

Rainbow matchings in k -partite hypergraphs

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Abstract

In this paper, we prove a conjecture of Aharoni and Howard on the existence of rainbow (transversal) matchings in sufficiently large families $\mathcal{F}_1, \dots, \mathcal{F}_s$ of tuples in $\{1, \dots, n\}^k$, provided $s \geq 470$.

1 Introduction

Let $[n] := \{1, \dots, n\}$. In this paper, we study families of k -tuples $\mathcal{F} \subset [n]^k$. We say that $F = (f_1, \dots, f_k)$ and $F' = (f'_1, \dots, f'_k)$, $F, F' \in [n]^k$, *intersect* iff for some $1 \leq i \leq k$ we have $f_i = f'_i$. Note that $[n]^k$ can be seen as the complete k -partite k -uniform hypergraph with parts of size n . The following conjecture was made by Aharoni and Howard [1]:

Conjecture 1. *Let n, s and k be positive integers. If $\mathcal{F}_1, \dots, \mathcal{F}_s \subset [n]^k$ satisfy $|\mathcal{F}_i| > (s-1)n^{k-1}$ for all i then there exist $F_1 \in \mathcal{F}_1, \dots, F_s \in \mathcal{F}_s$, such that $F_i \cap F_j = \emptyset$ for any $1 \leq i < j \leq s$.*

We call any such collection of F_i a *rainbow s -matching*. If true, the bound on $|\mathcal{F}_i|$ in the conjecture is best possible: consider the families $\mathcal{F}_1 = \dots = \mathcal{F}_s = \{F = (f_1, \dots, f_k) \in [n]^k : f_1 \in [s-1]\}$. In their paper, Aharoni and Howard proved this conjecture for $k = 2, 3$. Later, Lu and Yu [12] proved it for $n > 3(s-1)(k-1)$.

The main result of this paper is the proof of Conjecture 1 for all $s \geq s_0$.

Theorem 1. *There exists s_0 such that Conjecture 1 is true for any $s \geq s_0$.*

Although we haven't put much effort into optimizing s_0 , our proof allows us to take $s_0 = 470$.

The proof of Theorem 1 relies on the idea that intersection of any family with a random matching is highly concentrated around its expectation. This idea was introduced in the paper of Frankl and the second author [7] in the context of the Erdős Matching Conjecture (Erdős, [4]). Let $\binom{[n]}{k}$ stand for the collection of all k -element subsets of $[n]$. The EMC states that the largest family $\mathcal{F} \subset \binom{[n]}{k}$ with no s pairwise disjoint sets for $n \geq sk$ has size at most $f(n, k, s) := \max \left\{ \binom{n}{k} - \binom{n-s+1}{k}, \binom{sk-1}{k} \right\}$. The conjecture is proven in some ranges of parameters.

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For somewhat large n , the best results on the conjecture is due to Frankl [5], who showed the validity of the conjecture for roughly $n > 2sk$ and Frankl and the second author [7], who showed the conjecture is valid for $n > \frac{5}{3}sk$ and $s > s_0$. For n close to sk , the only nontrivial result is due to Frankl [6], who showed the validity of the conjecture for $n < s(k + \epsilon)$, where $\epsilon = \epsilon(k)$ is roughly k^{-k} . A more general problem was studied in [9].

Interestingly, the 'one-family' version of the conjecture of Aharoni and Howard is almost trivial. Indeed, let us show that any family $\mathcal{F} \subset [n]^k$ of size greater than $(s - 1)n^{k-1}$ has a matching of size s . Consider a perfect matching in $[n]^k$ taken uniformly at random. Obviously, it consists of n edges. Thus, via simple averaging, the expected intersection of \mathcal{F} with this matching is $n \cdot \frac{|\mathcal{F}|}{n^k} > s - 1$. Therefore, there exists a perfect matching that has at least s sets in the intersection with \mathcal{F} . This should explain the intuition behind using the concentration for the intersections of families with a random matching. Indeed, if for each of the families $\mathcal{F}_1, \dots, \mathcal{F}_s$ their intersection with a perfect matching is either of size $s - 1$ or s , then essentially the same argument as above will imply that we can find a rainbow s -matching in a fixed perfect matching, as long as one of the families intersects it in s sets.

