Girth-reducibility and the algorithmic barrier for coloring

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Abstract

All known efficient algorithms for constraint satisfaction problems are stymied by random instances. For example, no efficient algorithm is known that can q-color a random graph with average degree $(1+\epsilon)q\ln q$, even though random graphs remain q-colorable for average degree up to $(2-o(1))q\ln q$. Similar failure to find solutions at relatively low constraint densities is known for random CSPs such as random k-SAT and other hypergraph-based problems. The constraint density where algorithms break down for each CSP is known as the "algorithmic barrier" and provably corresponds to a phase transition in the geometry of the space of solutions [Achlioptas and Coja-Oghlan 2008]. In this paper we aim to shed light on the following question: Can algorithmic success up to the barrier for each CSP be ascribed to some simple deterministic property of the inputs?

We answer this question positively for graph coloring by identifying the property of *girth-reducibility*. We prove that every girth-reducible graph of average degree $(1-o(1))q\ln q$ is efficiently q-colorable and that the threshold for girth reducibility of random graphs coincides with the algorithmic barrier. Thus, we link the tractability of graph coloring up to the algorithmic barrier to a single deterministic property. Our main theorem actually extends to coloring k-uniform hypergraphs. As such, we believe that it is an important first step towards discovering the structural properties behind the tractability of arbitrary k-CSPs for constraint densities up to the algorithmic barrier.

*This material is based upon work directly supported by the IAS Fund for Math and indirectly supported by the National Science Foundation Grant No. CCF-1900460. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

1 Introduction

Due to intense research in the past couple of decades, for many random constraint satisfaction problems (CSPs), such as random graph coloring, random k-SAT, and other hypergaph-based problems, we are now aware of asymptotically tight [5], and in some cases even exact [16], estimates for the largest constraint density for which typical instances have solutions. At the same time though, current state-of-the-art algorithms stop finding solutions efficiently at much lower constraint densities than those for which the existence of solutions is guaranteed. For example, no efficient algorithm is known that is able to q-color a random graph with average degree $(1+\epsilon)q\ln q$, even though random graphs remain q-colorable for average degree up to $(2-o(1))q\ln q$. (Equivalently, while random graphs of bounded average degree d are known to be almost surely $(\frac{1}{2}+o(1))\frac{d}{\ln d}$-colorable [4, 14, 31], all known efficient algorithms require twice as many colors.)

This is not a coincidence, as the point where all known algorithms stop corresponds precisely to a phase transition in the geometry of the space of solutions known as the *shattering threshold* [1, 39], often referred to as the "algorithmic barrier" [1]. In particular, Achlioptas and Coja-Oghlan [1] proved that while the set of solutions for low densities forms a giant well-connected cluster, at some critical threshold this "huge ball" shatters into an exponential number pieces (clusters of solutions), where each cluster is well-connected and any two clusters are well-separated by huge "energy barriers". This phenomenon is a large part of the "1-step Replica Symmetry Breaking" hypothesis [29, 32] in statistical physics, and finding an efficient algorithm to solve any CSP for constraint-densities beyond the algorithmic barrier is a major open problem. For example, it seems that local algorithms are bound to fail beyond this threshold [3, 12, 13].

In this paper we aim to shed light on the question:

Can algorithmic success up to the barrier for each CSP be ascribed to some simple deterministic property of the inputs?

In other words, given a CSP of interest, is there a structural property P such that (i) the family of instances that admit property P are tractable; and (ii) almost all instances of constraint density up to the algorithmic barrier admit P? Besides its theoretical importance, an answer to this question would allow the design of algorithms that are *robust* enough to apply to *deterministic* instances, while at the same time matching the performance of the best known algorithms for random models.

In this paper we study this question for the problem of coloring graphs and, more generally, k-uniform hypergraphs. To simplify the exposition, we first focus on the case of graphs.

Our contribution is to identify a family of graphs which we call *girth-reducible* and prove the following theorem regarding their chromatic number and their relation to sparse random graphs.

Theorem 1.1 (Informal statement)**.** *There exists an efficient deterministic algorithm that properly colors any girth-reducible graph of average degree d using $(1+o(1))d/\ln d$ colors. Additionally, almost every graph of bounded average degree is girth-reducible.*

The bound on the chromatic number in Theorem 1.1 matches the algorithmic barrier [1, 39] for coloring sparse random graphs. Thus, Theorem 1.1 links the tractability of graph q-coloring up to the algorithmic barrier to a single deterministic property. (To see this, say that a graph has the *q-girth-reducibility* property if it is of average degree $(1-o(1))q\ln q$ and is girth-reducible. Theorem 1.1 implies that any such graph is efficiently q-colorable, and that the threshold for q-girth-reducibility of random graphs coincides with the algorithmic barrier.) Note also that Theorem 1.1 extends to the more general list-coloring problem and, as a consequence, we obtain the first efficient deterministic algorithm for list-coloring sparse random graphs that works up to the algorithmic barrier. To the best of our knowledge, the currently best list-coloring algorithm [3] for sparse random graphs is randomized.

Roughly speaking, a graph of average degree d is girth-reducible if it can be treated as a graph of girth 5 and of maximum degree d for the purposes of coloring. This means that its vertex set can be seen as the union of two parts: A "low-degeneracy" part, which contains all vertices of degree more than d, and a "high-girth" part, which induces a graph of maximum degree roughly d and girth 5. (Recall that a graph is κ-degenerate if its vertices can be ordered so that every vertex has at most κ neighbors greater than itself. Thus, any such graph can be greedily colored with $\kappa + 1$ colors.)

Definition 1.2 (Informal definition)**.** *We say that graph $G(V, E)$ of average degree d is* girth-reducible *if its vertex set can be partitioned in two sets, U and $V \setminus U$, such that:*

(a) subgraph $G[U]$ is $\frac{d}{\ln d}$-degenerate;

(b) subgraph $G[V \setminus U]$ has maximum degree at most $(1 + o(1))d$ and is of girth at least 5;

(c) every vertex in $V \setminus U$ has $o\left(\frac{d}{\ln d}\right)$ neighbors in U.

As we will see, there exists a simple and efficient procedure that decides whether a given graph is girth-reducible, which also outputs the promised partition in case it is. Furthermore, the definition of girth-reducibility naturally extends to k-uniform hypergraphs (see Section 1.1, Definition 1.5). In fact, Theorem 1.1 is a special case of the following more general result, which gives a bound on the chromatic number of k-uniform hypergraphs that is within a factor of $(k - 1)$ of the algorithmic barrier [9, 21].

Theorem 1.3 (Informal statement)**.** *There exists an efficient deterministic algorithm that properly colors any girth-reducible k-uniform hypergraph of average degree d using $(1 + o(1))(k - 1)\,(d/\ln d)^{1/(k-1)}$ colors. Additionally, almost every k-uniform hypergraph of bounded average degree is girth-reducible.*

As we discuss in Section 1.2, dealing with constraints of large arity is highly non-trivial when the input is deterministic. Thus, we believe that Theorem 1.3 is an important first step towards discovering the structural properties behind the tractability of arbitrary k-CSPs for constraint densities up to the algorithmic barrier.

It is also worth noting that the definition of girth-reducibility emerges quite naturally in the light of a simple observation regarding the ways in which we can properly color the neighborhood of any fixed vertex in a high-girth k-uniform hypergraph. To describe the observation we will again focus for simplicity on the case of graphs, but an analogous phenomenon takes place for any $k \geq 2$.

Given a triangle-free graph fix one of its vertices, say v, and let d_v be its degree. Consider now all the possible ways to color the neighborhood of v using q colors. Say that a color is "available" for v in such a coloring if assigning it to v does not create any monochromatic edge. A simple argument reveals that if $q \leq (1 - \epsilon)d_v/\ln d_v$, then the majority of ways to properly color the neighbors of v leaves it with no available colors while, if $q \geq (1 + \epsilon)d_v/\ln d_v$, then the number of available colors for v is non-vanishing as d_v grows. (We give the details in Section 1.2.) Our key insight is that this local phenomenon suffices to explain the tractability of coloring up to the algorithm barrier. To get a feeling for why this is the case, at first recall that our best algorithms are not able to efficiently color a random graph of bounded average degree d using $(1 - \epsilon)d/\ln d$ colors, i.e., this point corresponds to the algorithmic barrier for coloring. Further, it is well-known that sparse random graphs are "locally tree-like", namely a random vertex does not participate in any cycle of constant length and, in particular, in any triangles. Therefore, our initial observation implies that the algorithmic barrier *coincides* with the point at which a typical vertex of a random graph is most likely left with no available colors after a random coloring of its neighborhood. ("Typical" here means a vertex of degree at most d that is not contained in short cycles.) This is too striking to be a mere coincidence, and indeed we show that it is not. At the very least, it suggests that in order to efficiently color any deterministic graph of average degree d using $(1 + o(1))d/\ln d$ colors, we should be able to color the vertices of degree higher than d *before* the rest of the graph. For otherwise, the vast majority of ways to color the neighborhood of any such vertex leaves it with no available colors. Now the most straightforward way to color these vertices is via a greedy algorithm, and this is going to work on a deterministic instance only if the subgraph induced by these vertices is of low degeneracy.

Indeed, this is precisely the strategy we follow to color a girth-reducible hypergraph. That is, we first color its low-degeneracy part using the greedy algorithm. The remaining vertices form a high-girth hypergraph that can be efficiently colored by a generalization of a classical result of Kim [26] which we develop in this paper.

Theorem 1.4 (Informal statement)**.** *Every k-uniform hypergaph of degree Δ and girth at least 5 is efficiently $(1+o(1))(k-1)(\Delta/\ln\Delta)^{1/(k-1)}$-list colorable via a deterministic algorithm.*

Theorem 1.4 is of independent interest as it implies the classical theorem of Ajtai-Komlós-Pintz-Spencer-Szemerédi [6] regarding the independence number of k-uniform hypergraphs of degree Δ and girth 5. The latter is a seminal result in combinatorics, with applications in geometry and coding theory [27, 28, 30]. Further, Theorem 1.4 is tight up to a constant [10]. Note also that, without the girth assumption, the best possible bound [17] on the chromatic number of k-uniform hypergraphs is $O(\Delta^{1/(k-1)})$, i.e., it is asymptotically worse than the one of Theorem 1.4. For example, there exist graphs of degree Δ whose chromatic number is exactly $\Delta+1$. We further discuss the relation of Theorem 1.4 to past results in Section 1.1.

We remark that our work is inspired by a recent paper of Molloy [33], who showed that triangle-free graphs of maximum degree Δ can be efficiently properly colored using at most $(1+o(1))\frac{\Delta}{\ln\Delta}$ colors, and pointed out that this bound matches the algorithmic barrier for coloring random *regular* graphs of bounded degree [39]. Molloy's result is an extension of the one of Kim [26], who showed the same bound for graphs of degree Δ and girth at least 5. Remarkably, the results of Kim and Molloy imply that the tractability of coloring sparse regular graphs for densities up to the algorithmic barrier boils down to a very simple property, namely the absence of short cycles. (Random regular graphs of bounded degree are essentially high-girth graphs: We can almost surely remove a matching containing only a few edges to get a graph of girth 5. This modification changes the chromatic number of the graph by at most one.) Here we show that this phenomenon extends to general, i.e., not necessarily regular, sparse k-uniform hypergraphs and that the crucial structural property is girth-reducibility.

To conclude, we stress that the technique of Molloy does not seem to easily extend to hypergraphs even though it is significantly simpler than the one of Kim. Indeed, our main technical contribution is dealing with constraints of large arity and our approach is based on the so-called *semi-random method*. We further discuss the technical aspect of our work in Section 1.2.

1.1 Statement of results

In hypergraph coloring one is given a hypergraph $H(V,E)$ and the goal is to find an assignment of one of q colors to each vertex $v \in V$ so that no hyperedge is monochromatic. In the more general *list-coloring* problem, a list of q allowed colors is specified for each vertex. A graph is q-list-colorable if it has a list-coloring no matter how the lists are assigned to each vertex. The *list chromatic number*, $\chi_\ell(H)$, is the smallest q for which H is q-list colorable. To formally describe our results, we need some notation.

