of graphs are known to be 2-common and even fewer are known to be k -common for $k \geq 3$. The well-known Goodman Bound [15] implies that K_3 is 2-common; another proof was given by Lorden [20]. This result led Erdős [8] to conjecture that every complete graph is 2-common and Burr and Rosta [1] to extend the conjecture to all graphs. We now know that 2-common graphs are far more scarce than Erdős, Burr and Rosta had anticipated. Sidorenko [28] disproved the Burr–Rosta Conjecture by showing that a triangle with a pendant edge is not 2-common. Around the same time, Thomason [32] showed that K_p is not 2-common for any $p \geq 4$, thereby disproving the original conjecture of Erdős [8]. Additional constructions showing that K_p is not 2-common for $p \geq 4$ have since been found [11, 12, 33]. Determining the asymptotics of the minimum number of monochromatic copies of K_4 in 2-edge-colorings of large complete graphs continues to attract a good amount of attention [14, 24, 30] and remains one of the most mysterious problems in extremal graph theory (with no conjectured answer).

Jagger, Štoviček and Thomason [18, Theorem 12] extended the result from [32] by showing that no graph containing a copy of K_4 is 2-common. On the positive side, Sidorenko [28] showed that all odd cycles are 2-common and Jagger, Štoviček and Thomason [18, Theorem 8] that all even wheels are 2-common. Additional examples of 2-common graphs can be obtained by certain gluing operations [18, 27]. However, these operations do not increase the chromatic number and, for a long time, no examples of 2-common graphs with chromatic number greater than three were known. Only in 2012, the 5-wheel, which has chromatic number four, was shown to be 2-common [17] using Razborov’s flag algebra method [25]; this result settled a problem of [18].

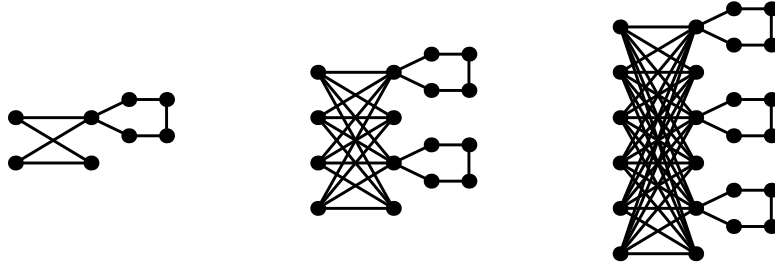


Figure 1: Examples of graphs from the statement of Theorem 1 for $n = 1, 2, 3$. The three graphs are denoted by $K_{2,2,C_5}$, $K_{4,4,C_5}$ and $K_{6,6,C_5}$ in Section 3.

Much less is known about k -common graphs for $k \geq 3$. Cummings and Young [6] proved that every 3-common graph is triangle-free, which implies that the same is true for k -common graphs for any $k \geq 3$ (see Section 2 for details). The only known examples of k -common graphs for $k \geq 3$ are bipartite graphs that are known to be Sidorenko. Jagger, Št'ovíček and Thomason [18, Section 5] asked about the existence of non-bipartite k -common graphs; no examples of such graphs are known, even for $k = 3$. We resolve this by showing the following.

Theorem 1. *For every $k \geq 2$, there exists n_k such that, for every $n \geq n_k$, the graph obtained from $K_{2n,2n}$ by pasting a copy of C_5 on every second vertex in one of the two parts of $K_{2n,2n}$ is k -common.*

Examples of graphs described in the statement of Theorem 1 can be found in Figure 1. We remark that one of the key ingredients in the proof of Theorem 1 is establishing that such graphs are k -common in a certain “local” sense (see Lemma 12), which is proved using spectral arguments.

As we have already mentioned, there is a close connection between k -common graphs and Sidorenko graphs. We say that a graph H is *Sidorenko* if the number of copies of H in a graph with edge density d is asymptotically minimized by the random graph with edge density d . Sidorenko’s Conjecture [26, 29] famously asserts that every bipartite graph H is Sidorenko; an equivalent conjecture was made earlier by Erdős and Simonovits [9]. It is easy to show that every Sidorenko graph is bipartite and k -common for every $k \geq 2$. There are now many families of bipartite graphs that are known to be Sidorenko, see, e.g., [2–5, 16, 19, 31]; prior to our work, these graphs were the only known examples of k -common graphs for any fixed $k \geq 3$.

The following simple construction of [18, Theorem 14] shows that, for every non-bipartite graph H , there exists $k \geq 2$ such that H is not k -common. Split the vertices of K_N into 2^{k-1} sets of roughly equal size, indexed by $0, \dots, 2^{k-1} - 1$. Color the edges between the i -th and j -th sets with the color corresponding to the first bit on which i and j differ in their binary representations and color the edges inside each set with the color k . Since H is non-bipartite, the only monochromatic

copies of H are inside the sets and thus their number is $(1+o(1))N^{|H|}2^{-(k-1)(|H|-1)}$. Thus, if Sidorenko’s Conjecture is true, then Sidorenko graphs are precisely the graphs that are k -common for every $k \geq 2$. We prove this without the assumption that Sidorenko’s Conjecture holds.

Theorem 2. *A graph H is k -common for all $k \geq 2$ if and only if it is Sidorenko.*

We also establish the variant of Theorem 2 in the local setting, i.e., when the edge-coloring is “close” to the random edge-coloring. The notion of locally k -common graphs is formally defined in Section 2. Recall that the *girth* of a graph is the length of its shortest cycle.

Theorem 3. *The following holds for every $k \geq 3$: if a graph H has odd girth, then H is not locally k -common.*

Since a theorem of Fox and the last author [10] asserts that all forests and graphs of even girth are locally Sidorenko, Theorem 3 implies for every $k \geq 3$ that a graph H is locally k -common if and only if H is locally Sidorenko. We remark that Theorem 3 strengthens the result of Cummings and Young [6] that no graph containing a triangle is 3-common by showing that such graphs are not even locally 3-common.

2 Preliminaries

In this section, we fix the notation used throughout the paper and present basic properties of k -common graphs. We also introduce the terminology of the theory of graph limits. While, strictly speaking, everything in this paper can be phrased in terms of finite graphs, the language of graph limits has the great advantage that it allows us to eliminate almost all discussion of “small order” asymptotic terms. Our notation and terminology mainly follows that of the monograph of Lovász [22], and we refer the reader to [22] for a more thorough introduction.

We write \mathbb{N} for the set of all positive integers and $[k]$ for the set of the first k positive integers, i.e., $[k] = \{1, \dots, k\}$. We work with the Borel measures on \mathbb{R}^d throughout the paper and if $A \subseteq [0, 1]^d$ is a measurable subset of \mathbb{R}^d , we write $|A|$ for its measure. Graphs that we consider in this paper are finite and simple. If G is a graph, then its vertex set is denoted by $V(G)$ and its edge set by $E(G)$; the cardinalities of $V(G)$ and $E(G)$ are denoted by $|G|$ and $\|G\|$, respectively. A *homomorphism* from a graph H to a graph G is a function $f : V(H) \rightarrow V(G)$ such that $f(u)f(v) \in E(G)$ whenever $uv \in E(H)$. The *homomorphism density* of H in G is the probability that a random function from $V(H)$ to $V(G)$ is a homomorphism, i.e., it is the number of homomorphisms from H to G divided by $|G|^{\|H\|}$. We denote the homomorphism density of H in G by $t(H, G)$.

A *graphon* is a measurable function $W : [0, 1]^2 \rightarrow [0, 1]$ that is symmetric, i.e., $W(x, y) = W(y, x)$ for all $(x, y) \in [0, 1]^2$. Intuitively, a graphon can be thought

of as a continuous variant of the adjacency matrix of a graph. The graphon that is equal to $p \in [0, 1]$ everywhere is called the p -constant graphon; when there will be no confusion, we will just use p to denote such a graphon. A graphon W is a *step graphon* if there exist a partition of $[0, 1]$ into non-null subsets A_1, \dots, A_m such that W is constant on each of the sets $A_i \times A_j$, $i, j \in [m]$. The sets A_i , $i \in [m]$, are called *parts* of the step graphon W ; the sets $A_i \times A_j$, $i, j \in [m]$, are *tiles* and those with $i = j$ are *diagonal tiles*.