Huang, Loh and Sudakov [10], as well as Aharoni and Howard [1] suggested the s -family analogue of the EMC: any s families $\mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ with $\min_i |\mathcal{F}_i| > f(n, k, s)$ contain a rainbow s -matching. (We refer to it as to *rainbow EMC* later on.) Huang, Loh and Sudakov proved it in the very same range: for $n > 3sk^2$. (Later, their approach was transposed in the aforementioned paper of Lu and Yu [12] to progress on Conjecture 1. Note here that the bound $n > 3sk$ in the case of $[n]^k$ corresponds to the bound $n > 3sk^2$ in the case of $\binom{[n]}{k}$ since the ground set has size nk in the former case.) The approaches of [5] and [7], unfortunately, do not seem to work for the rainbow version of the EMC. Using a junta method, Keller and Lifshitz [11] showed the validity of the rainbow EMC for $n > f(s)k$, where $f(s)$ is an unspecified and a very quickly growing function of s . Together with Frankl [8], the second author managed to prove a much better junta approximation for shifted families and to show the validity of the rainbow EMC for $n > 12sk(2 + \log s)$. We also note that the validity of the conjecture for $n > Csk$ with some large unspecified C is announced by Keevash, Lifshitz, Long and Minzer.

In the next section, we give the necessary preliminaries and prove the aforementioned concentration inequality. In Section 3, we give the proof of Theorem 1.

2 Concentration for intersections with a random matching

We start with some results concerning the eigenvalues of graphs and their relation to their quasirandomness properties, as well as concentration results for martingales.

Consider the graph $PG_{n,k}$ on the vertex set $[n]^k$ and with the edge set consisting of all pairs of disjoint sets. It is not difficult to see that $PG_{n,k}$ is the direct product of k copies of K_n , the complete graph on n vertices. Thus, the adjacency matrix of $PG_{n,k}$ is the k -th Kronecker power of $(J_n - E_n)$, the adjacency matrix of K_n . For two $n \times n$ matrices A and B with eigenvalues λ_i and μ_j , respectively, the eigenvalues of their Kronecker product $A \otimes B$ are $\lambda_i \mu_j$.

This implies the following proposition.

Proposition 2. *The largest eigenvalue of $PG_{n,k}$ is equal to $D := (n - 1)^k$ and the second largest absolute value of an eigenvalue is equal to $\lambda_2 := (n - 1)^{k-1}$.*

Proof. This readily follows from the paragraph above and the fact that the adjacency matrix of K_n has the form $J_n - E_n$ and its eigenvalues are $(n - 1)$, with multiplicity 1, and -1 , with multiplicity $(n - 1)$. \square

Fix integers n, k . Let $\mathcal{G} \subset [n]^k$ be a family and set $\alpha := |\mathcal{G}|/n^k$. Take two disjoint sets $S_1, S_2 \in [n]^k$ uniformly at random and let A_i be the event that $S_i \in \mathcal{G}$. Now we will need the following result of Alon and Chung [2]:

Theorem 3. *Let $G = (V, E)$ be a D -regular graph on m vertices, let $\lambda_2(G)$ be the second largest absolute value of the eigenvalues of G and let S be a subset of vertices G of cardinality $|S| = \alpha m$. Then the following holds:*

$$\left| \frac{2e(S)}{Dm} - \alpha^2 \right| \leq \frac{\lambda_2(G)\alpha(1 - \alpha)}{D},$$

where $e(S)$ is the number of edges which join two vertices of S .

We apply Theorem 3 to the graph $PG_{n,k}$. Note that, in our case, $\mathbb{P}[A_1 \cap A_2]$ is exactly the proportion of all edges of $PG_{n,k}$ that are contained in \mathcal{G} . We also use that $\mathbb{P}[\bar{A}_1 \cap A_2] = \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2] = \alpha - \mathbb{P}[A_1 \cap A_2]$. Putting this together, we get the following.

Proposition 4. $|\mathbb{P}[A_1 \cap A_2] - \alpha^2|, |\mathbb{P}[\bar{A}_1 \cap A_2] - \alpha(1 - \alpha)| \leq \frac{\lambda_2\alpha(1-\alpha)}{D} = \frac{\alpha(1-\alpha)}{n-1}$.

Next, we state a result on martingales that will be used. In their survey, Chung and Lu proved the following concentration inequality (cf. [3, Theorems 7.1 and 7.2]):

Theorem 5. *Suppose that a nonnegative martingale X , associated with a filter \mathbf{F} , satisfies*

$$\text{Var}(X_i | \mathcal{F}_{i-1}) \leq \phi_i X_{i-1}$$

and

$$|X_i - X_{i-1}| \leq M$$

for $1 \leq i \leq n$. Here, ϕ_i and M are nonnegative constants. Then for $\delta \in \{-1, 1\}$ we have

$$\mathbb{P}[\delta \cdot (X_n - \mathbb{E}X) \geq \lambda] \leq \exp\left(-\frac{\lambda^2}{2((\mathbb{E}X + \lambda)(\sum_{i=1}^n \phi_i) + M\lambda/3)}\right).$$

2.1 The concentration

Fix integers n, k . Let $\mathcal{G} \subset [n]^k$ be a family and set $\alpha = |\mathcal{G}|/n^k$. Let η be the random variable $|\mathcal{G} \cap \mathcal{M}|$, where $\mathcal{M} = (M_1, \dots, M_n)$ is chosen uniformly at random out of all ordered perfect matchings. Clearly, we have $\mathbb{E}\eta = \alpha n$.