A hypergraph is is k-*uniform* if every hyperedge contains exactly k variables. An i-cycle in a k-uniform hypergraph is a collection of i distinct hyperedges spanned by at most $i(k-1)$ vertices. We say that a k-uniform hypergraph has girth at least g if it contains no i-cycles for $2 \le i < g$. Note that if a k-uniform hypergraph has girth at least 3 then every two of its hyperedges have at most one vertex in common.

A k-uniform hypergraph H is κ-*degenerate* if the induced subhypergraph of all subsets of its vertex set has a vertex of degree at most κ. The *degeneracy* of a hypergraph H is the smallest value of κ for which H is κ-degenerate. Note that it is known that κ-degenerate hypergraphs are $(\kappa+1)$-list colorable and that the degeneracy of a hypergraph can be computed efficiently by an algorithm that repeatedly removes minimum degree vertices. Indeed, to list-color a κ-degenerate hypergraph we repeatedly find a vertex with (remaining) degree at most κ, assign to it a color that does not appear in any of its neighbors so far, and remove it from the hypergraph. Clearly, if the lists assigned to each vertex are of size at least $\kappa + 1$ this procedure always terminates successfully.

Definition 1.5. *For $\delta \in (0,1)$, we say that a k-uniform hypergraph $H(V,E)$ of average degree d is δ-*girth-reducible *if its vertex set can be partitioned in two sets, U and $V \setminus U$, such that:*

(a) subhypergraph $H[U]$ is $\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$-degenerate;

(b) subhypergraph $H[V \setminus U]$ has maximum degree at most $(1+\delta)d$ and is of girth at least 5;

(c) every vertex in $V \setminus U$ has at most $\delta \left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ neighbors in U.

Note that given a δ-girth-reducible hypergraph we can efficiently find the promised partition $(U, V \setminus U)$ as follows. We start with $U := U_0$, where U_0 is the set of vertices that either have degree at least $(1+\delta)d$, or they are contained in a cycle of length at most 4. Let ∂U denote the vertices in $V \setminus U$ that violate property (c). While $\partial U \neq \emptyset$, update U as $U := U \cup \partial U$. The correctness of the process lies in the fact that in each step we add to the current U a set of vertices that must be in the low-degeneracy part of the hypergraph. Observe also that this process allows us to efficiently check whether a hypergraph is δ-girth-reducible.

Theorem 1.6. *For any constants $\delta \in (0,1)$ and $k \geq 2$, there exists $d_{\delta,k} > 0$ such that if H is a δ-girth-reducible, k-uniform hypergraph of average degree d, then*

$$\chi_\ell(H) \leq (1+\epsilon)(k-1)\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}},$$

where $\epsilon = 4\delta = O(\delta)$. Furthermore, if H is a hypergraph on n vertices then there exists a deterministic algorithm that constructs such a coloring in time polynomial in n.

As we have already discussed, girth-reducibility is a pseudo-random property which is admitted by almost all sparse k-uniform hypregraphs. To establish this fact formally, we need some further notation.

The random k-uniform hypergraph $H(k,n,p)$ is obtained by choosing each of the $\binom{n}{k}$ k-element subsets of a vertex set V ($|V| = n$) independently with probability p. The chosen subsets are the hyperedges of the hypergraph. Note that for $k = 2$ we have the usual definition of the random graph $G(n,p)$. We say that $H(k,n,p)$ has a certain property A *almost surely* or *with high probability*, if the probability that $H \in H(k,n,p)$ has A tends to 1 as $n \to \infty$.

Theorem 1.7. *For any constants $\delta \in (0,1)$, $k \geq 2$, there exists $d_{\delta,k} > 0$ such that for every constant $d \geq d_{\delta,k}$, almost surely, the random hypergraph $H(k,n,d/\binom{n}{k-1})$ is δ-girth-reducible.*

Remark 1.1. *Note that, for k,d constants, a very standard argument reveals that $H(k,n,d/\binom{n}{k-1})$ is essentially equivalent to $\mathbb{H}(k,n,kdn)$, namely the uniform distribution over k-uniform hypergraphs with n vertices and exactly kdn hyperedges. Thus, Theorem 1.7 extends to that model as well.*

Theorem 1.7 follows by simple, although somewhat technical, considerations on properties of sparse random hypergraphs, which are mainly inspired by the the results of Alon, Krivelevich and Sudakov [8] and Łuczak [31]. Notice that for constant d, the random hypergraph $H(k,n,d/\binom{n}{k-1})$ is far from regular as its average degree is roughly d, while its maximum degree is in the order of $\ln n / \ln \ln n$. This is the main reason why the results of Kim and Molloy match the algorithmic barrier only for regular graphs. It is worth mentioning though that, as soon as $d = \Omega(\log n)$, the maximum degree of $G(n,d/n)$ also becomes approximately d, which allows recent extensions [2, 15] of Molloy's result regarding the chromatic number of (deterministic) locally spare graphs to successfully color $G(n,d/n)$ using $(1+o(1))d/\ln d$ colors for values of d that range from $\Omega(\log n)$ to at least $(n \ln n)^{\frac{1}{3}}$.[1] We are not aware of analogous results for k-uniform hypergraphs (it appears that the techniques of [2, 15, 33] do not easily extend to $k > 2$).

Theorem 1.6 is derived almost immediately by the following extension of the classical result of Kim [26] to hypergraphs.

Theorem 1.8. *Let H by any k-uniform hypergraph, $k \geq 2$, of maximum degree Δ and girth at least 5. For all $\epsilon > 0$, there exist a positive constant $\Delta_{\epsilon,k}$ such that if $\Delta \geq \Delta_{\epsilon,k}$, then*

$$\chi_\ell(H) \leq (1+\epsilon)(k-1)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}} \quad . \tag{1}$$

Furthermore, if H is a hypergraph on n vertices then there exists a deterministic algorithm that constructs such a coloring in time polynomial in n.

The bound of Theorem 1.8 is within a factor of $k-1$ of the algorithmic barrier for coloring random regular k-uniform hypergraphs [9, 21], while it holds for every hypergraph of girth at least 5. Further, when it applies, it improves upon a theorem of Frieze and Mubayi [20] for list-coloring simple, triangle-free, k-uniform hypergraphs, who showed (1) with an unspecified large leading constant (of order at least $\Omega(k^4)$). We remark that the techniques of Frieze and Mubayi are based on the proof of Johansson [22] for coloring triangle-free graphs of maximum degree Δ using $O(\Delta/\ln \Delta)$ colors, which is already suboptimal with respect to the algorithmic barrier for random regular graphs. Therefore, it is highly unlikely that their result can be significantly improved to (nearly) match the algorithmic barrier, even for small k.

1.2 Technical overview

As we show in the end of this section, Theorem 1.6 follows fairly easily from Theorem 1.8, so we focus on describing the main approach for proving the latter.

The intuition behind the proof of Theorem 1.8 comes from the following observation, which we explain in terms of graph coloring for simplicity. Let G be a triangle-free graph of degree Δ, and assume that each of its vertices is assigned an arbitrary list of q colors. Fix a vertex v of G, and consider the random experiment in which the neighborhood of v is properly list-colored randomly. Since G contains no triangles, this amounts to assigning to each neighbor of v a color from its list randomly and independently. Assuming that $q \geq q^* := (1+\epsilon)\Delta/\ln \Delta$, the expected number of *available* colors for v, i.e., the colors from the list of v that do not appear in any of its neighbors, is at least $q(1-1/q)^\Delta = \omega(\Delta^{\epsilon/2})$. In fact, a simple concentration argument reveals that the number of available colors for v in the end of this experiment is at least $\Delta^{\epsilon/2}$ with probability that goes to 1 as Δ grows. To put it differently, as long as $q \geq q^*$, the vast majority of valid ways to list-color the neighborhood of v "leaves enough room" to color v without creating any monochromatic edges.

A completely analogous observation regarding the ways to properly color the neighborhood of a vertex can be made for k-uniform hypergraphs. In order to exploit it we employ the so-called *semi-random method*, which is the main tool behind some of the strongest graph coloring results, e.g., [22, 23, 24, 25, 34, 38], including the one of Kim [26]. The idea is to gradually color the hypergraph in iterations until we reach a point where we can finish the coloring with a simple, e.g., greedy, algorithm. In its most basic form, each iteration consists of the following simple procedure (using graph vertex coloring as a canonical example): Assign to each vertex a color chosen uniformly at random; then uncolor any vertex that receives the same color as one of its neighbors. Using the Lovász Local Lemma [17] and concentration inequalities, one typically shows that, with positive probability, the resulting partial coloring has useful properties that allow for the continuation of the argument in the next iteration. (In fact, using the Moser-Tardos algorithm [36]

[1] However, the "shattering" phenomenon [1] has only been rigorously established for constant d.

this approach yields efficient, and often times deterministic [11], algorithms.) Specifically, one keeps track of certain parameters of the current partial coloring and makes sure that, in each iteration, these parameters evolve almost as if the coloring was totally random. For example, recalling the heuristic experiment of the previous paragraph, one of the parameters we would like to keep track of in our case is a lower bound on the number of available colors of each vertex in the hypergraph: If this parameter evolves "randomly" throughout the process, then the vertices that remain uncolored in the end are guaranteed to have a non-trivial number of available colors.

Applications of the semi-random method tend to be technically intense and this is even more so in our case, where we have to deal with constraints of large arity. Large constraints introduce several difficulties, but the most important one is that our algorithm has to control many parameters that interact with each other. Roughly, in order to guarantee the properties that allow for the continuation of the argument in the next iteration, for each uncolored vertex v, each color c in the list of v, and each integer $r \in [k-1]$, we should keep track of a lower bound on the number of adjacent to v hyperedges that have r uncolored vertices and $k-1-r$ vertices colored c. Clearly, these parameters are not independent of each other throughout the process, and so the main challenge is to design and analyze a coloring procedure in which all of them, simultaneously, evolve essentially randomly.

We conclude this section with a proof of Theorem 1.6, based on Theorem 1.8.

Proof of Theorem 1.6. Let $\epsilon = 4\delta$. Given lists of colors of size $(1+\epsilon)(k-1)\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ for each vertex of H, we first color the vertices of U using the greedy algorithm which exploits the low degeneracy of $H[U]$. Now each vertex in $V-U$ has at most $\delta\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ forbidden colors in its list as it has at most that many neighbors in U. We delete these colors from the list. Observe that if we manage to properly color the induced subgraph $H[V \setminus U]$ using colors from the updated lists, then we are done since every hyperedge with vertices both in U and $V \setminus U$ will be automatically "satisfied", i.e., it cannot be monochromatic. Notice now that the updated list of each vertex still contains at least $(1+3\delta)(k-1)\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ colors, for sufficiently large d. Since the induced subgraph $H[V \setminus U]$ is of girth at least 5 and of maximum degree at most $(1+\delta)d$, it is efficiently $(1+\delta)(k-1)\left(\frac{(1+\delta)d}{\ln((1+\delta)d)}\right)^{\frac{1}{k-1}}$-list-colorable for sufficiently large d per Theorem 1.8. This concludes the proof since $(1+\delta)(1+\delta)^{\frac{1}{k-1}} < (1+3\delta)$.

□

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we present the necessary background. In Section 3 we present the algorithm and state the key lemmas for the proof of Theorem 1.8, while in Section 4 we give the full details. Finally, in Section 5 we prove Theorem 1.7.

2 Background and preliminaries

In this section we give some background on the technical tools that we will use in our proofs.

2.1 The Lovász Local Lemma

We will find useful the so-called *lopsided* version of the Lovász Local Lemma [17, 18].