The notion of *homomorphism density* extends to graphons by setting

$$t(H, W) := \int_{[0,1]^{V(H)}} \prod_{uv \in E(H)} W(x_u, x_v) dx_{V(H)} \quad (1)$$

for a graph H and graphon W . Define the *density* of a graphon W to be $t(K_2, W)$. The quantity $t(H, W)$ has a natural interpretation in terms of sampling a random graph according to W : for an integer n , choose n independent uniform random points x_1, \dots, x_n from the interval $[0, 1]$ and create a graph with the vertex set $[n]$ by joining the vertices i and j with probability $W(x_i, x_j)$. The graph constructed in this way is called a *W -random graph* and denoted by $G_{n,W}$. If $H = [n]$, then $t(H, W)$ is precisely the probability that $G_{n,W}$ is H . It can be shown that the following holds for every graph H with probability one:

$$\lim_{n \rightarrow \infty} t(H, G_{n,W}) = t(H, W).$$

A sequence $(G_i)_{i \in \mathbb{N}}$ of graphs is *convergent* if the sequence $(t(H, G_i))_{i \in \mathbb{N}}$ converges for every graph H . A simple diagonalization argument implies that every sequence of graphs has a convergent subsequence. We say that a graphon W is a *limit* of a convergent sequence $(G_i)_{i \in \mathbb{N}}$ of graphs if

$$\lim_{i \rightarrow \infty} t(H, G_i) = t(H, W)$$

for every graph H . One of the crucial results in graph limits, due to Lovász and Szegedy [23], is that every convergent sequence of graphs has a limit. Hence, a graph H is Sidorenko if and only if $t(H, W) \geq t(K_2, W)^{\|H\|}$ for every graphon W . Similarly, the property of being k -common translates to the language of graph limits as follows. A graph H is k -common if

$$t(H, W_1) + \dots + t(H, W_k) \geq \frac{1}{k^{\|H\|-1}}$$

for any graphons W_1, \dots, W_k such that $W_1 + \dots + W_k = 1$.

We pause the exposition of graph limit theory to demonstrate how the just introduced notions are convenient for establishing some basic properties of k -common graphs. Jagger, Šťovíček and Thomason [18, Theorem 13] observed that if H is not k -common, then H is not ℓ -common for any $\ell \geq k$. We now

present their argument in the language of graph limits. Suppose that H is not k -common, i.e., there exists graphons W_1, \dots, W_k such that $W_1 + \dots + W_k = 1$ and $t(H, W_1) + \dots + t(H, W_k) < k^{-\|H\|+1}$. Consider an integer $\ell > k$. We set $W'_i = \frac{k}{\ell}W_i$ for $i \in [k]$ and $W'_i = 1/\ell$ for $i \in [\ell] \setminus [k]$. Observe that

$$\begin{aligned} t(H, W'_1) + \dots + t(H, W'_\ell) &= \left(\frac{k}{\ell}\right)^{\|H\|} (t(H, W_1) + \dots + t(H, W_k)) + \frac{\ell - k}{\ell^{\|H\|}} \\ &< \frac{k}{\ell^{\|H\|}} + \frac{\ell - k}{\ell^{\|H\|}} = \ell^{-\|H\|+1}, \end{aligned}$$

which implies that H is not ℓ -common. Hence, we can define $\kappa(H)$ to be the smallest integer k such that H is not k -common; if no such integer exists, we set $\kappa(H) = \infty$. That is, H is k -common if and only if $2 \leq k < \kappa(H)$. In particular, Theorem 2 asserts that H is Sidorenko if and only if $\kappa(H) = \infty$. In Section 1, for any non-bipartite graph H , we exhibited a k -edge-coloring of K_N from [18] which has $(1 + o(1))N^{|H|}2^{-(k-1)(|H|-1)}$ monochromatic copies of H . It follows that H is not k -common for any k satisfying $2^{-(k-1)(|H|-1)} < k^{-\|H\|+1}$. In particular, if H is a non-bipartite connected graph with average degree d , then $\kappa(H) \leq \lceil d \log_2 d \rceil$. If the chromatic number of H is larger than three, a better upper bound on $\kappa(H)$, where the base of the logarithm is replaced with $\chi(H) - 1$, can be obtained by considering the edge-coloring obtained by splitting vertices of K_N to $(\chi(H) - 1)^{k-1}$ roughly equal parts and defining the edge-coloring based on the base $(\chi(H) - 1)$ representations of the indices of the parts.

Let us return to our brief introduction to notions from the theory of graph limits that we use in this paper. A graphon W can be thought of as an operator on $L_2[0, 1]$ where the image of a function $f \in L_2[0, 1]$ is given by

$$\int_0^1 W(x, y)f(y) dy.$$

Every such operator is compact and so its spectrum $\sigma(W)$ is either finite or countably infinite, the only accumulation point of $\sigma(W)$ can be zero and every non-zero element of $\sigma(W)$ is an eigenvalue of W [22, Section 7.5]. In addition, all elements of $\sigma(W)$ are real and the largest is at least the density of W . We define $\hat{\sigma}(W)$ to be the multiset containing all non-zero elements λ of $\sigma(W)$, with multiplicity equal to the dimension of the kernel of $(W - \lambda)$, which is finite. In the graph case, the trace of the n -th power of the adjacency matrix of a graph G , which is equal to the sum of the n -th powers of the eigenvalues of the matrix, is the number of homomorphisms from C_n to G , i.e., it is $t(C_n, G)|G|^n$ [22, Equation (5.31)]. We will need the analogous statement for graphons, which we now state as a proposition.

Proposition 4 (Lovász [22, Equation (7.22)]). *Let W be a graphon. It holds for every $n \geq 3$:*

$$t(C_n, W) = \sum_{\lambda \in \hat{\sigma}(W)} \lambda^n.$$

There are several useful metrics on graphons. One of the most important from the perspective of graph limit theory is the metric induced by the cut norm. A *kernel* is a bounded symmetric measurable function from $[0, 1]^2$ to \mathbb{R} ; a kernel can be thought of as a continuous variant of the adjacency matrix of an edge-weighted graph. We define the *cut norm* of a kernel U to be

$$\|U\|_{\square} := \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} U(x, y) \, dx \, dy \right|$$

where the supremum is over all measurable subsets S and T of $[0, 1]$. The *cut distance* of graphons W and W' is the cut norm $\|W - W'\|_{\square}$ of their difference. If two graphons have small cut distance, then their homomorphism densities do not differ substantially, as the next lemma shows.

Lemma 5 (Lovász [22, Lemma 10.23]). *Let W and W' be two graphons and H a graph. It holds that $|t(H, W) - t(H, W')| \leq \|H\| \cdot \|W - W'\|_{\square}$.*

Lemma 5 asserts that two graphons which are close in the cut distance have similar homomorphism densities. The next lemma allows us to find a step graphon of bounded complexity that is close in cut distance to any graphon.

Lemma 6 (Frieze and Kannan [13]; see also [22, Lemma 9.3]). *For every $\varepsilon > 0$, there exists an integer $M \in \mathbb{N}$ such that for every graphon W , there exists a step graphon W' with at most M parts, all of equal sizes, such that the densities of W and W' are the same and $\|W - W'\|_{\square} \leq \varepsilon$.*

The homomorphism density function extends naturally to kernels U by also setting $t(H, U)$ to be the integral in (1) with W replaced by U . A graphon W which is close to the p -constant graphon is naturally reparameterized as $p + \varepsilon U$ for some kernel U and small $\varepsilon > 0$. The following proposition provides a useful expansion for expressions of the form $t(H, p + \varepsilon U)$, which implicitly appeared in [21, 28]; we use the formulation from the proof of Proposition 16.27 in [22].

Proposition 7. *Let U be a kernel, H a graph and $p \in [0, 1]$. It holds that*

$$t(H, p + \varepsilon U) = \sum_{F \subseteq E(H)} t(H[F], U) p^{|H| - |F|} \varepsilon^{|F|}$$

where $H[F]$ is the spanning subgraph of H with the edge set F .