The goal of this section is to prove the following theorem.

Theorem 6. *For any $\lambda > 0$ and $\delta \in \{-1, 1\}$ the following holds.*

$$\mathbb{P}[\delta \cdot (\eta - \alpha n) \geq 2\lambda] \leq 2 \exp\left(-\frac{\lambda^2}{\alpha n/2 + 2\lambda}\right). \quad (1)$$

Moreover, for any $\mu > 0$, $\lambda \geq 2\sqrt{\alpha n} + 1 \geq 9$ and $\delta \in \{-1, 1\}$, we have

$$\mathbb{P}[\delta \cdot (\eta - \alpha n) > 2\lambda + 2\mu] \leq \mathbb{P}[\delta \cdot (\eta - \alpha n) \geq 1] \cdot 4 \exp\left(-\frac{\mu^2}{(\alpha n + 2\lambda)/2 + 2\mu}\right). \quad (2)$$

While the first part of the statement is a large deviations statement for η , the second part of the statement is a 'conditional large deviations' result, which tells us that 'very large deviations should be much less probable than moderately large deviations'. This part of the statement is technical, but required in the proof of the main theorem.

Put $\eta := \eta_1 + \dots + \eta_n$ and $\tilde{\eta} := \eta_1 + \dots + \eta_{n/2}$, where η_i is the indicator function of the event A_i that the set M_i belongs to \mathcal{G} . (In what follows, we assume that n is divisible by 2. This simplifying assumption only slightly affects the calculations to follow and does not affect the result.)

Although ultimately we are interested in the behaviour of η , for technical reasons we need to deal with $\tilde{\eta}$. To this end, we may relate them as follows.

Proposition 7. *Let $a > 0$ be a real number, $\delta \in \{-1, 1\}$. Then $\mathbb{P}[\delta \cdot (\eta - \mathbb{E}\eta) \geq 2a] \leq 2\mathbb{P}[\delta \cdot (\tilde{\eta} - \mathbb{E}\tilde{\eta}) \geq a]$.*

Proof. The proof of the inequality for both choices of δ follows from the easy fact that $\eta = \tilde{\eta} + \tilde{\eta}'$, where $\tilde{\eta}'$ is a copy of $\tilde{\eta}$. (Note that $\tilde{\eta}$ and $\tilde{\eta}'$ are, in general, dependent.) \square

Let $X_0, \dots, X_{n/2}$ and $Y_0, \dots, Y_{n/2}$ be the following exposure martingales:

$$X_i = \mathbb{E}[\tilde{\eta} \mid \eta_1, \dots, \eta_i] \quad \text{and} \quad Y_i = \mathbb{E}[\eta_n \mid \eta_1, \dots, \eta_i].$$

Note that $X_i = \sum_{j=1}^i \eta_j + (n/2 - i)Y_i$.

Proposition 8. *The following holds for any $i \leq n - 2$:*

1. $\text{Var}[\mathbb{E}[\eta_n \mid \eta_{i+1}, M_1, \dots, M_i] \mid M_1, \dots, M_i] \leq \frac{\mathbb{E}[\eta_n \mid M_1, \dots, M_i]}{(n-i-1)^2}$.
2. $|\mathbb{E}[\eta_n \mid \eta_{i+1}, M_1, \dots, M_i] - \mathbb{E}[\eta_n \mid M_1, \dots, M_i]| \leq \frac{1}{n-i-1}$.

Note that in both 1. and 2. we are comparing some functions pointwise, i.e., the inequalities should hold for any choices of M_1, \dots, M_i and η_{i+1} .

Proof of Proposition 8. Fix M_1, \dots, M_i and consider the graph $PG_{n',k}$ on $V' := [n]^k \setminus \bigcup_{j=1}^i M_j \cong [n']^k$, where $n' := n - i$. Put $\mathcal{G}' := \mathcal{G} \cap V'$ and $\alpha' := |\mathcal{G}'|/(n')^k$. Consider the intersection of \mathcal{G}' with a randomly chosen n' -matching $\mathcal{M}' = (M_{i+1}, \dots, M_n)$ of sets from $[n']^k$. Let η_j and A_j be defined as before.