Theorem 2.1. *Consider a set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of (bad) events. For each $B \in \mathcal{B}$, let $D(B) \subseteq \mathcal{B}\backslash\{B\}$ be such that $\Pr[B \mid \bigcap_{C\in S} \overline{C}] \le \Pr[B]$ for every $S \subseteq \mathcal{B}\backslash(D(B)\cup\{B\})$. If there is a function $x : \mathcal{B} \to (0,1)$ satisfying*

$$\Pr[B] \le x(B) \prod_{C\in D(B)} (1 - x(C)) \quad \textit{for all } B \in \mathcal{B} \ , \tag{2}$$

then the probability that none of the events in $\mathcal{B}$ occurs is at least $\prod_{B\in\mathcal{B}}(1 - x(B)) > 0$.

In particular, we will need the following two corollaries of Theorem 2.1. For their proofs, the reader is referred to Chapter 19 in [35].

Corollary 2.2. *Consider a set $\mathcal{B} = \{B_1, \ldots, B_m\}$ of (bad) events. For each $B \in \mathcal{B}$, let $D(B) \subseteq \mathcal{B} \setminus \{B\}$ be such that $\Pr[B \mid \bigcap_{C\in S} \overline{C}] \le \Pr[B]$ for every $S \subseteq \mathcal{B} \setminus (D(B) \cup \{B\})$. If for every $B \in \mathcal{B}$:*

(a) $\Pr[B] \le \frac{1}{4}$;

(b) $\sum_{C\in D(B)} \Pr[C] \le \frac{1}{4}$,

then the probability that none of the events in $\mathcal{B}$ occurs is strictly positive.

Corollary 2.3. *Consider a set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of (bad) events such that for each $B \in \mathcal{B}$:*

(a) $\Pr[B] \le p < 1$;

(b) B is mutually independent of a set of all but at most Δ of the other events.

If $4p\Delta \le 1$ then with positive probability, none of the events in $\mathcal{B}$ occur.

2.2 Talagrand's inequality

We will also need the following version of Talagrand's inequality [37] whose proof can be found in [35].

Theorem 2.4. *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$, and satisfying the following for some $c, r > 0$:*

1. *changing the outcome of any trial can affect X by at most c, and*
2. *for any s, if $X \ge s$ then there is a set of at most ws trials whose outcomes certify that $X \ge s$,*

then for any $0 \le t \le \mathbb{E}[X]$,

$$\Pr[|X - \mathbb{E}[X]| > t + 60c\sqrt{w\mathbb{E}[X]}] \le 4\mathrm{e}^{-\frac{t^2}{8c^2 w\mathbb{E}[X]}} \ .$$

3 List-coloring high-girth hypergraphs

In this section we describe the algorithm of Theorem 1.8. As we already explained, our approach is based on the semi-random method. For an excellent exposition both of the method and Kim's result the reader is referred to [35].

We assume without loss of generality that $\epsilon < \frac{1}{10}$. Also, it will be convenient to define the parameter $\delta := (1+\epsilon)(k-1) - 1$, so that the list of each vertex initially has at least $(1+\delta)(\frac{\Delta}{\ln\Delta})^{\frac{1}{k-1}}$ colors.

We analyze each iteration of our procedure using a probability distribution over the set of (possibly improper) colorings of the uncolored vertices of H where, additionally, each vertex is either activated or deactivated. We call a pair of coloring and activation bits assignments for the uncolored vertices of hypergraph H a *state*.

Let V_i denote the set of uncolored vertices in the beginning of the the i-th iteration. (Initially, all vertices are uncolored.) For each $v \in V_i$ we denote by $L_v = L_v(i)$ the list of colors of v. Further, we say that a color $c \in L_v$ is *available for* v in a state σ if assigning c to v does not cause any hyperedge whose initially uncolored vertices are all activated in σ to be monochromatic.

For each vertex v, color $c \in L_v$ and iteration i, we define a few quantities of interest that our algorithm will attempt to control. Let $\ell_i(v)$ be the size of L_v. Further, for each $r \in [k]$, let $D_{i,r}(v,c)$ denote the set of hyperedges h that contain v and (i) exactly r vertices $\{u_1, \ldots, u_r\} \subseteq h \setminus \{v\}$ are uncolored and $c \in L_{u_j}$ for every $j \in [r]$; (ii) the rest $k-1-r$ vertices other than v are colored c. We define $t_{i,r}(v,c) = |D_{i,r}(v,c)|$.

As it is common in the applications of the semi-random method, we will not attempt to keep track of the values of $\ell_i(v)$ and $t_{i,r}(v,c)$, $r \in [k-1]$, for every vertex v and color c but, rather, we will focus on their extreme values. In particular, we will define appropriate $L_i, T_{i,r}$ such that we can show that, for each i, the following property holds at the beginning of iteration i:

Property P(i): For each vertex $v \in V_i$, color $c \in L_v$ and $r \in [k-1]$,

$$\begin{aligned} \ell_i(v) &\geq L_i, \\ t_{i,r}(v,c) &\leq T_{i,r}. \end{aligned}$$

As a matter of fact, it would be helpful for our analysis (though not necessary) if the inequalities defined in $P(i)$ were actually tight. Given that $P(i)$ holds, we can always enforce this stronger property in a straightforward way as follows. First, for each vertex v such that $\ell_i(v) > L_i$ we choose arbitrarily $\ell_i(v) - L_i$ colors from its list and remove them. Then, for each vertex v and color $c \in L_i$ such that $t_{i,r}(v,c) < T_{i,r}$ we add to the hypergraph $T_{i,r} - t_{i,r}(v,c)$ new hyperedges of size $r+1$ that contain v and r new "dummy" vertices. (As it will be evident from the proof, we can always assume that $L_i, T_{i,r}$ are integers, since our analysis is robust to replacing $L_i, T_{i,r}$ with $\lfloor L_i \rfloor$ and $T_{i,r}$ with $\lceil T_{i,r} \rceil$.) We assign each dummy vertex a list of L_i colors: $L_i - 1$ of them are new and do not appear in the list of any other vertex, and the last one is c.

Remark 3.1. *Dummy vertices are only useful for the purposes of our analysis and can be removed at the end of the iteration. Indeed, one could use the technique of "equalizing coin flips" instead. For more details see e.g., [35].*

Overall, without loss of generality, at each iteration i our goal will be to guarantee that $P(i+1)$ holds assuming $Q(i)$.

Property Q(i): For each vertex $v \in V_i$, color $c \in L_v$ and $r \in [k-1]$,

$$\begin{aligned} \ell_i(v) &= L_i, \\ t_{i,r}(v,c) &= T_{i,r}. \end{aligned}$$

An iteration. For the i-th iteration we will apply the Local Lemma with respect to the probability distribution induced by assigning to each vertex $v \in V_i$ a color chosen uniformly at random from L_v and activating v with probability $\alpha = \frac{K}{\ln \Delta}$, where $K = (100k^{3k})^{-1}$.

The partial coloring of the hypergraph, set V_{i+1}, and the lists of colors for each uncolored vertex in the beginning of iteration $i+1$ are induced as follows. Let σ be the output state of the i-th iteration. The list of each vertex $v \in V_{i+1} \subseteq V_i$, $L_v(i+1)$, is induced from $L_v(i)$ by removing every non-available color $c \in L_v(i)$ for v in σ. We obtain the partial coloring ϕ for the hypergraph and set V_{i+1} for the beginning of iteration $i+1$ by removing the color from every vertex $v \in V_i$ which is either deactivated or is assigned a non-available for it color in σ.

Controlling the parameters of interest. Next we describe the recursive definitions for L_i and $T_{i,r}$ which, as we already explained, will determine the behavior of the parameters $\ell_i(v)$ and $t_{i,r}(v,c)$, respectively.

Initially, $L_1 = (1+\delta)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}}$, $T_{1,k-1} = \Delta$ and $T_{1,r} = 0$ for every $r \in [k-2]$. Letting

$$\mathrm{Keep}_i = \prod_{r=1}^{k-1} \left(1 - \left(\frac{\alpha}{L_i}\right)^r\right)^{T_{i,r}}, \tag{3}$$

we define

$$L_{i+1} = L_i \cdot \mathrm{Keep}_i - L_i^{2/3}, \tag{4}$$

$$\begin{aligned} T_{i+1,r} &= \sum_{j=r}^{k-1} \left(T_{i,j} \cdot \binom{j}{r} \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^r \left(\frac{\alpha \mathrm{Keep}_i}{L_i}\right)^{j-r} \right) \\ &\quad + 3k^r \alpha^{-r+1} L_i^r \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}} + \left(\sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_i^{j-r}} \right)^{2/3}. \end{aligned} \tag{5}$$

To get some intuition for the recursive definitions (4), (5), observe that Keep_i is the probability that a color $c \in L_v(i)$ is present in $L_v(i+1)$ as well. Note further that this implies that the expected value of $\ell_{i+1}(v,c)$ is $L_i \cdot \mathrm{Keep}_i$, a fact which motivates (4). Calculations of similar flavor for $\mathbb{E}[t_{i+1,r}(v,c)]$ motivate (5).

The key lemmas. We are almost ready to state the main lemmas that will guarantee that our procedure eventually reaches a partial list-coloring of H with favorable properties that will allow us to extend it to a full list-coloring. Before doing so, we need to settle a subtle issue that has to do with the fact that $t_{i+1,r}(v,c)$ is not sufficiently concentrated around its expectation. To see this, notice for example that $t_{i+1,1}(v,c)$ drops to zero if v is assigned c. (Similarly, for $r \in \{2,\dots,k-1\}$, if v is assigned c then $t_{i+1,r}(v,c)$ can be affected by a large amount.) To deal with this problem we will focus instead on variable $t'_{i+1,r}(v,c)$, i.e., the number of hyperedges h that contain v and (i) exactly $k-r-1$ vertices of $h \setminus \{v\}$ are colored c in the end of iteration i; (ii) the rest r vertices of $h \setminus \{v\}$ did not retain their color during iteration i and, further, c would be available for them if we ignored the color assigned to v. Observe that if c is not assigned to v then $t_{i+1,r}(v,c) = t'_{i+1,r}(v,c)$ and $t'_{i+1,r}(v,c) \geq t_{i+1,r}(v,c)$ otherwise.

The first lemma that we prove estimates the expected value of the parameters at the end of the i-th iteration. Its proof can be found in Section 4.

Lemma 3.1. *Let $S_i = \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}}$ and $Y_{i,r} = \sum_{j=r}^{k-1} \frac{T_{i,j}}{L_i^j}$. If $Q(i)$ holds and for all $1 < j < i, r \in [k-1], L_j \geq (\ln \Delta)^{20(k-1)}, T_{i,r} \geq (\ln \Delta)^{20(k-1)}$, then, for every vertex $v \in V_{i+1}$ and color $c \in L_v$:*

(a) $\mathbb{E}[\ell_{i+1}(v)] = \ell_i(v) \cdot \mathrm{Keep}_i$;

(b)

$$\begin{aligned} \mathbb{E}[t'_{i+1,r}(v,c)] \leq &\sum_{j=r}^{k-1} \left(T_{i,j}(v,c) \cdot \binom{j}{r} \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^r \left(\frac{\alpha \mathrm{Keep}_i}{L_i}\right)^{j-r} \right) \\ &+ 3k^r \alpha^{-r+1} L_i^r S_i + O(Y_i). \end{aligned}$$

The next step is to prove strong concentration around the mean for our random variables per the following lemma. Its proof can be found in Section 4.

Lemma 3.2. *If $Q(i)$ holds and $L_i, T_{i,r} \geq (\ln\Delta)^{20(k-1)}$, $r \in [k-1]$, then for every vertex $v \in V_{i+1}$ and color $c \in L_v$,*

(a) $\Pr\left[|\ell_{i+1}(v) - \mathbb{E}[\ell_{i+1}(v)]| < L_i^{2/3}\right] < \Delta^{-\ln\Delta}$*;*

(b) $\Pr\left[t'_{i+1,r}(v,c) - \mathbb{E}[t'_{i+1,r}(v,c)] > \frac{1}{2}\left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right] < \Delta^{-\ln\Delta}$.

Armed with Lemmas 3.1, 3.2, a straightforward application of the symmetric Local Lemma, i.e., Corollary 2.3, reveals the following.