A natural weakening of Sidorenko's Conjecture is to ask whether it holds locally, which has been considered in [21] and in [22, Chapter 16]. Here, we consider a stronger notion discussed in [10]: a graph H is *locally Sidorenko* if there exists $\varepsilon_0 > 0$ such that for every graphon W with density p such that $\|W - p\|_{\square} \leq \varepsilon_0 p$ and $\|W - p\|_{\infty} \leq p$, it holds that $t(H, W) \geq p^{|H|}$. The following theorem characterized locally Sidorenko graphs.

Theorem 8 (Fox and Wei, see [10]). *A graph H is locally Sidorenko if and only if H is forest or its girth is even.*

Similarly, we say that a graph H is *locally k -common* if for every $k \geq 2$, there exists $\varepsilon_0 > 0$ such that

$$t(H, W_1) + \cdots + t(H, W_k) \geq k^{-\|H\|+1}$$

for all graphons W_1, \dots, W_k such that $W_1 + \cdots + W_k = 1$, $\|W_i - 1/k\|_{\square} \leq \varepsilon_0/k$ and $\|W_i - 1/k\|_{\infty} \leq 1/k$ for all $i \in [k]$.

We next define a notion of a subgraphon which is somewhat more involved than just restricting a graphon to a measurable subset of $[0, 1]$ and rescaling. This will be used in the proof of Theorem 1 to apply induction on k within a “sparse” part of one of the graphons W_1, W_2, \dots, W_k . Fix a graphon W and a real $\delta > 0$ and consider the set $\mathcal{A}(W, \delta)$ of all measurable functions $h : [0, 1] \rightarrow [0, 1]$ such that

$$\int_{[0,1]^2} h(x)W(x,y)h(y) \, dx \, dy \leq \delta \|h\|_1^2.$$

Intuitively, for $\|h\|_1 > 0$, one can think of h as a weight function on $[0, 1]$ with the property that, if x and y are chosen independently at random according to the probability measure induced by $h/\|h\|_1$, then the expected value of $W(x, y)$ is at most δ . We define the δ -independence ratio of W to be

$$\alpha_{\delta}(W) := \sup_{h \in \mathcal{A}(W, \delta)} \|h\|_1.$$

Let $h : [0, 1] \rightarrow [0, 1]$ be a measurable function such that $\|h\|_1 > 0$ and let $f : [0, \|h\|_1] \rightarrow [0, 1]$ be a measurable function defined by

$$f(z) := \inf \left\{ t \in [0, 1], \int_{[0,t]} h(x) \, dx \geq z \right\}.$$

Observe that

$$\int_A h(x) \, dx = |f^{-1}(A)|$$

for every measurable subset A of $[0, 1]$. The subgraphon of W induced by h , which is denoted by $W[h]$, is the graphon defined by

$$W[h](x, y) := W(f(x \cdot \|h\|_1), f(y \cdot \|h\|_1))$$

for every $(x, y) \in [0, 1]^2$. One way to think of the graphon $W[h]$ is as follows. Choose n points independently at random according to the probability measure induced by $h/\|h\|_1$ and form a graph $G_{n,W,h}$ with vertex set $[n]$ by joining vertices i

and j with probability $W(x_i, x_j)$. Then $W[h]$ is a limit of the sequence $(G_{n,W,h})_{n \in \mathbb{N}}$ with probability one. The definition of $W[h]$ implies that

$$t(H, W[h]) = \frac{1}{\|h\|_1^{|H|}} \int_{[0,1]^{V(H)}} \prod_{u \in V(H)} h(u) \prod_{uv \in E(H)} W(x_u, x_v) dx_{V(H)} \quad (2)$$

for every graph H . In particular, $t(H, W)$ is at least $\|h\|_1^{|H|} \cdot t(H, W[h])$.

We conclude this section by relating certain “reflection operations” to homomorphism densities. The arguments of this kind are standard in the area; however, we have decided to provide a self-contained exposition for completeness. Let H be a graph and let $U \subseteq V(H)$ be an independent set of vertices of H . For a graphon W , we define a function $t_W^H : [0, 1]^U \rightarrow \mathbb{R}$ as follows:

$$t_W^H(x_U) = \int_{[0,1]^{V(H) \setminus U}} \prod_{vv' \in E(H)} W(x_v, x_{v'}) dx_{V(H) \setminus U};$$

note that the function t_W^H depends on the choice of the set U , however, the choice of the set U will always be made clear and so we have decided not to include the set U to keep the notation simple. Informally speaking, the function $t_W^H(x_U)$ counts the number of homomorphic copies of H rooted at x_U . Observe that

$$t(H, W) = \int_{[0,1]^U} t_W^H(x_U) dx_U.$$

We now state a proposition, which gives a lower bound on the homomorphism density of a graph obtained by reflecting H along the set U .

Proposition 9. *Let H be a graph, $n \in \mathbb{N}$ and $U \subseteq V(H)$ an independent set of its vertices. Further, let H^n be the graph obtained by taking n copies of H and identifying the corresponding vertices of the set U , i.e., the graph H^n has $n|H| - (n-1)|U|$ vertices. The following holds for every graphon W :*

$$t(H^n, W) \geq t(H, W)^n.$$

Proof. Fix a graphon W . We consider both graphs H and H^n with the set U and note that $t_W^{H^n}(x_U) = t_W^H(x_U)^n$ for every $x_U \in [0, 1]^U$. Hence, it follows that

$$t(H^n, W) = \int_{[0,1]^U} t_W^{H^n}(x_U) dx_U \geq \left(\int_{[0,1]^U} t_W^H(x_U) dx_U \right)^n = t(H, W)^n$$

by Jensen’s Inequality. \square

The same argument translates to the rooted setting, which we formulate here for future reference but omit the proof as it is completely analogous to the proof of Proposition 9.

Proposition 10. *Let H be a graph, $n \in \mathbb{N}$, $U \subseteq V(H)$ an independent set of its vertices and $U' \subseteq V(H)$ an independent set that is a superset of U . Further, let H^n be the graph obtained from H taking n copies of H and identifying the corresponding vertices of the set U' . The following holds for every graphon W and every $x_U \in [0, 1]^U$:*

$$t_W^{H^n}(x_U) \geq t_W^H(x_U)^n.$$

The following is a particularly simple application of Proposition 9.

Proposition 11. *The following holds for every graphon W and every $n \in \mathbb{N}$:*

$$t(K_{2n,2n}, W) \geq t(K_{2,2}, W)^{n^2}.$$

Proof. For any positive integers a, b and m , the graph obtained by taking m copies of $K_{a,b}$ and identifying the vertices in the part of the bipartition of cardinality a is precisely $K_{a,mb}$. Thus, the proposition follows from two applications of Proposition 9: firstly, with $H = K_{2,2}$ and U being any part of the bipartition and secondly with $H = K_{2,2n}$ and U being the largest part of the bipartition. \square

3 Non-bipartite k -common graphs

This section is devoted to the proof of Theorem 1. For $a, b \geq 1$, we let $K_{2a,2b,C_5}$ be the graph obtained from $K_{2a,2b}$ by adding b disjoint copies of C_5 and identifying one vertex of each of these copies with a vertex in the part of $K_{2a,2b}$ containing $2b$ vertices, where all of these vertices are distinct. In particular, $K_{2n,2n,C_5}$ is the graph from the statement of Theorem 1. We start with proving that $K_{2n,2,C_5}$ is locally Sidorenko in a certain strong sense; note that the assumptions here are weaker than in the local Sidorenko property discussed in Section 2 since we do not require any bound on $\|W - p\|_\infty$. Moreover, we establish that the same cut distance threshold holds for all values of n , which does not automatically follow from Theorem 8.