Note that $\mathbb{E}\eta_n = \mathbb{E}\eta_{i+1} = \mathbb{P}[A_{i+1}] = \alpha'$. In these terms, we need to show

$$\text{Var}[\mathbb{E}[\eta_n \mid \eta_{i+1}]] \leq \frac{\alpha'}{(n' - 1)^2} \quad \text{and} \quad |\mathbb{E}[\eta_n \mid \eta_{i+1}] - \alpha'| \leq \frac{1}{n' - 1} \quad \text{for } i \leq n - 2. \quad (3)$$

Note that $\mathbb{E}[\eta_n \mid \eta_{i+1}]$ and $\text{Var}[\eta_n \mid \eta_{i+1}]$ are both random variables taking two values depending on the value of η_{i+1} . Let us first consider the case $\eta_{i+1} = 1$.

$$\mathbb{E}[\eta_n \mid \eta_{i+1} = 1] = \mathbb{E}[\eta_n \mid A_{i+1}] = \mathbb{P}[A_n \mid A_{i+1}] = \frac{\mathbb{P}[A_n \cap A_{i+1}]}{\mathbb{P}[A_{i+1}]} = \frac{\mathbb{P}[A_n \cap A_{i+1}]}{\alpha'}.$$

Then, using Proposition 4, we conclude that

$$|\mathbb{E}[\eta_n \mid \eta_{i+1} = 1] - \alpha'| \leq \frac{1 - \alpha'}{n' - 1}. \quad (4)$$

Next, let us consider the case $\eta_{i+1} = 0$.

$$\mathbb{E}[\eta_n \mid \eta_{i+1} = 0] = \frac{\mathbb{P}[A_n \cap \bar{A}_{i+1}]}{\mathbb{P}[\bar{A}_{i+1}]} = \frac{\mathbb{P}[A_n \cap \bar{A}_{i+1}]}{1 - \alpha'}.$$

Similarly, using Proposition 4, we conclude that

$$|\mathbb{E}[\eta'_n \mid \eta'_{i+1} = 0] - \alpha'| \leq \frac{\alpha'}{n' - 1}. \quad (5)$$

From (4) and (5) we conclude that the second part of (3) and thus the proposition holds. Since $\mathbb{P}[\eta_{i+1} = 1] = \alpha'$ and $\mathbb{P}[\eta_{i+1} = 0] = 1 - \alpha'$, combining (4) and (5), we conclude that

$$\text{Var}[\eta_n \mid \eta_{i+1}] = \alpha' \cdot (\mathbb{E}[\eta_n \mid \eta_{i+1} = 1] - \alpha')^2 + (1 - \alpha') \cdot (\mathbb{E}[\eta_n \mid \eta_{i+1} = 0] - \alpha')^2 \leq \frac{\alpha'(1 - \alpha')}{(n' - 1)^2} \leq \frac{\alpha'}{(n' - 1)^2}. \quad \square$$

Now let us note that it is easy to deduce the following corollary:

Corollary 9. *The following holds for any $i \leq n - 2$:*

1. $\text{Var}[Y_{i+1} \mid \eta_1, \dots, \eta_i] \leq \frac{Y_i}{(n - i - 1)^2}.$

2. $|Y_{i+1} - Y_i| \leq \frac{1}{n - i - 1}.$

Proof. Informally, this corollary follows from Proposition 8 via averaging over all choices of M_1, \dots, M_i that give the corresponding values of η_1, \dots, η_i and using the convexity of variance. Let us give a formal proof. Again, we need to show that a certain inequality between functions holds pointwise. In what follows, we fix a choice of η_1, \dots, η_i . It is easy to see that

$$Y_{i+1} = t^{-1} \sum \mathbb{E}[\eta_n \mid \eta_{i+1}, M_1 = M'_1, \dots, M_i = M'_i],$$

$$Y_i = t^{-1} \sum \mathbb{E}[\eta_n \mid M_1 = M'_1, \dots, M_i = M'_i],$$

where both sums are over all tuples (M'_1, \dots, M'_i) of disjoint k -sets such that $M'_i \in \mathcal{G}$ if and only if $\eta_j = 1$, and t is the number of such tuples. This immediately implies the second part of the corollary. As for the first part, we derive it below. We apply Jensen's inequality to the variance to get the first inequality below and use the first part of Proposition 8 to get the second inequality:

$$\begin{aligned} \text{Var}[Y_{i+1} \mid \eta_1, \dots, \eta_i] &= \text{Var}\left[t^{-1} \sum \mathbb{E}[\eta_n \mid \eta_{i+1}, M_1 = M'_1, \dots, M_i = M'_i] \mid \eta_1, \dots, \eta_i\right] \\ &\leq t^{-1} \sum \text{Var}\left[\mathbb{E}[\eta_n \mid \eta_{i+1}, M_1 = M'_1, \dots, M_i = M'_i] \mid \eta_1, \dots, \eta_i\right] \\ &\leq t^{-1} \sum \frac{\mathbb{E}[\eta_n \mid M_1 = M'_1, \dots, M_i = M'_i]}{(n - i - 1)^2} \\ &= \frac{Y_i}{(n - i - 1)^2}. \end{aligned} \quad \square$$