Lemma 3.3. *With positive probability, $P(i)$ holds for every i such that for all $1 < j < i : L_j, T_{j,r} \geq (\ln\Delta)^{20(k-1)}$ and $T_{j,k-1} \geq \frac{1}{10k^2}L_j^{k-1}$.*

The proof of Lemma 3.3 can be found in Section 4.

In analyzing the recursive equations (4), (5), it would be helpful if we could ignore the "error terms". The next lemma shows that this is indeed possible. Its proof can be found in Section 4.

Lemma 3.4. *Define $L'_1 = (1+\delta)\left(\frac{\Delta}{\ln\Delta}\right)^{\frac{1}{k-1}}$, $T'_{1,k-1} = \Delta$, $T'_{1,r} = 0$ for $r \in [k-2]$, and recursively define*

$$
\begin{aligned}
L'_{i+1} &= L'_i \cdot \mathrm{Keep}_i, \\
T'_{i+1,r} &= \sum_{j=r}^{k-1}\left(T'_{i,j}\cdot\binom{j}{r}(\mathrm{Keep}_i\cdot(1-\alpha\mathrm{Keep}_i))^r\left(\frac{\alpha\mathrm{Keep}_i}{L'_i}\right)^{j-r}\right) \\
&\quad +3k^r\alpha^{-r+1}L_i^r\sum_{\ell=1}^{k-1}\frac{T_{i,\ell}}{L_i^{2\ell}(\ln\Delta)^{2\ell}}.
\end{aligned}
$$

If for all $1 < j < i$, $L_j \geq (\ln\Delta)^{20(k-1)}$, $T_{j,r} \geq (\ln\Delta)^{20(k-1)}$ for every $r \in [k-1]$, and $T_{j,k-1} \geq \frac{L_j^{k-1}}{10k^2}$, then

(a) $|L_i - L'_i| \leq (L'_i)^{\frac{5}{6}}$*;*

(b) $|T_{i,r} - T'_{i,r}| \leq (T'_{i,r})^{\frac{100r}{100r+1}}$.

Remark 3.2. *Note that* Keep_i *in Lemma 3.4 is still defined in terms of $L_i, T_{i,r}$ and not $L'_i, T'_{i,r}$. Note also that in the definition of $T'_{i+1,r}$, the second summand is a function of $T_{i,\ell}, L_i$, $\ell \in [r-1]$, and not $T'_{i,\ell}, L'_i$.*

Using Lemma 3.4 we are able to prove the following in Section 4.

Lemma 3.5. *There exists $i^* = O(\ln\Delta\ln\ln\Delta)$ such that*

(a) For all $1 < i \leq i^*, T_{i,r} > (\ln\Delta)^{20(k-1)}, L_i \geq \Delta^{\frac{1}{2(k-1)}}$, and $T_{i,k-1} \geq \frac{1}{10k^2}L_i^{k-1}$;

(b) $T_{i^*+1,r} \leq \frac{1}{10k^2}L_{i^*+1}^r$, for every $r \in [k-1]$ and $L_{i^*+1} \geq \Delta^{\frac{\epsilon/3}{(k-1)(1+\epsilon/2)}}$.

Lemmas 3.3, 3.5 and 3.6 imply Theorem 1.8.

Lemma 3.6. *Let σ be the state promised by Lemma 3.5. Given σ, we can find a full list-coloring of H in polynomial time in the number of vertices of H.*

Proof of Theorem 1.8. We carry out i^* iterations of our procedure. If $P(i)$ fails to hold for any iteration i, then we halt. By Lemmas 3.3 and 3.5, $P(i)$ (and, therefore, $Q(i)$) holds with positive probability for each iteration and so it is possible to perform i^* iterations. Further, the fact that our LLL application is within the scope of the so-called *variable setting* [36] implies that the deterministic version of the Moser-Tardos algorithm [36, 11] applies and, thus, we can perform i^* iterations in polynomial time.

After i^* iterations we can apply the algorithm of Lemma 3.6 and complete the list-coloring of the input hypergraph.

$\square$

3.1 Proof of Lemma 3.6

Let $\mathcal{U}_\sigma$ denote the set of uncolored vertices in σ, and $\mathcal{U}_\sigma(h)$ the subset of $\mathcal{U}_\sigma$ that belongs to a hyperedge h. Our goal is to color the vertices in $\mathcal{U}_\sigma$ to get a full list-coloring.

Towards that end, let $L_v = L_v(\sigma)$ denote the list of colors for v at σ, and $D_r(v,c) := D_{i^*+1,r}(v,c)$ the set of hyperedges (of size $t_{i^*+1,r}(v,c)$) with r uncolored vertices in σ whose vertices "compete" for c with v, and recall the conclusion of Lemma 3.5. Let μ be the probability distribution induced by giving each vertex $v \in \mathcal{U}_\sigma$ a color from L_v uniformly at random. For every hyperedge h and color $c \in \bigcap_{u\in h} L_u$ we define $A_{h,c}$ to be the event that all vertices of h are colored c. Let $\mathcal{A}$ be the family of these (bad) events, and observe that for every $A_{h,c} \in \mathcal{A}$:

$$\mu\left(A_{h,c}\right) \le \frac{1}{\prod_{v\in\mathcal{U}_\sigma(h)} |L_v(\sigma)|} < \frac{1}{4}$$

for large enough Δ, since $L_{i^*+1} = L_{i^*+1}(\Delta) \xrightarrow{\Delta\to\infty} \infty$.

Moreover, let $I(A_{h,c})$ denote the set of all bad events $A_{h',c'}$, where $h' \neq h$, such that either $\mathcal{U}_\sigma(h) \cap \mathcal{U}_\sigma(h') = \emptyset$, or c' is not in the list of colors of the (necessarily unique) uncolored vertex that h and h' share. Notice that conditioning on any the non-occurrence of any set $S \subseteq I(A_{h,c})$ does not increase the probability of $A_{h,c}$.

Let $D(A_{h,c}) := \mathcal{A} \setminus I(A_{h,c})$. Lemma 3.6 follows from Corollary 2.2 (and can be made constructive using the deterministic version of the Moser-Tardos algorithm [36, 11]) as, for every $A_{h,c} \in \mathcal{A}$:

$$\begin{aligned}
\sum_{A\in D(A_{h,c})} \mu(A) &\le \sum_{v\in\mathcal{U}_\sigma(h)} \sum_{c'\in L_v} \sum_{r=1}^{k-1} \sum_{h'\in D_r(v,c')} \mu\left(A_{h',c'}\right) \\
&= \sum_{v\in\mathcal{U}_\sigma(h)} \sum_{c'\in L_v} \sum_{i=1}^{k-1} \sum_{h'\in D_r(v,c')} \frac{1}{\prod_{u\in\mathcal{U}_\sigma(h')} |L_u|} \\
&\le \max_{v\in\mathcal{U}_\sigma(h)} \frac{k}{|L_v|} \sum_{c'\in L_v} \sum_{r=1}^{k-1} \frac{|D_r(v,c')|}{L_{i^*+1}^r} && (6)\\
&\le \frac{k}{10k^2} \max_{v\in\mathcal{U}_\sigma(h)} \frac{L_{i^*+1}^r \cdot |L_v|}{|L_v| \cdot L_{i^*+1}^r} && (7)\\
&\le \frac{1}{10} < \frac{1}{4}\ , && (8)
\end{aligned}$$

for large enough Δ, concluding the proof. Note that in (6) we used the facts that every hyperedge has at most k vertices and $L_{i^*+1} \ge \Delta^{\frac{\epsilon/3}{(k-1)(1+\epsilon/2)}}$, and in (7) we used the fact that $|D_r(v,c')| \le T_{i^*+1}^r \le \frac{1}{10k^2} L_{i^*+1}^r$.

4 Hypergraph list-coloring proofs

In this section we prove Lemmas 3.1, 3.2, 3.3, 3.4, 3.5.

We start by showing a couple of important lemmas that will be helpful for these proofs. It will be convenient to define $R_{i,r} = \frac{T_{i,r}}{L_i^r}$, $R'_{i,r} = \frac{T'_{i,r}}{(L'_i)^r}$ for every $r \in [k-1]$.

Lemma 4.1. *If for all* $1 < j < i, r \in [k-1], L_j, T_{j,r} \geq (\ln \Delta)^{20(k-1)}$*, then*

$$R_{i,r} \leq k^{2(k-1-r)} \ln \Delta.$$

Proof. We proceed by induction. The case $i = 1$ is straightforward to verify since $R_{1,r} = 0$ for every $r \in [k-2]$, while $R_{1,k-1} = \frac{\ln \Delta}{(1+\delta)^{k-1}}$. Therefore, we inductively assume the claim for i, and consider the case $i+1$. Note that the inductive hypothesis implies that $\mathrm{Keep}_i = \Omega(1)$ since $1 - \frac{1}{x} \geq \mathrm{e}^{-\frac{1}{x-1}}$ for every $x \geq 2$ and, thus,

$$\mathrm{Keep}_i \geq \exp\left(-\sum_{r=1}^{k-1} \frac{T_{i,r}}{(\alpha^{-1} L_i)^r - 1}\right) \geq \exp\left(-\frac{K k^{2(k-2)}}{1 - \frac{\delta}{100k}}\right), \tag{9}$$

for sufficiently large Δ.

$$\begin{aligned}
R_{i+1,r} &= \sum_{j=r}^{k-1} \left(\frac{T_{i,j}}{L_{i+1}^r} \cdot (\mathrm{Keep}_i (1 - \alpha \mathrm{Keep}_i))^r \binom{j}{r} \left(\frac{\alpha \mathrm{Keep}_i}{L_i}\right)^{j-r} \right) \\
&\quad + \frac{1}{L_{i+1}^r} \left(\sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,r}}{L_i^{j-r}} \right)^{2/3} + 3k^r \alpha^{-r+1} \left(\frac{L_i}{L_{i+1}}\right)^r \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}} \\
&= \sum_{j=r}^{k-1} \left(\frac{T_{i,j}}{L_i^r \left(\mathrm{Keep}_i - L_i^{-1/3}\right)^r} \cdot (\mathrm{Keep}_i (1 - \alpha \mathrm{Keep}_i))^r \binom{j}{r} \left(\frac{\alpha \mathrm{Keep}_i}{L_i}\right)^{j-r} \right) \\
&\quad + O\left((\ln \Delta)^{-\frac{20(k-1)r}{3} + \frac{2}{3}}\right) \\
&= \sum_{j=r}^{k-1} \left(R_{i,j} \cdot \frac{(1 - \alpha \mathrm{Keep}_i)^r}{\left(1 - \frac{L_i^{-1/3}}{\mathrm{Keep}_i}\right)^r} \binom{j}{r} (\alpha \mathrm{Keep}_i)^{j-r} \right) + O\left((\ln \Delta)^{-\frac{20(k-1)r}{3} + \frac{2}{3}}\right) &(10) \\
&\leq \sum_{j=r}^{k-1} \left(R_{i,j} \cdot \left(1 - \frac{\alpha \mathrm{Keep}_i}{2}\right)^r \binom{j}{r} (\alpha \mathrm{Keep}_i)^{j-r} \right) + O\left((\ln \Delta)^{-\frac{20(k-1)r}{3} + \frac{2}{3}}\right) &(11) \\
&\leq \left(1 - \frac{\alpha \mathrm{Keep}_i}{2}\right)^r \left(R_{i,r} + \sum_{j=r+1}^{k-1} \binom{j}{r} R_{i,j} \alpha^{j-r} \right) + O\left((\ln \Delta)^{-\frac{20(k-1)r}{3} + \frac{2}{3}}\right) \\
&\leq \left(1 - \frac{\alpha \mathrm{Keep}_i}{2}\right) \left(k^{2(k-1-r)} \ln \Delta + \sum_{j=r+1}^{k-1} \binom{j}{r} \frac{k^{2(k-1-j)} K^{j-r}}{(\ln \Delta)^{j-r-1}} \right) + O\left((\ln \Delta)^{-\frac{20(k-1)r}{3} + \frac{2}{3}}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \left(1-\frac{\alpha\mathrm{Keep}_i}{2}\right)\left(k^{2(k-1-r)}\ln\Delta + k^{2(k-1-(r+1))}(r+1)K\right) + O\left(\frac{1}{\ln\Delta}\right)\\
&\leq k^{2(k-1-r)}\ln\Delta - K\left(\frac{\mathrm{Keep}_i k^{2(k-1-r)}}{2} - k^{2(k-1-(r+1))}(r+1)\right) + O\left(\frac{1}{\ln\Delta}\right)\\
&\leq k^{2(k-1-r)}\ln\Delta, \qquad (12)
\end{aligned}$$

for sufficiently large Δ, concluding the proof. Note that in (11) we used the facts that $\mathrm{Keep}_i = \Omega(1)$, $L_i, T_{i,r} \geq (\ln\Delta)^{20(k-1)}$. In deriving (12) we used the fact that $K = (100k^{3k})^{-1}$ and, therefore, the second term is a negative constant, the inductive hypothesis, and that $L_i \geq (\ln\Delta)^{20(k-1)}$ and $\mathrm{Keep}_i = \Omega(1)$. ∎

A straightforward corollary of Lemma 4.1 is the following.