Lemma 12. *For every $p_0 \in (0, 1)$, there exist $\varepsilon_0 \in (0, 1)$ such that the following holds. If W is a graphon with density $p \geq p_0$ such that $t(K_{2,2}, W) \leq p^4 + \varepsilon_0$, then $t(K_{2n,2,C_5}, W) \geq p^{4n+5}$ for all $n \in \mathbb{N}$.*

Proof. We show that the statement of the lemma holds for $\varepsilon_0 = p_0^7/16$. Throughout the proof, fix a graphon W with density $p \geq p_0$ such that $t(K_{2,2}, W) - p^4 = \varepsilon \leq \varepsilon_0$. If the set $\widehat{\sigma}(W)$ is finite, then set $I = [|\widehat{\sigma}(W)|]$ and set $I = \mathbb{N}$ otherwise. Let λ_i , $i \in I$, be the elements of $\widehat{\sigma}(W)$ listed in the decreasing order of their absolute value and if there are positive and negative eigenvalues with the same absolute value, then the positive ones are listed first. Further, let $g_i : [0, 1] \rightarrow \mathbb{R}$ be an eigenfunction corresponding to λ_i . Without loss of generality, we assume

that $\|g_i\|_2 = 1$ for all $i \in I$ and that the eigenfunctions are orthogonal to one another, i.e.,

$$\int_{[0,1]} g_i(x)g_{i'}(x) dx = 0$$

for any two distinct i and i' from I . Since it holds that

$$\lambda_1 = \max_{\substack{f \in L_2[0,1] \\ \|f\|_2=1}} \int_{[0,1]^2} f(x)W(x,y)f(y) dx dy,$$

it follows $\lambda_1 \geq p$. In particular, $\lambda_1 \geq p_0$.

For every $x \in [0, 1]$, we define a measurable function $f_x : [0, 1] \rightarrow [0, 1]$ by setting $f_x(y) = W(x, y)$ for all $y \in [0, 1]$, i.e., f_x describes the “neighborhood” of x in the graphon W . We next define functions α_i such that $\alpha_i(x)$ would be the coordinate of f_x with respect to g_i , $i \in I$, for an orthonormal basis extending g_i , $i \in I$, i.e.,

$$\alpha_i(x) = \int_{[0,1]} g_i(y)f_x(y) dy.$$

Since the L_2 -norm of f_x is at most one and the functions g_i , $i \in I$, are orthogonal and have L_2 -norm one, we obtain that

$$\sum_{i \in I} \alpha_i(x)^2 \leq 1 \tag{3}$$

for every $x \in [0, 1]$. Next consider a cycle C_k and let U consist of any single vertex of C_k . Using the facts that the functions g_i , $i \in I$, are orthogonal, have L_2 -norm equal to one and are eigenfunctions for W , we get that

$$\begin{aligned} t_W^{C_k}(x) &= \int_{[0,1]^{k-1}} f_x(y_1)W(y_1, y_2)W(y_2, y_3) \cdots W(y_{k-2}, y_{k-1})f_x(y_{k-1}) dy_1 \cdots y_{k-1} \\ &= \sum_{i \in I} \lambda_i^{k-2} \alpha_i(x)^2 \end{aligned} \tag{4}$$

holds for every $k \geq 3$ and $x \in [0, 1]$. It follows that

$$t(C_k, W) = \int_{[0,1]} t_W^{C_k}(x) dx = \sum_{i \in I} \lambda_i^{k-2} \int_{[0,1]} \alpha_i(x)^2 dx. \tag{5}$$

On the other hand, Proposition 4 tells us that

$$t(C_k, W) = \sum_{i \in I} \lambda_i^k. \tag{6}$$

In particular, we obtain for $k = 4$ that

$$\varepsilon = t(K_{2,2}, W) - p^4 = \sum_{i \in I} \lambda_i^4 - p^4 \geq \sum_{i \in I \setminus \{1\}} \lambda_i^4,$$

which implies that $|\lambda_i| \leq \varepsilon^{1/4}$ for every $i \in I \setminus \{1\}$. In particular, λ_1 has multiplicity one.

In order for the right side of (5) to equal the right side of (6) for every $k \geq 3$ simultaneously, it must hold that, for every $\lambda \in \sigma(W)$:

$$\sum_{\substack{i \in I \\ \lambda_i = \lambda}} \int_{[0,1]} \alpha_i(x)^2 dx = |\{i \in I, \lambda_i = \lambda\}| \cdot \lambda^2.$$

Since λ_1 has multiplicity one, we obtain that

$$\int_{[0,1]} \alpha_1(x)^2 dx = \lambda_1^2. \quad (7)$$

Our aim is to estimate $t_W^{K_{2n,2}, C_5}(x)$ where U is the set consisting of the vertex shared by $K_{2n,2}$ and C_5 . Observe that

$$t_W^{K_{2n,2}, C_5}(x) = t_W^{K_{2n,2}}(x) \cdot t_W^{C_5}(x) \quad (8)$$

where U is the set consisting of one of the vertices of the 2-vertex part in the case of $K_{2n,2}$. We start by rewriting the identity (4) for $k = 4$ and $k = 5$:

$$t_W^{C_4}(x) = \lambda_1^2 \alpha_1^2(x) + \sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \quad (9)$$

$$t_W^{C_5}(x) = \lambda_1^3 \alpha_1^2(x) + \sum_{i \in I \setminus \{1\}} \lambda_i^3 \alpha_i^2(x). \quad (10)$$

Note that all of the terms on the right sides of these two expressions are non-negative, except for possibly the summation in (10). By Proposition 10 and (9),

$$\begin{aligned} t_W^{K_{2n,2}}(x) &\geq t_W^{C_4}(x)^n \\ &= \left(\lambda_1^2 \alpha_1^2(x) + \sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right)^n \\ &\geq \lambda_1^{2n} \alpha_1^{2n}(x) + \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \end{aligned} \quad (11)$$

Our next goal is to show that, unless f_x is almost completely orthogonal to g_1 , the homomorphism density of $K_{2n,2}, C_5$ rooted at x is at least its expected average value. Specifically, we will set $\pi = p_0^2/2$ and show that if $\alpha_1^2(x) \geq \pi$, then

$$t_W^{K_{2n,2}, C_5}(x) \geq \lambda_1^{2n+3} \alpha_1^{2n+2}(x). \quad (12)$$

To this end, we substitute (10) and (11) into (8) to obtain

$$t_W^{K_{2n,2}, C_5}(x) \geq \left(\lambda_1^{2n} \alpha_1^{2n}(x) + \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \\ \times \left(\lambda_1^3 \alpha_1^2(x) + \sum_{i \in I \setminus \{1\}} \lambda_i^3 \alpha_i^2(x) \right).$$

Multiplying out, we obtain four terms. One of them is the right side of (12) and the remaining three terms are as follows:

$$\lambda_1^{2n+1} \alpha_1^{2n}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right), \\ \lambda_1^{2n} \alpha_1^{2n}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^3 \alpha_i^2(x) \right) \text{ and} \\ \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^3 \alpha_i^2(x) \right).$$

So, to establish (12), we need to show that the sum of these three terms is non-negative. We first consider the sum of half of the first term and the whole of the second term. Since $p_0 \leq \lambda_1$ and $\lambda_i \leq \varepsilon^{1/4}$ for all $i \in I \setminus \{1\}$, we get

$$\frac{1}{2} \lambda_1^{2n+1} \alpha_1^{2n}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) + \lambda_1^{2n} \alpha_1^{2n}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^3 \alpha_i^2(x) \right) \\ \geq \left(\frac{p_0}{2} - \varepsilon^{1/4} \right) (\lambda_1^{2n} \alpha_1^{2n}(x)) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \geq 0.$$

Next, we estimate the sum of half of the first term and the third term as follows:

$$\begin{aligned}
& \frac{1}{2} \lambda_1^{2n+1} \alpha_1^{2n}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \\
& + \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^3 \alpha_i^2(x) \right) \\
& \geq \left(\frac{1}{2} \lambda_1^3 \alpha_1^2(x) - \sum_{i \in I \setminus \{1\}} |\lambda_i|^3 \alpha_i^2(x) \right) \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \\
& \geq \left(\frac{p^3 \pi}{2} - \varepsilon^{3/4} \sum_{i \in I \setminus \{1\}} \alpha_i^2(x) \right) \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right) \\
& \geq \left(\frac{p_0^5}{4} - \varepsilon^{3/4} \right) \lambda_1^{2n-2} \alpha_1^{2n-2}(x) \left(\sum_{i \in I \setminus \{1\}} \lambda_i^2 \alpha_i^2(x) \right).
\end{aligned}$$

The last inequality follows from (3). The final expression is non-negative (with room to spare) by the choice of ε_0 .