Corollary 10. For any $i \leq n/2 - 1$, we have

$$|X_{i+1} - X_i| \leq \frac{3}{2} \quad \text{and} \quad \text{Var}[X_{i+1} \mid \eta_1, \dots, \eta_i] \leq \frac{X_i}{n}.$$

Proof. We have $0 \leq Y_i, \eta_{i+1} \leq 1$ and thus $-(n/2 - i - 1)|Y_{i+1} - Y_i| - Y_i \leq |X_{i+1} - X_i| \leq \eta_{i+1} + (n/2 - i - 1)|Y_{i+1} - Y_i|$, which, using part 2 of Corollary 9, implies $|X_{i+1} - X_i| \leq \frac{3}{2}$. As for the second inequality, using Corollary 9, we can expand it as follows.

$$\text{Var}[X_{i+1} \mid \eta_1, \dots, \eta_i] = (n/2 - i)^2 \text{Var}[Y_{i+1} \mid \eta_1, \dots, \eta_i] \leq \frac{(n/2 - i)^2}{(n - i - 1)^2} Y_i \leq \frac{n/2 - i}{(n - i - 1)^2} X_i \leq \frac{X_i}{n}.$$

We note that getting the last inequality is essentially the only reason why we had to deal with $\tilde{\eta}$ instead of η itself. \square

Now let us apply Theorem 5 to $X_{n/2} = \tilde{\eta}$. If we choose $\phi_i = \frac{1}{n}$, $M = \frac{3}{2}$, then we have

$$\mathbb{P}[\delta \cdot (\tilde{\eta} - \alpha n/2) \geq \lambda] \leq \exp\left(-\frac{\lambda^2}{\alpha n/2 + 2\lambda}\right).$$

Combining this with Proposition 7, we get (1).

Let us deduce (2) from (1). Assume that $\delta = 1$ and that, for some choice of (M_1, \dots, M_n) , we have $\eta - \mathbb{E}\eta > 2\mu + 2\lambda$. Let Z_1, \dots, Z_n be the martingale associated with η , where

$$Z_i := \mathbb{E}[\eta \mid \eta_1, \dots, \eta_i].$$

Then, for some i , we have $2\lambda - 2 \leq Z_i - \alpha n \leq 2\lambda$. The latter follows since, as it is not difficult to see, $|Z_{i+1} - Z_i| \leq 2$. To see that, one just needs to repeat the proof of Corollary 10 with the obvious changes that we get when passing from $\tilde{\eta}$ to η .

Take any collection of disjoint edges (M_1, \dots, M_i) such that i is the first such step. Let us restrict to the subgraph as in the proof of Proposition 8. Note that, within this restriction, Z_i is a constant from the interval $[2\lambda - 2 + \alpha n, 2\lambda + \alpha n]$. We apply the first part of Theorem 6 with $Z_i - \sum_{j=1}^i \eta_j = \mathbb{E}Z_n - \sum_{j=1}^i \eta_j \leq \alpha n + 2\lambda$ playing the role of αn . We get that

$$\mathbb{P}[Z_n - \alpha n \leq 0] \leq \mathbb{P}[Z_n - Z_i \leq -2\lambda + 2] \leq 2 \exp\left(-\frac{(\lambda - 1)^2}{(\alpha n + 2\lambda)/2 + 2(\lambda - 1)}\right).$$

Recall that $\lambda = 2\sqrt{\alpha n} + 1$ and that $\sqrt{\alpha n} \geq 4$. Then $(\alpha n + 2\lambda)/2 + 2\lambda - 2 \leq \frac{8}{3}\alpha n$ and the expression in the right hand side above is at most $2 \exp(-\frac{4\alpha n}{8\alpha n/3}) = 2e^{-3/2} < \frac{1}{2}$. The same holds for any bigger λ . Recall that Z_n has only integer values. Therefore, in these assumptions,

$$\mathbb{P}[Z_n - \alpha n \geq 1] \geq \frac{1}{2}.$$

On the other hand, we can similarly see that

$$\mathbb{P}[Z_n - \alpha n > 2\lambda + 2\mu] \leq \mathbb{P}[Z_n - Z_i > 2\mu] \leq 2 \exp\left(-\frac{\mu^2}{(\alpha n + 2\lambda)/2 + 2\mu}\right).$$

Since both displayed formulas are valid for any choice of i and (M_1, \dots, M_i) for which the event ' $\eta - \alpha n > 3\lambda + 2\mu$ ' has non-zero probability, we can combine the two displayed formulas, sum them over all possible choices of (M_1, \dots, M_i) and get that

$$\mathbb{P}[\eta - \alpha n > 2\lambda + 2\mu] \leq \mathbb{P}[\eta - \alpha n \geq 1] \cdot 4 \exp\left(-\frac{\mu^2}{(\alpha n + 2\lambda)/2 + 2\mu}\right)$$

for any $\lambda \geq 2\sqrt{\alpha n} + 1 \geq 9$. The case $\delta = -1$ is absolutely analogous.