Corollary 4.2. *If $L_i, T_{i,r} \geq (\ln\Delta)^{20(k-1)}$ and $R_{i,k-1} \geq \frac{1}{10k^2}$, then*

$$C := \exp\left(-\frac{Kk^{2(k-2)}}{1-\frac{\delta}{100k}}\right) \leq \mathrm{Keep}_i \leq 1 - \frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}}.$$

Proof. The lower bound follows directly from (9). The upper bound follows from our assumption that $R_{i,k-1} \geq \frac{1}{10k^2}$ which implies that

$$\mathrm{Keep}_i \leq \mathrm{e}^{-\sum_{r=1}^{k-1}\alpha^r R_{i,r}} \leq \mathrm{e}^{-\alpha^{k-1}R_{i,k-1}} \leq \mathrm{e}^{-\frac{K^{k-1}}{10k^2(\ln\Delta)^{k-1}}} < 1 - \frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}},$$

for sufficiently large Δ. ∎

Lemma 4.3. *If $L_j, T_{j,r} \geq (\ln\Delta)^{20(k-1)}$ for all $1 < j \leq i$, then for every $r \in [k-1]$:*

$$R'_{i,r} \leq (1-\alpha C)^{r(i-1)}\ln\Delta \cdot \frac{(1+\frac{\delta}{k^{100}})^{k-1-r}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}C^{k-1-r}}\prod_{p=r}^{k-2}(p+1).$$

Proof of Lemma 4.3. We proceed by induction. The base cases are easy to verify since $R'_{1,r} = 0$ for every $r \in [k-2]$ and $R'_{1,k-1} = \frac{\ln\Delta}{(1+\delta)^{k-1}}$.

We first focus on the case $r = k-1$. We assume that the claim is true for $i-1$ and consider i. Note that the inductive hypothesis, the fact that $L_j \geq (\ln\Delta)^{20(k-1)}$ and $\mathrm{Keep}_j = \Omega(1)$, imply that

$$\frac{3k^r\alpha^{-r+1}L_{i-1}^r}{L_i^r}\sum_{\ell=1}^{k-1}\frac{T_{i-1,\ell}}{(L_{i-1})^{2\ell}(\ln\Delta)^{2\ell}} \leq \frac{1}{(\ln\Delta)^{10(k-1)}}, \qquad (13)$$

for sufficiently large Δ, for every $r \in [k-1]$ and $j < i$. Therefore,

$$\begin{aligned}
R'_{i,k-1} &\leq R'_{i-1,k-1}(1-\alpha\mathrm{Keep}_{i-1})^{k-1} + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&\leq R'_{i-2,k-1}(1-\alpha\mathrm{Keep}_{i-2})^{k-1}(1-\alpha\mathrm{Keep}_{i-1})^{k-1} + \frac{(1-\alpha\mathrm{Keep}_{i-1})^{k-1}}{(\ln\Delta)^{10(k-1)}} + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&\leq R'_{i-2,k-1}(1-\alpha C)^{2(k-1)} + \frac{(1-\alpha C)^{k-1}}{(\ln\Delta)^{10(k-1)}} + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&\leq \dots \\
&\leq (1-\alpha C)^{(i-1)(k-1)}R_{1,k-1} + \frac{1}{(\ln\Delta)^{10(k-1)}}\sum_{\ell=0}^{i-1}(1-\alpha C)^{(k-1)\ell} \\
&\leq (1-\alpha C)^{(i-1)(k-1)}\frac{\ln\Delta}{(1+\delta)^{k-1}} + \frac{1}{(\ln\Delta)^{5(k-1)}} \\
&\leq (1-\alpha C)^{(i-1)(k-1)}\frac{\ln\Delta}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}},
\end{aligned}$$

for sufficiently large Δ, concluding the proof for the case $r = k-1$.

We now focus on $r \in [k-2]$. As the first step, we observe that

$$\begin{aligned}
R'_{2,r} &\leq \sum_{j=r}^{k-1}\left(\frac{T'_{1,j}}{(L'_1)^r}\cdot\left(\mathrm{Keep}_1\left(1-\alpha\mathrm{Keep}_1\right)\right)^r\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_1}{L'_1}\right)^{j-r}\right) + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&= R'_{1,k-1}\cdot\left(\mathrm{Keep}_1\left(1-\alpha\mathrm{Keep}_1\right)\right)^r\binom{k-1}{r}(\alpha\mathrm{Keep}_1)^{k-1-r} + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&\leq \frac{(\ln\Delta)^{-(k-2-r)}}{(1+\delta)^{k-1}}K^{k-1-r}\binom{k-1}{r} + \frac{1}{(\ln\Delta)^{10(k-1)}} \qquad (14) \\
&\leq \frac{(1+\frac{\delta}{k^{100}})^{k-1-r}(\ln\Delta)^{-(k-2-r)}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}}K^{k-1-r}\prod_{p=r}^{k-2}(p+1),
\end{aligned}$$

concluding the proof of the base cases.

Assume that the claim holds for all pairs (r', i'), where $r' \in \{r, \dots, k-1\}$ and $i' \leq i-1$. It suffices to prove that it also holds for the pair (r, i), where $i > 2$ and $r \in [k-2]$. To see this, observe that

$$\begin{aligned}
R'_{i,r} &\leq \sum_{j=r}^{k-1}\left(\frac{T'_{i-1,j}}{(L'_i)^r}\cdot\left(\mathrm{Keep}_{i-1}\left(1-\alpha\mathrm{Keep}_{i-1}\right)\right)^r\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_{i-1}}{L'_{i-1}}\right)^{j-r}\right) + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&= \sum_{j=r}^{k-1}\left(\frac{T'_{i-1,j}}{(L'_{i-1})^j}\cdot\mathrm{Keep}_{i-1}^{j-r}\left(1-\alpha\mathrm{Keep}_{i-1}\right)^r\binom{j}{r}\alpha^{j-r}\right) + \frac{1}{(\ln\Delta)^{10(k-1)}} \\
&= \left(1-\alpha\mathrm{Keep}_{i-1}\right)^r\sum_{j=r}^{k-1}\left(R'_{i-1,j}\binom{j}{r}(\alpha\mathrm{Keep}_{i-1})^{j-r}\right) + \frac{1}{(\ln\Delta)^{10(k-1)}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad (1-\alpha C)^r \sum_{j=r}^{k-1} \left(R'_{i-1,j} \binom{j}{r} \alpha^{j-r} \right) + \frac{1}{(\ln \Delta)^{10(k-1)}} \\
&\leq \quad (1-\alpha C)^r \, R'_{i-1,r} \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} (1-\alpha C)^{j(i-2)+r} \frac{(1+\frac{\delta}{k^{100}})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1) \\
&\qquad\qquad + \frac{1}{(\ln \Delta)^{10(k-1)}} \\
&\leq \quad (1-\alpha C)^{2r} \, R'_{i-2,r} \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} (1-\alpha C)^{j(i-2)+r} \frac{(1+\frac{\delta}{k^{100}})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1) \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} (1-\alpha C)^{j(i-3)+2r} \frac{(1+\frac{\delta}{k^{100}})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1) \\
&\quad + \frac{1+(1-\alpha C)^r}{(\ln \Delta)^{10(k-1)}} \qquad (15) \\
&\leq \quad (1-\alpha C)^{3r} \, R'_{i-3,r} \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} (1-\alpha C)^{j(i-2)+r} \frac{(1+\frac{\delta}{k^{100}})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1) \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} (1-\alpha C)^{j(i-3)+2r} \frac{(1+\frac{\delta}{k^{100}})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1) \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} (1-\alpha C)^{j(i-4)+3r} \frac{(1+\frac{\delta}{k^{100}})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1) \\
&\quad + \frac{1+(1-\alpha C)^r+(1-\alpha C)^{2r}}{(\ln \Delta)^{10(k-1)}} \qquad (16) \\
&\leq \quad \ldots \\
&\leq \quad (1-\alpha C)^{(i-1)r} \, R'_{2,r} \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} \prod_{p=j}^{k-2} (p+1) \frac{(1+\delta/k^{100})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1} C^{k-1-j}} \sum_{\ell=1}^{i-2} (1-\alpha C)^{j(i-\ell-1)+\ell r} \\
&\quad + \frac{\sum_{\ell=0}^{i-3} (1-\alpha C)^{r(\ell-1)}}{(\ln \Delta)^{10(k-1)}} \qquad (17) \\
&\leq \quad (1-\alpha C)^{(i-1)r} \frac{(\ln \Delta)^{-(k-2-r)}}{(1+\delta)^{k-1}} K^{k-1-r} \binom{k-1}{r} + O\left(\frac{1}{(\ln \Delta)^{5(k-1)}} \right) \\
&\quad + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r} (\ln \Delta)^{1-(j-r)} \prod_{p=j}^{k-2} (p+1) \frac{(1+\delta/k^{100})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}} \sum_{\ell=1}^{i-2} (1-\alpha C)^{j(i-\ell-1)+\ell r} \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{K^{k-1-r}(\ln\Delta)^{-(k-2-r)}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}}(1-\alpha C)^{(i-1)r}\binom{k-1}{r}+\\
&\quad+\sum_{j=r+1}^{k-1}\binom{j}{r}K^{j-r}(\ln\Delta)^{1-(j-r)}\prod_{p=j}^{k-2}(p+1)\frac{(1+\delta/k^{100})^{k-1-j}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}C^{k-1-j}}\sum_{\ell=1}^{i-2}(1-\alpha C)^{(i-1)r+(i-\ell-1)(j-r)}\\
&\le \frac{(1-\alpha C)^{(i-1)r}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}}\left(\sum_{j=r+1}^{k-1}\binom{j}{r}K^{j-r}(\ln\Delta)^{1-(j-r)}\prod_{p=j}^{k-2}(p+1)\frac{(1+\delta/k^{100})^{k-1-j}}{C^{k-1-j}}\sum_{\ell\ge 0}(1-\alpha C)^{\ell(j-r)}\right)\\
&= \frac{(1-\alpha C)^{(i-1)r}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}}\left(\sum_{j=r+1}^{k-1}\binom{j}{r}K^{j-r}(\ln\Delta)^{1-(j-r)}\frac{\prod_{p=j}^{k-2}(p+1)}{C^{k-1-j}}\frac{(1+\delta/k^{100})^{k-1-j}}{1-(1-\alpha C)^{j-r}}\right)\\
&\le (1-\alpha C)^{r(i-1)}\ln\Delta\cdot\frac{(1+\frac{\delta}{k^{100}})^{k-1-r}}{(1+\delta-\frac{\delta}{k^{99}})^{k-1}C^{k-1-r}}\prod_{p=r}^{k-2}(p+1), \qquad (19)
\end{aligned}
$$

for sufficiently large Δ, concluding the proof. Note that in order to get (15) we upper bound $R'_{i-1,r}$ in the same way we upper bounded $R'_{i,r}$. We keep using the same idea to bound $R'_{i-2,r}, R'_{i-3,r}, \ldots$ until we get (17), and we obtain (18) by using (14).