The statement would now follow from (7), (12) and a convexity argument if $\alpha_1^2(x) \geq \pi$ held for almost all $x \in [0, 1]$. If this is not the case, then we show that the “neighborhoods” of $x \in [0, 1]$ differ substantially and the homomorphism density of C_4 (and so $K_{2n,2}$) rooted at each $x \in [0, 1]$ with $\alpha_1^2(x) \geq \pi$ must be significantly higher than expected. We now quantify this intuition. Let X_1 be the set of $x \in [0, 1]$ such that $\alpha_1^2(x) \geq \pi$ and let $\delta = 1 - |X_1|$. By (7), we have

$$\int_{X_1} \alpha_1^2(x) dx = \int_{[0,1]} \alpha_1^2(x) dx - \int_{[0,1] \setminus X_1} \alpha_1^2(x) dx \geq \lambda_1^2 - \delta \pi.$$

The right side of this expression is positive which, in particular, implies that $|X_1| = 1 - \delta > 0$. Using Jensen’s Inequality, we have

$$\begin{aligned}
\int_{X_1} \alpha_1^{2n+2}(x) dx & \geq \frac{(\lambda_1^2 - \delta \pi)^{n+1}}{(1 - \delta)^n} \\
& = \left(\frac{\lambda_1^2 - \delta \pi}{1 - \delta} \right)^{n-1} \cdot \frac{\lambda_1^4 - 2\delta \pi \lambda_1^2 + \delta^2 \pi^2}{1 - \delta} \\
& \geq \lambda_1^{2n-2} \cdot \frac{\lambda_1^4 - 2\delta \pi \lambda_1^2 + \delta^2 \pi^2}{1 - \delta} \\
& = \lambda_1^{2n-2} \cdot \left(\lambda_1^4 + \frac{\delta \lambda_1^4 - 2\delta \pi \lambda_1^2 + \delta^2 \pi^2}{1 - \delta} \right) \\
& \geq \lambda_1^{2n+2} + \lambda_1^{2n-2} \cdot \frac{\delta \lambda_1^2 (\lambda_1^2 - 2\pi)}{1 - \delta} \geq \lambda_1^{2n+2}.
\end{aligned}$$

In the step between the second and third lines and in the last line, we used the fact that $2\pi = p_0^2 \leq p^2 \leq \lambda_1^2$. Since the estimate (12) holds for every $x \in X_1$, we obtain that

$$t(K_{2n,2,C_5}, W) \geq \int_{X_1} t_W^{K_{2n,2,C_5}}(x) dx \geq \int_{X_1} \lambda_1^{2n+3} \alpha_1^{2n+2}(x) dx > \lambda_1^{4n+5} \geq p^{4n+5}.$$

This concludes the proof of the lemma. \square

The next lemma follows from Lemma 12 using Proposition 10 for the graph $H = K_{2n,2,C_5}$ and the set U being the part of $K_{2n,2}$ with $2n$ vertices.

Lemma 13. *For every $p_0 \in (0, 1)$, there exists $\varepsilon_0 > 0$ such that the following holds. If W is a graphon with density $p \geq p_0$ such that $t(K_{2,2}, W) \leq p^4 + \varepsilon_0$, then $t(K_{2n,2n,C_5}, W) \geq p^{4n^2+5n}$ for all $n \in \mathbb{N}$.*

The second ingredient in the proof of Theorem 1 is the following lemma, which, informally speaking, says that if the cut distance of W from the p -constant graphon is large, then the homomorphism density of $K_{2n,2n,C_5}$ is at least as in the p -constant graphon unless the graphon W contains a large sparse part.

Lemma 14. *For every $p_0 \in (0, 1)$ and every $\varepsilon_0 \in (0, 1)$, there exist $n_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that the following holds for every graphon W with density $p \geq p_0$ such that $t(K_{2,2}, W) \geq p^4 + \varepsilon_0$:*

- $t(K_{2n,2n,C_5}, W) \geq p^{4n^2+5n}$ for every $n \geq n_0$, or
- $\alpha_{p_0}(W) \geq \delta_0$.

Proof. Set $\delta_0 := p_0\varepsilon_0/16$ and set $d_0 := \delta_0$. The reason that we let δ_0 and d_0 to represent the same quantity is that they play different roles in the proof; δ_0 is the lower bound on the p_0 -independence ratio in the statement of the theorem whereas d_0 is the threshold for considering a point $x \in [0, 1]$ to have “small degree” in a graphon W . Choose n_0 to be large enough so that

$$(1 + \varepsilon_0/2)^{n_0} d_0^4 p_0^3 \geq 1.$$

Fix a graphon W with density $p \geq p_0$ such that $t(K_{2,2}, W) \geq p^4 + \varepsilon_0$. We iteratively define sets A_i , $i \in \mathbb{N}$, such that A_i is the set of all $x \in [0, 1]$ with “small degree” when disregarding neighbours in A_{i-1} . Formally, we let $A_0 = \emptyset$ and let A_i , $i \in \mathbb{N}$, be the set of all $x \in [0, 1]$ such that

$$\int_{[0,1] \setminus A_{i-1}} W(x, y) dy \leq d_0.$$

Note that $A_{i-1} \subseteq A_i$ for every $i \in \mathbb{N}$. Let A be the union of all sets A_i , $i \in \mathbb{N}$, and observe that, for every $x \in [0, 1] \setminus A$,

$$\int_{[0,1] \setminus A} W(x, y) dy = \lim_{i \rightarrow \infty} \int_{[0,1] \setminus A_{i-1}} W(x, y) dy.$$

In particular, it holds that

$$\int_{[0,1] \setminus A} W(x, y) \, dy \geq d_0$$

for every $x \in [0, 1] \setminus A$.

We divide the proof into cases depending on the measure of A . Consider first the case that $|A| \geq \varepsilon_0/2$. The integral $\int_{A^2} W(x, y) \, dx \, dy$ can be written as

$$\begin{aligned} & \sum_{i \in \mathbb{N}} \int_{(A_i \setminus A_{i-1})^2} W(x, y) \, dx \, dy + 2 \sum_{i \in \mathbb{N}} \int_{(A_i \setminus A_{i-1}) \times (A \setminus A_i)} W(x, y) \, dx \, dy \\ & \leq \sum_{i \in \mathbb{N}} \int_{(A_i \setminus A_{i-1})^2} W(x, y) \, dx \, dy + 2 \sum_{i \in \mathbb{N}} \int_{(A_i \setminus A_{i-1}) \times ([0,1] \setminus A_i)} W(x, y) \, dx \, dy \\ & \leq 2 \sum_{i \in \mathbb{N}} \int_{(A_i \setminus A_{i-1}) \times ([0,1] \setminus A_{i-1})} W(x, y) \, dx \, dy \\ & \leq 2 \sum_{i \in \mathbb{N}} |A_i \setminus A_{i-1}| d_0 \leq 2|A|d_0. \end{aligned}$$

It follows that

$$\frac{\int_{A^2} W(x, y) \, dx \, dy}{|A|^2} \leq \frac{2d_0}{|A|} = \frac{p_0 \varepsilon_0}{2|A|} \leq p_0.$$

Thus, the characteristic function of A is a certificate for $\alpha_{p_0}(W) \geq \varepsilon_0/2 \geq \delta_0$.

In the rest of the proof, we assume that $|A| \leq \varepsilon_0/2$. Roughly speaking, what we aim to show is that the homomorphism density of $K_{2n,2n,C_5}$ is large enough to obtain the first alternative in the theorem even if we disregard the points contained in A . We next estimate the homomorphism density $K_{2n,2n}$ in W' using Proposition 11 as follows:

$$\begin{aligned} t(K_{2n,2n}, W') & \geq (t(K_{2,2}, W'))^{n^2} \\ & \geq (t(K_{2,2}, W) - 4|A|)^{n^2} \\ & \geq (p^4 + \varepsilon_0 - \varepsilon_0/2)^{n^2} = (p^4 + \varepsilon_0/2)^{n^2} \end{aligned}$$

The goal now is to combine these copies of $K_{2n,2n}$ with copies of C_5 rooted at $x \in [0, 1] \setminus A$. Consider $x \in [0, 1] \setminus A$ and let $h(y) = W'(x, y)$. Note that

$$\int_{[0,1]} h(y) \, dy = \int_{[0,1]} W'(x, y) \, dy = \int_{[0,1] \setminus A} W(x, y) \, dy \geq d_0 = \delta_0.$$

Since $h(y) = 0$ for $y \in A$, we obtain that

$$\int_{[0,1]^2} h(y) W'(y, z) h(z) \, dy \, dz = \int_{[0,1]^2} h(y) W(y, z) h(z) \, dy \, dz. \quad (13)$$

If the integral in (13) is less than $p_0 \|h\|_1^2$, then $\alpha_{p_0}(W) \geq \delta_0$, which is the second conclusion of the lemma.