3 Proof of Theorem 1

We argue indirectly. Assume that there exist s families $\mathcal{F}_1, \dots, \mathcal{F}_s$ in $[n]^k$, each of size exactly $(s-1)n^{k-1} + 1$, and with no rainbow s -matching. Consider the following random variables:

$$\zeta_i(\mathcal{M}) := |\mathcal{F}_i \cap \mathcal{M}| - s + 1,$$

where \mathcal{M} is a uniformly random perfect matching of k -element sets.

For a fixed perfect matching $\mathcal{M} = \{M_1, \dots, M_n\}$, consider the graph $G(\mathcal{M})$ with parts $A(\mathcal{M}) = \{M_1, \dots, M_n\}$ and $B := \{\mathcal{F}_1, \dots, \mathcal{F}_s\}$, where a family is connected to a set from the matching iff the set belongs to the family. The condition that there is no rainbow s -matching in $\mathcal{F}_1, \dots, \mathcal{F}_s$ translates into the condition that $G(\mathcal{M})$ does not have a matching of the part B . Thus Hall's necessary and sufficient condition for the existence of a matching of B must be violated. An easy consequence of that is that there exists some $j_{\mathcal{M}} \in \{0, \dots, s-1\}$, such that $\zeta_i(\mathcal{M}) \leq -j_{\mathcal{M}}$ for at least $s - j_{\mathcal{M}}$ indices $i \in [s]$. To make the definition of $j_{\mathcal{M}}$ unambiguous, choose $j_{\mathcal{M}}$ to be the largest out of all possible values. We treat $j_{\mathcal{M}}$ as a random variable depending on \mathcal{M} .

We have

$$\sum_{i=1}^s \zeta_i(\mathcal{M}) \cdot I[\zeta_i(\mathcal{M}) \leq 0] \leq -j_{\mathcal{M}}(s - j_{\mathcal{M}}) \quad (6)$$

due to the violation of Hall's condition.

First, let us show that

$$\mathbb{E}[\zeta_i \mid \zeta_i > 0] < 3.7\sqrt{s \log s}. \quad (7)$$

For that, we shall employ (2) with $\eta := \zeta_i + s - 1$ and $\alpha n = \frac{|\mathcal{F}_i|}{n^k}$. Put $\delta = 1$, $\lambda = 2\sqrt{s}$ and let $\mu \geq 0$. Then we have

$$\mathbb{P}[\zeta_i > 2\mu + 4\sqrt{s}] \leq \mathbb{P}[\zeta_i > 0] \cdot 4 \exp\left(-\frac{\mu^2}{(s + 4\sqrt{s})/2 + 2\mu}\right).$$

Now we rewrite $\mathbb{E}[\zeta_i \mid \zeta_i > 0]$ as follows for some $X > 0$:

$$\mathbb{E}[\zeta_i \mid \zeta_i > 0] = \mathbb{E}[\zeta_i \cdot I[\zeta_i < X] \mid \zeta_i > 0] + \mathbb{E}[\zeta_i \cdot I[\zeta_i \geq X] \mid \zeta_i > 0] \leq X + \sum_{i \geq X} \mathbb{P}[\zeta_i \geq i \mid \zeta_i > 0].$$

Then, putting $X = \sqrt{s \log s} + 4\sqrt{s}$, we get

$$\begin{aligned} \mathbb{E}[\zeta_i \mid \zeta_i > 0] &\leq 2\sqrt{s \log s} + 4\sqrt{s} + \sum_{t \geq \sqrt{s \log s}: 2t+4\sqrt{s} \in \mathbb{N}} (2t + 4\sqrt{s}) \cdot 4 \exp\left(-\frac{t^2}{(s + 4\sqrt{s})/2 + 2t}\right) \\ &\leq 2\sqrt{s \log s} + 4\sqrt{s} + 4 \leq 3.7\sqrt{s \log s}. \end{aligned}$$

(In the last two inequalities we use that $s \geq 470$ and calculated the corresponding expressions on computer.)