□

We are now ready to prove Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5.

4.1 Proof of Lemma 3.1

Proof of part (a). For every color $c \in L_v(i)$,

$$
\Pr[c\in L_v(i+1)] = \prod_{r=1}^{k-1}\prod_{h\in D_{i,r}(v,c)}\left(1-\prod_{u\in(h\setminus\{v\})\cap V_i}\frac{\alpha}{\ell_i(u)}\right) = \prod_{r=1}^{k-1}\left(1-\left(\frac{\alpha}{L_i}\right)^r\right)^{T_{i,r}} = \text{Keep}_i, \quad (20)
$$

where for the second equality we used our assumption that $Q(i)$ holds. Therefore, the proof of the first part of the lemma follows from linearity of expectation.

□

Proof of part (b) . Recall the definition of $t'_{i+1,r}(v,c)$ and note that only hyperedges in $\bigcup_{j=r}^{k-1} D_{i,j}(v,c)$ can be potentially counted by $t'_{i+1,r}(v,c)$. In particular, unless every uncolored vertex of an edge $h \in D_{i,j}(v,c)$, $j \ge r$, is assigned the same color with v in iteration i, then if h is counted by $t'_{i+1,r}(c)$, it is also counted by $t_{i+1,r}(c)$. Therefore,

$$
\mathbb{E}[t'_{i+1,r}(v,c)] \le \mathbb{E}[t_{i+1,r}(v,c)] + O\left(\sum_{j=r}^{k-1}\frac{T_{i,j}}{L_i^j}\right), \quad (21)
$$

and so we will focus on bounding $\mathbb{E}[t_{i+1,r}(v,c)]$.

Fix $h \in D_{i,j}(v,c)$, where $j \ge r$. Our goal will be to show that

$$
\begin{aligned}
\Pr[h\in D_{i+1,r}(v,c)] \le &\binom{j}{j-r}(\text{Keep}_i(1-\alpha\text{Keep}_i))^r\left(\frac{\alpha\text{Keep}_i}{L_i}\right)^{j-r}\\
&+4r\binom{j}{r}\frac{\text{Keep}_i^{j-1}\alpha^{j-r+1}S_i}{L_i^{j-r}}+O\left(\frac{1}{L_i^j}\right), \qquad (22)
\end{aligned}
$$

since combining (22) with (21) implies the lemma. To see this, observe that

$$\begin{aligned} T_{i,j} \cdot 4r\binom{j}{r}\frac{\mathrm{Keep}_i^{j-1}\alpha^{j-r+1}S_i}{L_i^{j-r}} &\leq 4\alpha r\binom{j}{r}\cdot\alpha^j\frac{T_{i,j}}{L_i^j}\mathrm{Keep}_i^{j-1}\cdot(\alpha^{-1}L_i)^r S_i \\ &\leq \frac{4\alpha\binom{j}{r}}{\mathrm{e}(j-1)}(\alpha^{-1}L_i)^r S_i, \end{aligned} \tag{23}$$

and, therefore,

$$\begin{aligned} \mathbb{E}[t_{i+1,r}(v,c)] &\leq \sum_{j=r}^{k-1} T_{i,j}\max_{h\in D_{i,j}(v,c)}\Pr[h\in D_{i+1,r}(v,c)] \\ &\leq \sum_{j=r}^{k-1}\left(T_{i,j}\cdot\binom{j}{r}(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i))^r\left(\frac{\alpha\mathrm{Keep}_i}{L_i}\right)^{j-r}\right) \\ &\quad +3k^r\alpha(\alpha^{-1}L_i)^r S_i + O\left(\sum_{j=r}^{k-1}\frac{T_{i,j}}{L_i^j}\right). \end{aligned}$$

Note that in (23) we first used that $1-x\leq \mathrm{e}^{-x}$ for every $x\geq 0$ (to bound Keep_i), and then that $\max_x x\mathrm{e}^{-\ell x}\leq\frac{1}{\ell\mathrm{e}}$ for every ℓ.

Towards proving (22), for any vertex $u\in h\setminus\{v\}$, consider the events

$$\begin{aligned} E_{u,1} &= \text{“}u \text{ does not retain its color and } c\in L_u(i+1)\text{”}, \\ E_{u,2} &= \text{“}u \text{ is assigned } c \text{ and retains its color”}. \end{aligned}$$

Let also B_c be the event that v and $j-1$ other uncolored vertices of h receive color c. Since we have assumed that our hypergraph is of girth at least 5, for any neighbor u of v and $j\in\{1,2\}$, event $E_{u,j}$ is mutually independent of all events $E_{u',\ell},\ell\in\{1,2\},u\neq u'$, conditional on B_c not occurring. Thus, if $\Pr[E_{u,\ell}\mid\overline{B_c}]\leq p_\ell$, $\ell\in\{1,2\}$, for every neighboring vertex u of v, we obtain

$$\Pr[h\in D_{i+1,r}(v,c)]\leq\binom{j}{r}p_1^r p_2^{j-r}+\frac{2^k}{L_i^j}, \tag{24}$$

since $\Pr[B_c]\leq 2^k L_i^{-j}$.

Assume now that we condition on an arbitrary assignment of colors to the uncolored vertices of h so that B_c does not hold. Then, for any neighboring vertex u of v, and sufficiently large Δ,

$$\Pr[E_{u,2}\mid\overline{B_c}]\leq\frac{\alpha\mathrm{Keep}_i}{L_i}+\frac{2}{(L_i\ln\Delta)^j}=: q_2+\delta_2, \tag{25}$$

since the probability (conditional on $\overline{B_c}$) that u is activated is α, it is assigned c with probability less than L_i^{-1}; and it retains c with probability at most

$$\prod_{r\in[k-1]\setminus\{j\}}\left(1-\frac{\alpha^r}{L_i^r}\right)^{T_{i,r}}\cdot\left(1-\frac{\alpha^j}{L_i^j}\right)^{T_{i,j}-1}\leq\mathrm{Keep}_i+\frac{2}{(L_i\ln\Delta)^j},$$

for sufficiently large Δ.

We now claim that

$$\Pr[E_{u,1} \mid \overline{B_c}] \le \mathrm{Keep}_i(1-\alpha \mathrm{Keep}_i) + \big(2\alpha S_i + (L_i \ln \Delta)^{-j}(3+4\alpha S_i)\big) =: q_1 + \delta_1. \tag{26}$$

To see this, at first observe that if u is not activated, then u will not be assigned a color, and the probability that $c \in L_u(i+1)$ conditional on $\overline{B_c}$ is at most $\mathrm{Keep}_i + 2(L_i \ln \Delta)^{-j}$. If u is activated and is assigned c, then the probability that $c \in L_u(i+1)$ and u does not retain c is zero.

Finally, suppose that u is activated and is assigned $\gamma \ne c$. For each $\gamma \in L_u(i) \setminus \{c\}$ we compute the probability that $\gamma \notin L_u(i+1)$ conditional on that u was activated and assigned γ, B_c did not occur and $c \in L_u(i+1)$.

For any $\ell \in [k-1]$ and any $g \in D_{i,\ell}(u,\gamma)$, we consider the probability that every vertex in $(g \setminus \{u\}) \cap V_i$ is activated and assigned γ, conditional on that u was activated and assigned γ, $c \in L_u(i+1)$ and $\overline{B_c}$. Since the color activations and color assignments are independent over different vertices, this equals the probability that every vertex in $(g \setminus \{u\}) \cap V_i$ is activated and assigned γ, conditional on the event A_g that not every vertex in $(g \setminus \{u\}) \cap V_i$ is activated and assigned c. The latter probability equals to

$$\begin{aligned}
&\frac{\Pr[(\text{every vertex in } g \cap V_i \setminus \{u\} \text{ is activated and assigned } \gamma) \wedge A_g]}{\Pr[A_g]} \\
=&\frac{\Pr[(\text{every vertex in } g \cap V_i \setminus \{u\} \text{ is activated and assigned } \gamma)]}{\Pr[A_g]} \\
\le&\frac{\alpha^\ell L_i^{-\ell}}{1-\alpha^\ell L_i^{-\ell}} \le \left(\frac{\alpha}{L_i}\right)^\ell + \frac{1}{L_i^{2\ell}(\ln \Delta)^{2\ell}},
\end{aligned}$$

for sufficiently large Δ, since $K < 1$. Therefore, the probability of $\gamma \notin L_u(i+1)$ given that u was activated and assigned γ, $c \in L_u(i+1)$ and $\overline{B_c}$ is at most

$$\begin{aligned}
&1 - \prod_{\ell=1}^{k-1} \prod_{g \in D_{i,\ell}(u,\gamma)} (1 - \Pr[\forall w \in (g \setminus \{u\}) \cap V_i \colon w \text{ is activated and assigned } \gamma \mid A_g])^{T_{i,\ell}(u,\gamma)} \\
\le& 1 - \prod_{\ell=1}^{k-1} \left(1 - \left(\frac{\alpha}{L_i}\right)^\ell - \frac{1}{L_i^{2\ell}(\ln \Delta)^{2\ell}}\right)^{T_{i,\ell}(u,\gamma)} \\
\le& 1 - \mathrm{Keep}_i + 2\sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}},
\end{aligned}$$

for sufficiently large Δ.

Overall, we have shown that $\Pr[E_{u,1} \mid \overline{B_c}]$ is at most

$$\begin{aligned}
&(1-\alpha)\mathrm{Keep}_i + 2(L_i \ln \Delta)^{-j} + \alpha \frac{L_i - 1}{L_i}(\mathrm{Keep}_i + 2(L_i \ln \Delta)^{-j}) \left(1 - \mathrm{Keep}_i + 2\sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}}\right), \\
\le& \mathrm{Keep}_i(1-\alpha \mathrm{Keep}_i) + \big(2\alpha S_i + (L_i \ln \Delta)^{-j}(3+4\alpha S_i)\big),
\end{aligned}$$

where recall that $S_i = \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}}$.

Combining (24), (25) and (26) we obtain

$$\begin{aligned}
\Pr[h \in D_{i+1,r}(v,c)] &\le \binom{j}{r} q_1^r q_2^{j-r} \left(1+\delta_1 q_1^{-1}\right)^r \left(1+\delta_2 q_2^{-1}\right)^{j-r} + \frac{2^k}{L_i^j} \\
&\le \binom{j}{r} q_1^r q_2^{j-r} \left(1+2r\delta_1 q_1^{-1}\right)\left(1+2(j-r)\delta_2 q_2^{-1}\right) + \frac{2^k}{L_i^j} \\
&\le \binom{j}{r} q_1^r q_2^{j-r} + 2r\binom{j}{r} q_1^{r-1} q_2^{j-r}\delta_1 + O\left(\frac{1}{L_i^j}\right) \\
&\le \binom{j}{r} q_1^r q_2^{j-r} + 4r\binom{j}{r}\frac{\mathrm{Keep}_i^{j-1}\alpha^{j-r+1}S_i}{L_i^{j-r}} + O\left(\frac{1}{L_i^j}\right),
\end{aligned}$$

concluding the proof. Note that in the second inequality above we used that Keep_i is bounded below by a constant according to the bound in Corollary 4.2 (which only requires the assumptions of Lemma 4.1). □

4.2 Proof of Lemma 3.2

Let $\mathrm{Bin}(n,p)$ denote the binomial random variable that counts the number of successes in n Bernoulli trials, where each trial succeeds with probability p. We will find the following lemma useful (see, e.g., Exercise 2.12 in [35]) :

Lemma 4.4. *For any c, k, n we have*

$$\Pr\left[\mathrm{Bin}\left(n, \frac{c}{n}\right) \ge k\right] \le \frac{c^k}{k!}.$$

Proof of Part (a). We will use Theorem 2.4 to show that that the number of colors, $\overline{\ell_v}$, which are removed from L_v during iteration i is highly concentrated.

Note that changing the assignment to any neighboring vertex of v can change $\overline{\ell_v}$ by at most 1, and changing the assignment to any other vertex cannot affect $\overline{\ell_v}$ at all.