Hence, we can assume that the integral in (13) is at least $p_0 \|h\|_1^2$ for every $x \in [0, 1] \setminus A$. Since the 3-edge path P_4 is Sidorenko, we conclude by considering the graphon $W[h]$, see (2), that

$$t_{W'}^{C_5}(x) \geq \|h\|_1^4 \cdot t(P_4, W[h]) \geq p_0^3 \geq d_0^4 p_0^3$$

for every $x \in [0, 1] \setminus A$. It follows that

$$\begin{aligned} t(K_{2n, 2n, C_5}, W) &\geq t(K_{2n, 2n, C_5}, W') \\ &\geq t(K_{2n, 2n}, W') \cdot (d_0^4 p_0^3)^n \\ &\geq (p^4 + \varepsilon_0/2)^{n^2} (d_0^4 p_0^3)^n \\ &\geq p^{4n^2} (1 + \varepsilon_0/2)^{n^2} (d_0^4 p_0^3)^n \\ &= p^{4n^2} ((1 + \varepsilon_0/2)^{n_0} d_0^4 p_0^3)^n \geq p^{4n^2} \geq p^{4n^2 + 5n}. \end{aligned}$$

Hence, the first conclusion of the lemma holds. \square

We are now ready to prove the main theorem of this section, which implies Theorem 1. In a certain sense, Theorem 15 is a variant of Theorem 1 where a very small proportion of the edges can be left uncolored. This additional flexibility is needed for an inductive argument which is used to obtain the result for all values of k .

Theorem 15. *For every $k \in \mathbb{N}$, there exist $n_k \in \mathbb{N}$ and $\delta_k > 0$ with the following property. If W_1, \dots, W_k are graphons such that $t(K_2, W_1 + \dots + W_k) \geq 1 - \delta_k$, then*

$$\sum_{i \in [k]} t(K_{2n, 2n, C_5}, W_i) \geq \frac{t(K_2, W_1 + \dots + W_k)^{4n^2 + 5n}}{k^{4n^2 + 5n - 1}}$$

for every $n \geq n_k$.

Proof. We proceed by induction on $k \in \mathbb{N}$. Suppose first that $k = 1$. Apply Lemma 13 with $p_0 = 3/4$ to get $\varepsilon_0 \in (0, 1]$. We show that the statement of the theorem is true for $n_1 = 1$ and $\delta_1 = \varepsilon_0/4$. Let W_1 be a graphon with density $p \geq 1 - \delta_1$; note $\delta_1 \leq 1/4$ and so $p \geq 3/4$. Observe that

$$t(K_{2,2}, W) - p^4 \leq 1 - p^4 \leq 1 - (1 - \delta_1)^4 \leq 4\delta_1 = \varepsilon_0.$$

Hence, Lemma 13 implies that

$$t(K_{2n, 2n, C_5}, W_1) \geq p^{4n^2 + 5n}.$$

This completes the proof in the base case $k = 1$.

Now, suppose that we have already established the existence of n_1, \dots, n_{k-1} and $\delta_1, \dots, \delta_{k-1}$. Choose $p_0 = \delta_{k-1}/4k$ and apply Lemma 13 to get ε_0 . We then apply Lemma 14 with p_0 and ε_0 to obtain n_0 and δ_0 . Set $\delta_k = \frac{\delta_{k-1}\delta_0^2}{4k}$. Finally, choose n_k such that $n_k \geq \max\{n_0, n_{k-1}\}$ and

$$\left(\frac{1}{k} + \frac{1}{2k(k-1)}\right)^{4n_k} \delta_0^8 \geq \frac{k}{k-1}.$$

The choice of n_k yields that the following holds for all $n \geq n_k$:

$$(k-1) \left(\frac{1}{k} + \frac{1}{2k(k-1)}\right)^{4n^2+5n} \delta_0^{8n} \geq k \geq k \left(\frac{1}{k}\right)^{4n^2+5n}.$$

Let graphons W_1, \dots, W_k satisfying the assumption of the theorem be given and let $n \geq n_k$. Further, let $p = t(K_2, W_1 + \dots + W_k)$ be the density of the graphon $W_1 + \dots + W_k$; note that $p \geq 1 - \delta_k$.

We distinguish two cases. Roughly speaking, in the first case, we assume that one of the graphons W_1, \dots, W_k contains a large sparse part. That is, suppose that there exists $i \in [k]$ such that $\alpha_{p_0}(W_i) \geq \delta_0$; note that this includes the case that the density of W_i is at most p_0 . By symmetry, we can assume that $\alpha_{p_0}(W_k) \geq \delta_0$. Let $h : [0, 1] \rightarrow [0, 1]$ be such that $\|h\|_1 \geq \delta_0$ and

$$\int_{[0,1]^2} h(x)W_k(x,y)h(y) \, dx \, dy \leq p_0 \|h\|_1^2.$$

Since it holds that

$$\sum_{i \in [k]} \int_{[0,1]^2} h(x)W_i(x,y)h(y) \, dx \, dy \geq \|h\|_1^2 - \delta_k,$$

we obtain that

$$\begin{aligned} \sum_{i \in [k-1]} \int_{[0,1]^2} h(x)W_i(x,y)h(y) \, dx \, dy &\geq \|h\|_1^2 - p_0 \|h\|_1^2 - \delta_k \\ &\geq \|h\|_1^2 \left(1 - \frac{\delta_{k-1}}{2k}\right) \\ &\geq \|h\|_1^2 \left(1 - \frac{1}{2k}\right). \end{aligned}$$

Since it holds that

$$t(K_2, W_1[h] + \dots + W_{k-1}[h]) \geq 1 - \frac{\delta_{k-1}}{2k} \geq 1 - \delta_{k-1},$$

we can apply induction to $W_1[h], \dots, W_{k-1}[h]$ and arrive at the following:

$$\begin{aligned}
\sum_{i \in [k-1]} t(K_{2n, 2n, C_5}, W_i) &\geq \|h\|_1^{8n} \sum_{i \in [k-1]} t(K_{2n, 2n, C_5}, W_i[h]) \\
&\geq \|h\|_1^{8n} (k-1) \left(\frac{1 - 1/2k}{k-1} \right)^{4n^2+5n} \\
&\geq \delta_0^{8n} (k-1) \left(\frac{1}{k} + \frac{1}{2k(k-1)} \right)^{4n^2+5n} \\
&\geq k \left(\frac{1}{k} \right)^{4n^2+5n} \geq \frac{p^{4n^2+5n}}{k^{4n^2+5n-1}}.
\end{aligned}$$

Hence, in the following, we assume that $\alpha_{p_0}(W_i) < \delta_0$ for every $i \in [k]$. In particular, we assume that $t(K_2, W_i) \geq p_0$ for every $i \in [k]$ and so we can apply Lemmas 13 and 14 to each of W_1, \dots, W_k .

Based on whether it holds that $t(K_{2,2}, W_i) \leq t(K_2, W_i)^4 + \varepsilon_0$ or not, Lemma 13 or Lemma 14, respectively, implies

$$t(K_{2n, 2n, C_5}, W_i) \geq t(K_2, W_i)^{4n^2+5n}$$

for every $i \in [k]$. Therefore, we obtain that

$$\sum_{i \in [k]} t(K_{2n, 2n, C_5}, W_i) \geq \sum_{i \in [k]} t(K_2, W_i)^{4n^2+5n}$$

which is at least $k \left(\frac{p}{k} \right)^{4n^2+5n}$ by convexity. This concludes the proof of the theorem. \square

4 Sidorenko and locally Sidorenko graphs

In this section, we prove that a graph is k -common for all $k \geq 2$ if and only if it is Sidorenko and that no graph of odd girth is locally k -common for any $k \geq 3$. Note that forests and graphs with even girth are locally k -common for every k by Theorem 8. We start with the former statement.