Recall that $|\mathcal{F}_i| > (s-1)n^{k-1}$, and thus $\mathbb{E}\zeta_i = \mathbb{E}[|\mathcal{F}_i \cap \mathcal{M}|] - s + 1 > 0$. Then we have

$$0 < \sum_{i=1}^s \mathbb{E}\zeta_i = \sum_{i=1}^s \mathbb{E}[\zeta_i \mid \zeta_i \leq 0] \cdot \mathbb{P}[\zeta_i \leq 0] + \sum_{i=1}^s \mathbb{E}[\zeta_i \mid \zeta_i > 0] \cdot \mathbb{P}[\zeta_i > 0].$$

For simplicity, let us put $x := \lfloor 3.7\sqrt{s \log s} \rfloor$. Combining this with (6) and (7), we get that

$$0 < -\mathbb{E}[j_{\mathcal{M}}(s - j_{\mathcal{M}})] + \sum_{i=1}^s x \mathbb{P}[\zeta_i > 0] \leq -\mathbb{E}[j_{\mathcal{M}}(s - j_{\mathcal{M}})] + x \mathbb{E}[j_{\mathcal{M}}] = -\mathbb{E}[j_{\mathcal{M}}(s - x - j_{\mathcal{M}})]. \quad (8)$$

At the same time, using that $j_{\mathcal{M}} < s$, we get

$$\mathbb{E}[j_{\mathcal{M}} \cdot I[j_{\mathcal{M}} < s - x]] = \mathbb{E}j_{\mathcal{M}} - \mathbb{E}[j_{\mathcal{M}} \cdot I[j_{\mathcal{M}} \geq s - x]] \geq \mathbb{E}j_{\mathcal{M}} - s \mathbb{P}[j_{\mathcal{M}} \geq s - x]. \quad (9)$$

From here, we get the following chain of inequalities (note that we use $j_{\mathcal{M}}(s - x - j_{\mathcal{M}}) \geq -sx$ for $j_{\mathcal{M}} \leq s$):

$$\begin{aligned} \mathbb{E}[j_{\mathcal{M}}(s - x - j_{\mathcal{M}})] &= \mathbb{E}[j_{\mathcal{M}}(s - x - j_{\mathcal{M}}) \cdot I[j_{\mathcal{M}} < s - x]] + \mathbb{E}[j_{\mathcal{M}}(s - x - j_{\mathcal{M}}) \cdot I[j_{\mathcal{M}} \geq s - x]] \\ &\geq \mathbb{E}[j_{\mathcal{M}} \cdot I[j_{\mathcal{M}} < s - x]] - sx \mathbb{P}[j_{\mathcal{M}} \geq s - x] \\ &\stackrel{(9)}{\geq} \mathbb{E}j_{\mathcal{M}} - s(x+1) \mathbb{P}[j_{\mathcal{M}} \geq s - x]. \end{aligned}$$

Together with (8), this gives

$$s(x+1) \mathbb{P}[j_{\mathcal{M}} \geq s - x] > \mathbb{E}j_{\mathcal{M}}. \quad (10)$$

Note that, by definition, $\mathbb{P}[j_{\mathcal{M}} \geq s - x] \leq \sum_{i=1}^s \mathbb{P}[\zeta_i \leq x - s]$. Combining this with (7) and (10), we get

$$\sum_{i=1}^s \mathbb{E}[\zeta_i \cdot I[\zeta_i > 0]] \stackrel{(7)}{\leq} x \mathbb{E}j_{\mathcal{M}} \stackrel{(10)}{<} sx(x+1) \mathbb{P}[j_{\mathcal{M}} \geq s - x] \leq sx(x+1) \sum_{i=1}^s \mathbb{P}[\zeta_i \leq x - s]. \quad (11)$$

We shall apply (2) to $\eta = \zeta_i + (s-1)$. This time, we have $\delta = -1$, $\mu = \frac{s-x-4\sqrt{s}}{2}$, $\lambda = 2\sqrt{s}$ we get that

$$\begin{aligned} \mathbb{P}[\zeta_i \leq x - s] &= \mathbb{P}[\eta - (s-1) \leq x - s] \leq \mathbb{P}[\eta \leq s - 2] \cdot 4 \exp\left(-\frac{(s-x-4\sqrt{s})^2}{6s-4x-8\sqrt{s}}\right) \\ &\leq \mathbb{P}[\zeta_i \leq -1] \cdot 4e^{-z(s)} \leq -4e^{-z(s)} \mathbb{E}[\zeta_i \cdot I[\zeta_i \leq 0]], \end{aligned} \quad (12)$$

where $z(s) = \frac{(s-x-4\sqrt{s})^2}{6s-4x-8\sqrt{s}} \sim \frac{s}{6}$ (up to terms of order $\sqrt{s \log s}$). Combining (11) with (12) and substituting the value of x , we get that

$$\sum_{i=1}^s \mathbb{E}[\zeta_i \cdot I[\zeta_i > 0]] \leq -sx(x+1) \cdot 4e^{-z(s)} \sum_{i=1}^s \mathbb{E}[\zeta_i \cdot I[\zeta_i \leq 0]].$$

Once $sx(x+1) \cdot 4e^{-z(s)} < 1$, this contradicts the assumption $\sum_{i=1}^s \mathbb{E}\zeta_i > 0$. This happens for any $s > 470$. It is easy to see that all the other inequalities used in the proof are valid for this choice of s . The theorem is proved.