If $\overline{\ell_v} \ge s$, there are at most s groups of at most $k-1$ neighbors of v, so that each vertex in each group received the same color, and each group corresponds to a different color from L_v. Thus, the color assignments and activation choices of these vertices certify that $\overline{\ell_v} \ge s$.

Since, according to Corollary 4.2, $\mathrm{Keep}_i = \Omega(1)$, applying Theorem 2.4 with $t = L_i^{\frac{1.9}{3}}$, $w = 2k$, $c = 1$, we obtain

$$\Pr\left[|\overline{\ell_v} - \mathbb{E}[\overline{\ell_v}]| > L_i^{2/3}\right] < \Delta^{-\ln\Delta},$$

for sufficiently large Δ.

Finally, $\mathbb{E}[\ell_{i+1}(v)] = \ell_i(v) - \mathbb{E}[\overline{\ell_v}]$ implies that

$$\Pr\left[|\ell_{i+1}(v) - \mathbb{E}[\ell_{i+1}(v)]| > L_i^{2/3}\right] = \Pr\left[|\overline{\ell_v} - \mathbb{E}[\overline{\ell_v}]| > L_i^{2/3}\right] < \Delta^{-\ln\Delta}.$$

□

Proof of Part (b). Recall the definition of $D_{i,r}(v,c)$ and let $Z_{i,r}(v,c) = \bigcup_{j=r}^{k-1} D_{i,j}(v,c)$. Let $X_{i+1,r}(v,c)$ denote the number of hyperedges in $Z_{i,r}(v,c)$ which (i) contain exactly r uncolored vertices other than v; and (ii) the rest of their vertices are assigned c in the end of the i-th iteration. Let also $Y_{i+1,r}(v,c)$ be the number of these hyperedges which they contain an uncolored vertex $u \ne v$ such that (i) $c \notin L_u(i+1)$; and (ii) c would still not be in $L_u(i+1)$ even if we ignored the color of v.

It is straightforward to verify that $t'_{i+1,r}(v,c) = X_{i+1,r}(v,c) - Y_{i+1,r}(v,c)$. Therefore, by the linearity of expectation, it suffices to show that $X_{i+1,r}(v,c)$ and $Y_{i+1,r}(v,c)$ are both sufficiently concentrated. This is because

$$\begin{aligned}
&\Pr\left[t'_{i+1,r}(v,c) - \mathbb{E}[t'_{i+1}(v,c)] > \frac{1}{2}\left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right], \\
=&\Pr\left[X_{i+1,r}(v,c) - \mathbb{E}[X_{i+1}(v,c)] - (Y_{i+1,r}(v,c) - \mathbb{E}[Y_{i+1,r}(v,c)]) > \frac{1}{2}\left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right],
\end{aligned}$$

and, therefore, it is sufficient to prove that

$$\Pr\left[X_{i+1,r}(v,c) - \mathbb{E}[X_{i+1,r}(v,c)] > \frac{1}{4}\left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right] \leq \frac{1}{2}\Delta^{-\ln\Delta}, \tag{27}$$

$$\Pr\left[Y_{i+1,r}(v,c) - \mathbb{E}[Y_{i+1,r}(v,c)] < -\frac{1}{4}\left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right] \leq \frac{1}{2}\Delta^{-\ln\Delta}. \tag{28}$$

We first focus on $X_{i+1,r}(v,c)$. Let $X'_{i+1,r}(v,c)$ denote the number of hyperedges in $Z_{i,r}(v,c)$ which (i) contain exactly r uncolored vertices other than v; and (ii) the rest of their vertices were activated and assigned c (but did not retain their color necessarily). Further, let $W^1_{i+1,r}(v,c)$ denote the random variable that counts all the hyperedges counted by $X'_{i+1,r}(v,c)$, except for those whose r uncolored vertices (other than v) were uncolored because they were activated and received the same color as v. Finally, let $W^2_{i+1,r}(v,c)$ be the number of hyperedges which (i) contain exactly r vertices that are activated and received the same color with v; and (ii) the rest of their $k-1-r$ vertices were activated and assigned c.

Observe that $X_{i+1,r}(v,c) \leq W^1_{i+1,r}(v,c) + W^2_{i+1,r}(v,c)$. The idea here is that we cannot directly apply Talagrand's inequality to $X_{i+1,r}(v,c)$ and so we consider $W^1_{i+1,r}(v,c), W^2_{i+1,r}(v,c)$ instead.

First, we consider $W^1_{i+1,r}(v,c)$. Since our hypergraph is of girth at least 5, changing a choice for some vertex of a hyperedge $h \in Z_{i,r}(v,c)$ can only affect whether or not the vertices of h remain uncolored, and thus affect $W^1_{i+1,r}(v,c)$ by at most 1. Furthermore, changing a choice for a vertex outside the ones that correspond to the hyperedges in $Z_{i,r}(v,c)$ can affect at most one vertex of at most one hyperedge in $Z_{i,r}(v,c)$ and, therefore, can affect $W^1_{i+1,r}(v,c)$ by at most 1.

We claim now that if $W^1_{i+1,r}(v,c) \geq s$, then there exist at most $2k^2 s$ random choices that certify this event. To see this, notice that if a hyperedge h is counted by $W^1_{i+1,r}(v,c)$, then for every $u \in h \setminus \{v\}$ that remained uncolored, it must be that either u was deactivated, or u is contained in a hyperedge $h' \neq h$ such that all the vertices in $(h' \setminus \{u\}) \cap V_i$ were activated and received the same color as u. Moreover, the event that a variable $u \in h \setminus \{v\}$ was activated and received c can be verified by the outcome of two random choices. So, overall, we can certify that h was counted by $W^1_{i+1,r}(v,c)$ by using the outcome of at most $2k^2$ random choices.

Finally, observe that $\mathbb{E}[W^1_{i+1,r}(v,c)] \leq \sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}$ and, thus, applying Theorem 2.4 with $c=1$,

$w = 2k^2$ and $t = \left(\sum_{j=r}^{k-1} \binom{j}{r} \frac{T_{i,r}}{L_i^{j-r}}\right)^{1.9/3}$ we obtain

$$\Pr\left[\left|W_{i+1,r}^1(v,c) - \mathbb{E}[W_{i+1,r}^1(v,c)]\right| > \frac{1}{8}\left(\sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right] \le \frac{1}{4}\Delta^{-\ln\Delta}, \tag{29}$$

for sufficiently large Δ.

As far as $W_{i+1,r}^2(v,c)$ is concerned, note that it can be bounded above by $\sum_{j=r}^{k-1} \mathrm{Bin}(T_{i,j}, \binom{j}{r} \frac{\alpha^j}{L_i^j})$ and observe that, according to Lemma 4.1, each of these binomial random variables has constant expectation. Since $T_{i,j} \ge (\ln\Delta)^{20(k-1)}$ for every $j \in [k-1]$, Lemma 4.4 implies that

$$\Pr\left[\left|W_{i+1,r}^2(v,c) - \mathbb{E}[W_{i+1,r}^2(v,c)]\right| > \frac{1}{8}\left(\sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right] \le \frac{1}{4}\Delta^{-\ln\Delta}. \tag{30}$$

Combining (29) and (30) implies (27).

We follow the same approach for $Y_{i+1,r}(v,c)$. Let $Y'_{i+1,r}(v,c)$ be the number of hyperedges counted by $X'_{i+1,r}(v,c)$ and which also contain an uncolored vertex $u \neq v$ such that (i) $c \notin L_u(i+1)$; and (ii) c would still not be in $L_u(i+1)$ even if we ignored the color of v. Further, let $Y''_{i+1,r}(v,c)$ be the random variable that counts all the hyperedges counted by $Y'_{i+1,r}(v,c)$, except for those whose r uncolored vertices were uncolored because they were activated and received the same color as v, and observe that $Y_{i+1,r}(v,c) \le Y'_{i+1,r}(v,c) \le Y''_{i+1,r}(v,c) + W_{i+1,r}^2(v,c)$.

Moreover, if $Y''_{i+1,r}(v,c) \ge s$, then there exist at most $(2k^2+2k)s$ random choices that certify this event. To see this, observe that for each hyperedge h counted by $Y_{i+1,r}(v,c)$, we need the output of at most $2k^2$ choices to certify that it is counted by $X'_{i+1,r}(v,c)$, and the output of at most $2k$ extra random choices to certify that there is a vertex $u \in h \setminus \{v\}$ for which $c \notin L_u(i+1)$, and c would still not be in $L_u(i+1)$ even if we ignored the color of v.

Finally, $\mathbb{E}[Y''_{i+1,r}(v,c)] \le \sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_i^{j-r}}$ and, therefore, an almost identical argument to the case for $X_{i+1,r}(v,c)$ implies (28).

□

4.3 Proof of Lemma 3.3

We will use induction on i. Property $P(1)$ clearly holds, so we assume that property $P(i)$ holds and we prove that with property $P(i+1)$ holds with positive probability. Recall our discussion in the previous section in which we argued that we can assume without loss of generality that property $Q(i)$ holds.

For every v and $c \in L_v$ let A_v be the event that $\ell_{i+1}(v) < L_{i+1}$ and $B_{v,c}^r$ to be the event that $t_{i+1,r}(v,c) > T_{i+1,r}$. Clearly, if we can avoid these bad events with positive probability, then $P(i+1)$ holds.

Recall that, according to Lemma 4.1, $R_{i,r} = O(\ln\Delta)$ for every $r \in [k-1]$. Recall further that for any vertex v and color c such that at the beginning of iteration $i+1$, v is uncolored and $c \in L_v$, we have $t_{i+1,r}(v,c) = t'_{i+1,r}(v,c)$ for every $r \in [k-1]$. Therefore by Lemma 3.1, if $B_{v,c}^r$ holds then $t'_{i+1,r}(v,c) - \mathbb{E}[t'_{i+1,r}(v,c)] > \frac{1}{2}\sum_{j=r}^{k-1}\left(\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_{i,j}}\right)^{2/3}$.

By Lemma 3.2, the probability of any of our bad events is at most $\Delta^{-\ln\Delta}$. Furthermore, each bad event $f_v \in \{A_v, B_{v,c}^r\}$ event is determined by the colors assigned to vertices of distance at most 3 from v. Therefore, f_v is mutually independent of all but at most $(k\Delta)^4(1+\delta)\left(\frac{\Delta}{\ln\Delta}\right)^{\frac{1}{k-1}} < \Delta^5$ other bad events. For Δ sufficiently large, $\Delta^{-\ln\Delta}\Delta^5 < \frac{1}{4}$ and so the proof is concluded by applying Corollary 2.3.

4.4 Proof of Lemma 3.4

Since $L_i < L'_i$, for the first part of the lemma it suffices to prove that $L'_i \leq L_i + (L'_i)^{5/6}$. Towards that end, at first we observe that for sufficiently large Δ, Corollary 4.2 and the fact that $K = \frac{1}{100k^{3k}}$ imply:

$$
\begin{aligned}
\mathrm{Keep}_i^{5/6} - \mathrm{Keep}_i &\geq \left(1 - \frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}}\right)^{5/6} - \left(1 - \frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}}\right) \\
&\geq \left(1 - \frac{5}{6}\cdot\frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}}\right) - \left(1 - \frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}}\right) = \frac{K^{k-1}}{72k^2(\ln\Delta)^{k-1}}. \quad (31)
\end{aligned}
$$

Note that in deriving the first inequality we used the fact that the function $x^{5/6} - x$ is decreasing on the interval $[C, 1]$ since K is sufficiently small. For the second one, we used the Taylor Series for $(1-y)^{5/6}$ around $y = 0$.