Proof of Theorem 2. We first show that if a graph H is Sidorenko, then it is k -common for every $k \in \mathbb{N}$. Fix a Sidorenko graph H and an integer $k \geq 2$. Let W_1, \dots, W_k be graphons such that $W_1 + \dots + W_k = 1$ and let p_1, \dots, p_k be their respective densities. Note that $p_1 + \dots + p_k = 1$. Since H is Sidorenko,

$$t(H, W_1) + \dots + t(H, W_k) \geq p_1^{\|H\|} + \dots + p_k^{\|H\|} \geq k \left(\frac{p_1 + \dots + p_k}{k} \right)^{\|H\|} = k^{-\|H\|+1}.$$

Therefore, H is k -common.

To complete the proof, we need to show that if a graph H is not Sidorenko, then there exists $k \geq 2$ such that H is not k -common. Fix a graph H that is not Sidorenko and let W be a graphon with density p such that $t(H, W) < p^{\|H\|}$. Set $\varepsilon = p^{\|H\|} - t(H, W)$. By Lemma 6, there exists a step graphon W' with density p such that the cut distance between W and W' is at most $\varepsilon/(2\|H\|)$. Lemma 5 implies that

$$t(H, W') \leq t(H, W) + \varepsilon/2 = p^{\|H\|} - \varepsilon/2.$$

By splitting each of the parts of W' into the same number of equal size smaller parts, we can assume that the number m of parts of W' satisfies

$$4\|H\| \leq m\varepsilon \quad \text{and} \quad p^{\|H\|} - \varepsilon/4 < (p - 1/m)^{\|H\|}.$$

Let A_1, \dots, A_m be the parts of W' and let d_{ij} , $i, j \in [m]$ be the value of W' on the tile $A_i \times A_j$. Further, let δ be the average of d_{ij} taken over all pairs i and j such that $1 \leq i < j \leq m$ and let W'' be the step graphon with the same m parts as W' obtained from W' by making each of the m diagonal tiles to be equal to δ . Note that the density of the whole graphon W'' is δ and $\delta \geq p - 1/m$. Since the cut distance between W' and W'' is at most $m/m^2 = 1/m$, Lemma 5 implies that

$$\begin{aligned} t(H, W'') &\leq t(H, W') + \|H\| \cdot \|W - W'\|_{\square} \\ &\leq t(H, W') + \varepsilon/4 \\ &\leq p^{\|H\|} - \varepsilon/4 < (p - 1/m)^{\|H\|} \leq \delta^{\|H\|}. \end{aligned}$$

Next choose an integer $\ell \in \mathbb{N}$ such that $1 \leq \delta\ell m!$ and set $k = \ell m!$. We next define k graphons that witness that H is not k -common; the k graphons will be indexed by pairs consisting of a permutation $\sigma \in S_m$ of order m and an integer $s \in [\ell]$. The graphon $W_{\sigma, s}$ for $\sigma \in S_m$ and $s \in [\ell]$ is the step graphon with m parts A_1, \dots, A_m such that the graphon $W_{\sigma, s}$ on a tile $A_i \times A_j$, $i, j \in [m]$, is equal to $1/k$ if $i = j$ and is equal to $\frac{d_{\sigma(i)\sigma(j)}}{k\delta}$ if $i \neq j$ (note that $\frac{d_{\sigma(i)\sigma(j)}}{k\delta} \leq 1$). Note that the density of each of the graphons $W_{\sigma, s}$ is $\frac{1}{k}$. Moreover, the average value of all the k graphons on any of the tiles is $\frac{1}{k}$. Consequently, the k graphons $W_{\sigma, s}$, $\sigma \in S_m$ and $s \in [\ell]$, sum to the 1-constant graphon. Since the homomorphism density of H in each of the graphon $W_{\sigma, s}$, $\sigma \in S_m$ and $s \in [\ell]$, is equal to $\frac{1}{(k\delta)^{\|H\|}} t(H, W'') < k^{-\|H\|}$, it follows that H is not k -common. \square

We next show that locally k -common graphs for any $k \geq 3$ are precisely locally Sidorenko graphs (cf. Theorem 8).

Proof of Theorem 3. Fix an integer $k \geq 3$ for the proof, and a graph H with girth ℓ where ℓ is odd.

Let $A_1, \dots, A_{2\ell}$ be any partition of the interval $[0, 1]$ to 2ℓ disjoint measurable sets, each of measure $(2\ell)^{-1}$. Consider a kernel U defined as follows (also see

0	+1	0	0	+1	0	-1	0	0	-1
+1	0	+1	0	0	-1	0	-1	0	0
0	+1	0	+1	0	0	-1	0	-1	0
0	0	+1	0	+1	0	0	-1	0	-1
+1	0	0	+1	0	-1	0	0	-1	0
0	-1	0	0	-1	0	+1	0	0	+1
-1	0	-1	0	0	+1	0	+1	0	0
0	-1	0	-1	0	0	+1	0	+1	0
0	0	-1	0	-1	0	0	+1	0	+1
-1	0	0	-1	0	+1	0	0	+1	0

Figure 2: The kernel U used in the proof of Theorem 3 for $\ell = 5$.

Figure 2):

$$U(x, y) = \begin{cases} +1 & \text{if } x \in A_i, y \in A_j, \lceil i/\ell \rceil = \lceil j/\ell \rceil \text{ and } i = (j \pm 1) \bmod \ell, \\ -1 & \text{if } x \in A_i, y \in A_j, \lceil i/\ell \rceil \neq \lceil j/\ell \rceil \text{ and } i = (j \pm 1) \bmod \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a graph that has a vertex v of degree one and let v' be the neighbor of v . Note that

$$\begin{aligned} t(G, U) &= \int_{[0,1]^{V(G)}} \prod_{uu' \in E(G)} U(x_u, x_{u'}) dx_{V(G)} \\ &= \int_{[0,1]^{V(G) \setminus \{v\}}} \prod_{\substack{uu' \in E(G) \\ uu' \neq vv'}} U(x_u, x_{u'}) \cdot \left(\int_{[0,1]} U(x_{v'}, x_v) dx_v \right) dx_{V(G) \setminus \{v\}} \\ &= \int_{[0,1]^{V(G) \setminus \{v\}}} \prod_{\substack{uu' \in E(G) \\ uu' \neq vv'}} U(x_u, x_{u'}) \cdot 0 dx_{V(G) \setminus \{v\}} = 0 \end{aligned}$$

We conclude that $t(G, U) = 0$ for every graph G with a vertex of degree one.

We next compute $t(C_\ell, U)$. Observe that the product $\prod_{i \in [\ell]} U(x_i, x_{(i+1) \bmod \ell})$ is non-zero for $x_1, \dots, x_\ell \in [0, 1]$ if and only if there exists $j \in [\ell]$ such that either $x_i \in A_{(i+j) \bmod \ell} \cup A_{(i+j) \bmod \ell + \ell}$ for every $i \in [\ell]$ or $x_i \in A_{(\ell-i+j) \bmod \ell} \cup A_{(\ell-i+j) \bmod \ell + \ell}$ for every $i \in [\ell]$; if it is non-zero, then it is equal to one. Hence, it follows that

$$t(C_\ell, U) = \int_{[0,1]^\ell} \prod_{i \in [\ell]} U(x_i, x_{(i+1) \bmod \ell}) dx_{[\ell]} = 2\ell \cdot \prod_{i \in [\ell]} |A_i \cup A_{i+\ell}| = \frac{2}{\ell^{\ell-1}}$$