4 Discussion

As we have mentioned in the introduction, the example $\mathcal{F}_1 = \dots = \mathcal{F}_s = [s-1] \times [n]^{k-1}$ shows that the uniform lower bound in Conjecture 1 is tight. In what follows, we explore what happens if we bound the size of each family separately.

Problem 11. *Let $n \geq s$ and k be positive integers. We call a sequence $f_1 \leq \dots \leq f_s$ satisfying if any $\mathcal{F}_1, \dots, \mathcal{F}_s \subset [n]^k$ with $|\mathcal{F}_i| > f_i$ contain a rainbow matching. Which sequences f_1, \dots, f_s are satisfying?*

Theorem 6 easily implies that, for some $C > 0$, the sequence a_1, \dots, a_s with $a_i := (i + C\sqrt{s \log s})n^{k-1}$ is satisfying. (In fact, it implies that with very large probability, for all i simultaneously the family \mathcal{F}_i intersects a random matching in at least i sets.) A modification of the proof of Theorem 1 implies that the sequence b_1, \dots, b_s with $b_i := \min(i + C\sqrt{s \log s}, s - 1 + \varepsilon)n^{k-1}$ is satisfying, where $\varepsilon > 0$ can be chosen arbitrarily small, and we need to either take $C \geq C_0(\varepsilon)$ or $s \geq s_0(\varepsilon)$.

It is tempting to suggest that the sequence $0, n^{k-1}, \dots, (s-1)n^{k-1}$ is satisfying. However, this is not the case. Indeed, even the sequence c_1, \dots, c_s with $c_1 = \dots = c_{s-1} = (s-1)n^{k-1} - (n-1)^{k-1} - 1$, $c_s = (s-1)n^{k-1}$, is not. Indeed, fix a set $F \in [s+1, n] \times [n]^{k-1}$ and consider the following families: $\mathcal{F}_1 = \dots = \mathcal{F}_{s-1} = [s-1] \times [n]^{k-1} \setminus \{F' \in \{1\} \times [n]^{k-1} : F \cap F' = \emptyset\}$, $\mathcal{F}_s = [s-1] \times [n]^{k-1} \cup \{F\}$.

Conjecture 2. *The sequences d_1, \dots, d_s with $d_i := \min(i + C\sqrt{s \log s}, s-1)n^{k-1}$ and e_1, \dots, e_s with $e_i := i \cdot n^{k-1}$ are satisfying.*

Note that the sequence d_1, \dots, d_s being satisfying implies Conjecture 1.

Finally, we note that we may ask a variant of Conjecture 1 for $\mathcal{F}_1, \dots, \mathcal{F}_s \subset [n_1] \times [n_2] \times \dots \times [n_k]$ for $n_1 \leq \dots \leq n_k$. There, a natural extremal example would be to fix $s-1$ coordinates in the first part, which is the smallest, and consider the family of all tuples that intersect the first part in one of those coordinates.

A simple induction argument allows to reduce this, seemingly more general, version of Conjecture 1, to the case $n_1 = \dots = n_k$. Let us give a sketch of that. Take families $\mathcal{F}_1, \dots, \mathcal{F}_s \subset [n_1] \times \dots \times [n_k]$ that do not contain a rainbow s -matching. First, using a standard argument, we may assume that the families are *shifted* in each part, i.e., if $(f_1, \dots, f_k) \in \mathcal{F}_i$ then $(f'_1, \dots, f'_k) \in \mathcal{F}_i$ for any f'_1, \dots, f'_k satisfying $f'_j \leq f_j$ for every j . Given this, we may proceed by induction on k for fixed s and on $\sum_i n_i$ for fixed s and k , as long as $n_1 < n_k$. We apply the induction hypothesis to $\mathcal{F}_i(\bar{n}_k) := \{(a_1, \dots, a_k) \in \mathcal{F}_i : a_k \neq n_k\}$. It is easy to see that $\mathcal{F}_1(\bar{n}_k), \dots, \mathcal{F}_s(\bar{n}_k)$ do not contain a rainbow s -matching, and thus $\min_i |\mathcal{F}_i| \leq (s-1) \cdot n_2 \cdot \dots \cdot n_{k-1} \cdot (n_k - 1)$. At the same time, for every i , shiftedness imply that the degree of the element $\{n_k\}$ in the k -th part is the smallest among $\{1, \dots, n_k\}$, and thus $|\mathcal{F}_i(\bar{n}_k)| \geq \frac{n_k-1}{n_k} |\mathcal{F}_i|$. Combining these two inequalities, we get that $\min_i |\mathcal{F}_i| \leq (s-1) \cdot n_2 \cdot \dots \cdot n_k$, as required.

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