We next consider the following graphons: $W_1 = W_2 = 1/k + \varepsilon U$, $W_3 = 1/k - 2\varepsilon U$ and $W_4 = \dots = W_k = 1/k$. We will estimate the homomorphism density of H in W_1, \dots, W_k using Proposition 7. Note that if F is a subset of edges of H such that $1 \leq |F| \leq \ell$, then $H[F]$ contains a vertex of degree one unless $H[F]$ is a union of a cycle of length ℓ and isolated vertices. In particular, $t(H[F], U) = 0$ for such a set F unless F is the edge set of a cycle of length ℓ . Using Proposition 7, we obtain that

$$\begin{aligned} t(H, W_1) + \dots + t(H, W_k) &= 2t(H, 1/k + \varepsilon U) + t(H, 1/k - 2\varepsilon U) \\ &\quad + (k - 3)t(H, 1/k) \\ &= k^{-\|H\|+1} + 2 \cdot \frac{2m_\ell}{\ell^{\ell-1}} \varepsilon^\ell - \frac{2^{\ell+1}m_\ell}{\ell^{\ell-1}} \varepsilon^\ell + O(\varepsilon^{\ell+1}) \\ &= k^{-\|H\|+1} - \frac{(2^{\ell+1} - 4)m_\ell}{\ell^{\ell-1}} \varepsilon^\ell + O(\varepsilon^{\ell+1}) \end{aligned}$$

where m_ℓ is the number of cycles of length ℓ in H . Since $2^{\ell+1} - 4 > 0$, there exists $\varepsilon_0 > 0$ such that

$$t(H, W_1) + \dots + t(H, W_k) < k^{-\|H\|+1}$$

for every $\varepsilon \in (0, \varepsilon_0)$. We conclude that H is not locally k -common, which completes the proof of the theorem. \square

5 Open problems

We conclude with two open problems. Theorem 1 provides an example of a non-bipartite k -common graph for every $k \geq 2$. A natural next question is whether there exist k -common graphs of arbitrary large chromatic number. Currently, the only known example of a 2-common graph of chromatic number greater than three is the 5-wheel [17] and so this question is interesting even in the case $k = 2$ and $\ell \geq 5$.

Problem 1. *For every $k \geq 2$ and $\ell \geq 4$, construct a k -common ℓ -chromatic graph.*

The second problem stems from Theorem 3 which characterizes locally k -common graphs for $k \geq 3$. Interestingly, we do not have a similar characterization of locally 2-common graphs.

Problem 2. *Characterize graphs that are locally 2-common.*

Locally 2-common graphs include forests, all graphs with even girth, the triangle and the 5-wheel in particular, since these are graphs are locally Sidorenko or

2-common. On the other hand, a construction of Franek and Rödl [12] of asymptotically $n/2$ -regular n -vertex graphs with density of K_4 and their complement less than $0.987314 \times 1/32$, yields the existence of a kernel U such that

$$t(K_4, 1/2 + U) + t(K_4, 1/2 - U) \leq 0.987314 \times \frac{1}{32} \quad \text{and} \quad \int_{[0,1]} U(x, y) \, dy = 0$$

for every $x \in [0, 1]$. We use this construction to show that K_4 is not locally 2-common. For $z \in (0, 1]$, define a kernel U_z as

$$U_z(x, y) = \begin{cases} U(x/z, y/z) & \text{if } (x, y) \in [0, z]^2, \\ 0 & \text{otherwise.} \end{cases}$$

Since the cut norm of U_z is at most z^2 and $t(K_4, 1/2 + U_z) + t(K_4, 1/2 - U_z) < 1/32$ (here, we use that the kernel U_z is “0-regular”), it follows that K_4 is not locally 2-common.

Acknowledgement

The authors would like to thank László Miklós Lovász for insightful comments on the graph limit notions discussed in this paper.

References

- [1] S. A. Burr and V. Rosta: *On the Ramsey multiplicities of graphs—problems and recent results*, J. Graph Theory **4** (1980), 347–361.
- [2] D. Conlon, J. Fox and B. Sudakov: *An approximate version of Sidorenko’s conjecture*, Geom. Funct. Anal. **20** (2010), 1354–1366.
- [3] D. Conlon, J. H. Kim, C. Lee and J. Lee: *Some advances on Sidorenko’s conjecture*, J. Lond. Math. Soc. **98** (2018), 593–608.
- [4] D. Conlon and J. Lee: *Finite reflection groups and graph norms*, Adv. Math. **315** (2017), 130–165.
- [5] D. Conlon and J. Lee: *Sidorenko’s conjecture for blow-ups* (2018), preprint arXiv:1809.01259.
- [6] J. Cummings and M. Young: *Graphs containing triangles are not 3-common*, J. Comb. **2** (2011), 1–14.
- [7] P. Erdős: *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294.

- [8] P. Erdős: *On the number of complete subgraphs contained in certain graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 459–464.
- [9] P. Erdős and M. Simonovits: *Cube-supersaturated graphs and related problems*, in: Progress in graph theory (Waterloo, Ont., 1982) (1984), 203–218.
- [10] J. Fox and F. Wei: *On the local approach to Sidorenko’s conjecture*, Electronic Notes in Discrete Mathematics **61** (2017), 459–465.
- [11] F. Franek: *On Erdős’s conjecture on multiplicities of complete subgraphs: lower upper bound for cliques of size 6*, Combinatorica **22** (2002), 451–454.
- [12] F. Franek and V. Rödl: *2-colorings of complete graphs with a small number of monochromatic K_4 subgraphs*, Discrete Math. **114** (1993), 199–203. Combinatorics and algorithms (Jerusalem, 1988).
- [13] A. Frieze and R. Kannan: *Quick approximation to matrices and applications*, Combinatorica **19** (1999), 175–220.
- [14] G. Giraud: *Sur le problème de goodman pour les quadrangles et la majoration des nombres de ramsey*, J. Combin. Theory Ser. B **27** (1979), 237–253.
- [15] A. W. Goodman: *On sets of acquaintances and strangers at any party*, Amer. Math. Monthly **66** (1959), 778–783.
- [16] H. Hatami: *Graph norms and Sidorenko’s conjecture*, Israel J. Math. **175** (2010), 125–150.
- [17] H. Hatami, J. Hladký, D. Král’, S. Norine and A. Razborov: *Non-three-colourable common graphs exist*, Combin. Probab. Comput. **21** (2012), 734–742.
- [18] C. Jagger, P. Štoviček and A. Thomason: *Multiplicities of subgraphs*, Combinatorica **16** (1996), 123–141.
- [19] J. L. Li and B. Szegedy: *On the logarithmic calculus and Sidorenko’s conjecture* (2011), preprint arXiv:1107.1153, accepted to Combinatorica.
- [20] G. Lorden: *Blue-empty chromatic graphs*, Amer. Math. Monthly **69** (1962), 114–120.
- [21] L. Lovász: *Subgraph densities in signed graphons and the local Simonovits-Sidorenko conjecture*, Electron. J. Combin. **18** (2011), Paper 127, 21.
- [22] L. Lovász: *Large networks and graph limits*, AMS Colloquium Publications, volume 60, 2012.

- [23] L. Lovász and B. Szegedy: *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), 933–957.
- [24] S. Nieß: *Counting monochromatic copies of K_4 : a new lower bound for the ramsey multiplicity problem* (2012), preprint arXiv:1207.4714.
- [25] A. A. Razborov: *Flag algebras*, J. Symbolic Logic **72** (2007), 1239–1282.
- [26] A. Sidorenko: *A correlation inequality for bipartite graphs*, Graphs Combin. **9** (1993), 201–204.
- [27] A. Sidorenko: *Randomness friendly graphs*, Random Structures Algorithms **8** (1996), 229–241.
- [28] A. F. Sidorenko: *Cycles in graphs and functional inequalities*, Mat. Zametki **46** (1989), 72–79, 104.
- [29] A. F. Sidorenko: *Inequalities for functionals generated by bipartite graphs*, Diskret. Mat. **3** (1991), 50–65.
- [30] K. Sperfeld: *On the minimal monochromatic K_4 -density* (2011), preprint arXiv:1106.1030.
- [31] B. Szegedy: *An information theoretic approach to Sidorenko’s conjecture* (2015), preprint arXiv:1406.6738.
- [32] A. Thomason: *A disproof of a conjecture of Erdős in Ramsey theory*, J. London Math. Soc. (2) **39** (1989), 246–255.
- [33] A. Thomason: *Graph products and monochromatic multiplicities*, Combinatorica **17** (1997), 125–134.