We now proceed by using induction. The base case is trivial, so assume that the statement is true for i, and consider $i+1$. Since, by our assumption, $L_i \geq (\ln\Delta)^{20(k-1)}$ we obtain

$$
\begin{aligned}
L'_{i+1} &= \mathrm{Keep}_i L'_i \\
&\leq \mathrm{Keep}_i \left(L_i + (L'_i)^{5/6}\right) \quad (32) \\
&= L_{i+1} + L_i^{2/3} + \mathrm{Keep}_i (L'_i)^{5/6} \\
&\leq L_{i+1} + L_i^{2/3} + \mathrm{Keep}_i^{5/6}(L'_i)^{5/6} - \frac{K^{k-1}}{72k^2(\ln\Delta)^{k-1}}(L'_i)^{5/6} \quad (33) \\
&\leq L_{i+1} + (L'_{i+1})^{5/6} + L_i^{2/3} - \frac{K^{k-1}}{72k^2(\ln\Delta)^{k-1}}(L'_i)^{5/6} \\
&< L_{i+1} + (L'_{i+1})^{5/6}, \quad (34)
\end{aligned}
$$

for sufficiently large Δ. Note that in deriving (32) we used the inductive hypothesis; for (33) we used (31); and for (34) the fact that $L_i \geq (\ln\Delta)^{20(k-1)}$ and the inductive hypothesis.

The proof of the second part of the lemma is similar. That is, we observe that it suffices to show that $T'_{i,r} \geq T_{i,r} - (T'_{i,r})^{\frac{100r}{100r+1}}$ and proceed by using induction. Again, the base case is trivial, so we assume the statement is true for i, and consider $i+1$. We obtain the following.

$$
\begin{aligned}
T'_{i+1,r} &= \sum_{j=r}^{k-1}\left(T'_{i,j}\cdot(\mathrm{Keep}_i(1-\alpha\mathrm{Keep}_i))^r \binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L'_i}\right)^{j-r}\right) + 3k^r\alpha^{-r+1}L_i^r\sum_{\ell=1}^{k-1}\frac{T_{i,\ell}}{L_i^{2\ell}(\ln\Delta)^{2\ell}} \\
&\geq \sum_{j=r}^{k-1}\left(\left(T_{i,j} - (T'_{i,j})^{\frac{100r}{100r+1}}\right)(\mathrm{Keep}_i(1-\alpha\mathrm{Keep}_i))^r\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L'_i}\right)^{j-r}\right) \\
&\qquad + 3k^r\alpha^{-r+1}L_i^r\sum_{\ell=1}^{k-1}\frac{T_{i,\ell}}{L_i^{2\ell}(\ln\Delta)^{2\ell}} \quad (35)
\end{aligned}
$$

$$
\begin{aligned}
&= T_{i+1,r} - \left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3} \\
&\quad - \sum_{j=r}^{k-1} T_{i,j}\left(\mathrm{Keep}_i(1-\alpha\mathrm{Keep}_i)\right)^r(\alpha\mathrm{Keep}_i)^{j-r}\left(\frac{1}{L_i^{j-r}} - \frac{1}{(L_i')^{j-r}}\right) \\
&\quad - \sum_{j=r}^{k-1}(T'_{i,j})^{\frac{100r}{100r+1}}\left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^r\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L_i'}\right)^{j-r} \\
&\geq T_{i+1,r} - \left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3} \\
&\quad - \sum_{j=r+1}^{k-1}\left(\mathrm{Keep}_i(1-\alpha\mathrm{Keep}_i)\right)^r(\alpha\mathrm{Keep}_i)^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\,O\left(L_i^{-\frac{1}{6}}\right) \\
&\quad - \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{r/6}(T'_{i+1,r})^{\frac{100r}{100r+1}} \qquad (36) \\
&\geq T_{i+1,r} - \left(1 - \frac{K^{k-1}}{200k^2(\ln\Delta)^{k-1}}\right)(T'_{i+1,r})^{\frac{100r}{100r+1}} - O\left(L_i^{r-\frac{1}{6}} + L_i^{2r/3} + T_{i,r}^{2/3}\right) \qquad (37) \\
&\geq T_{i+1,r} - (T'_{i+1,r})^{\frac{100r}{100r+1}}, \qquad (38)
\end{aligned}
$$

for sufficiently large Δ, concluding the proof of the lemma. Note that in deriving (36) we used the first part of Lemma 3.4, and that

$$
\begin{aligned}
&\sum_{j=r}^{k-1}(T'_{i,j})^{\frac{100r}{100r+1}}\left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^r\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L_i'}\right)^{j-r} \\
&= \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{\frac{r}{100r+1}}\sum_{j=r}^{k-1}(T'_{i,j})^{\frac{100r}{100r+1}}\left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{\frac{100r^2}{100r+1}}\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L_i'}\right)^{j-r} \\
&\leq \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{\frac{r}{100r+1}}\sum_{j=r}^{k-1}(T'_{i,j})^{\frac{100r}{100r+1}}\left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{\frac{100r^2}{100r+1}}\left(\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L_i'}\right)^{j-r}\right)^{\frac{100r}{100r+1}} \\
&\leq \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{\frac{r}{100r+1}}\left(\sum_{j=r}^{k-1}(T'_{i,j})\left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^r\binom{j}{r}\left(\frac{\alpha\mathrm{Keep}_i}{L_i'}\right)^{j-r}\right)^{\frac{100r}{100r+1}} \\
&= \left(\mathrm{Keep}_i\,(1-\alpha\mathrm{Keep}_i)\right)^{\frac{r}{100r+1}}(T'_{i,r+1})^{\frac{100r}{100r+1}},
\end{aligned}
$$

for sufficiently large Δ. In deriving (37) we used Lemma 4.1 and the fact that, using Corollary 4.2, we obtain

$$
\begin{aligned}
\left(\mathrm{Keep}_i(1-\alpha\mathrm{Keep}_i)\right)^{\frac{r}{100r+1}} \leq \left(1 - \frac{K^{k-1}}{12k^2(\ln\Delta)^{k-1}}\right)^{\frac{r}{100r+1}} &\leq \left(1 - \frac{K^{k-1}}{200k^2(\ln\Delta)^{k-1}}\right) \\
&= \left(1 - \frac{K^{k-1}}{200k^2(\ln\Delta)^{k-1}}\right).
\end{aligned}
$$

Finally, to derive (38) at first we observe that $R_{i,\ell} = \Omega\left(\frac{1}{(\ln\Delta)^{k-1-r}}\right)$, by definition (recall e.g., (10)) and our assumption that $R_{i,k-1} = \Omega(1)$, and, thus, $L_i = O\left(T_{i,r}^{\frac{1}{r}}(\ln\Delta)^{\frac{k-1-r}{r}}\right)$. Then, we use our assumption that $T_{i,\ell} \geq (\ln \Delta)^{20(k-1)}$ for every $\ell \in [k-1]$ and the inductive hypothesis to conclude that

$$\frac{(T'_{i+1,r})^{\frac{100r}{100r+1}}}{200k^2(\ln \Delta)^{k-1}} = \omega\left(L_i^{r-\frac{1}{6}} + L_i^{2r/3} + T_{i,r}^{2/3}\right).$$

4.5 Proof of Lemma 3.5

We proceed by induction. Let $\eta := \frac{\epsilon/3}{(k-1)(1+\epsilon/2)}$. We will assume that $L_j \geq \Delta^{\eta}, T_{j,r} \geq (\ln \Delta)^{20(k-1)}$ for all $2 \leq j \leq i < i^*$, and prove that $L_{i+1} \geq \Delta^{\eta}, T_{i+1,r} \geq (\ln \Delta)^{20(k-1)}$. Towards that end, it will be useful to focus on the family of ratios $R_{i,r}$, $r \in [k-1]$. Note that, according to Lemma 3.4, this family is well-approximated by the family $R'_{i,r}$, $r \in [k-1]$. In particular, recalling Lemma 4.3 and applying Lemma 3.4 we obtain:

$$\begin{aligned} R_{i,r} &\leq R'_{i,r} \cdot \frac{1 + (T'_{i,r})^{-\frac{1}{100r+1}}}{\left(1 - (L'_i)^{-1/6}\right)^r} \\ &\leq (1-\alpha C)^{r(i-1)} \ln \Delta \cdot \frac{\prod_{p=r}^{k-2}(p+1)}{(1+\delta - \frac{1.1\delta}{k^{99}})^{k-1} C^{k-1-r}}, \end{aligned} \tag{39}$$

for sufficiently large Δ, since $L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)}$.

Using (39) and the fact that $1 - \frac{1}{x} > \mathrm{e}^{-\frac{1}{x-1}}$ for $x \geq 2$ we can get an improved lower bound for Keep_i as follows.

$$\begin{aligned} \mathrm{Keep}_i &\geq \exp\left(-\frac{1}{(1-\frac{\delta}{k^{100k}})} \sum_{r=1}^{k-1} \alpha^r R_{i,r}\right) \\ &\geq \exp\left(-\frac{1}{\left(1+\delta-\frac{1.2\delta}{k^{99}}\right)^{k-1}} \sum_{r=1}^{k-1} (1-\alpha C)^{r(i-1)} \frac{K^r \prod_{p=r}^{k-2}(p+1)}{(\ln \Delta)^{r-1} C^{k-1-r}}\right), \end{aligned} \tag{40}$$

for sufficiently large Δ.

Using (40) we get

$$\begin{aligned} \prod_{j=1}^{i-1} \mathrm{Keep}_j &\geq \exp\left(-\frac{1}{\left(1+\delta-\frac{1.2\delta}{k^{99}}\right)^{k-1}} \sum_{r=1}^{k-1} \left(\frac{K^r \prod_{p=r}^{k-2}(p+1)}{(\ln \Delta)^{r-1} C^{k-1-r}} \sum_{j=1}^{i-1} (1-\alpha C)^{r(j-1)}\right)\right) \\ &\geq \exp\left(-\frac{1}{\left(1+\delta-\frac{1.2\delta}{k^{99}}\right)^{k-1}} \sum_{r=1}^{k-1} \left(\frac{K^r \prod_{p=r}^{k-2}(p+1)}{(\ln \Delta)^{r-1} C^{k-1-r}} \cdot \frac{1}{1-(1-\alpha C)^r}\right)\right) \\ &\geq \exp\left(-\frac{C^{-(k-2)}(1+\frac{\delta}{k^{100}})(k-1)! \ln \Delta}{\left(1+\delta-\frac{1.2\delta}{k^{99}}\right)^{k-1}}\right) \\ &\geq \exp\left(-\frac{\ln \Delta}{(1+\frac{\epsilon}{2})(k-1)}\right) \end{aligned} \tag{41}$$

for sufficiently large Δ.

Using (41) we can now bound L'_i as follows.

$$L'_i = L'_1 \prod_{j=1}^{i-1} \mathrm{Keep}_j \geq (1+\delta)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}} \Delta^{-\frac{1}{(1+\frac{\epsilon}{2})(k-1)}} \geq \Delta^{\eta}, \tag{42}$$

for sufficiently large Δ. Thus, L_i' never gets too small for the purposes of our analysis. Lemma 3.4 implies that neither does L_i.

The proof is concluded by observing that (39) implies that $R_{i,r}$, $r \in [k-1]$, becomes smaller than $\frac{1}{10k^2}$ for $i = O(\ln \Delta \ln \ln \Delta)$.

5 A sufficient pseudo-random property for coloring

In this section we present the proof of Theorem 1.7. To do so, we build on ideas of Alon, Krivelevich and Sudakov [8] and show that the random hypergraph $H(k, n, d/\binom{n}{k-1})$ almost surely admits a few useful features.

The first lemma we prove states that all subgraphs of $H(k, n, d/\binom{n}{k-1})$ with not too many vertices are sparse and, therefore, of small degeneracy.

Lemma 5.1. *For every constant $k \geq 2$, there exists $d_k > 0$ such that for any constant $d \geq d_k$, the random hypergraph $H(k, n, d/\binom{n}{k-1})$ has the following property almost surely: Every $s \leq nd^{-\frac{1}{k-1}}$ vertices of H span fewer than $\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}} s$ hyperedges. Therefore, any subhypergraph of H induced by a subset $V_0 \subset V$ of size $|V_0| \leq nd^{-\frac{1}{k-1}}$, is $k\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$-degenerate.*

Proof. Letting $r = \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$, we see that the probability that there exists a subset $V_0 \subset V$ which violates the statement of the lemma is at most

$$\begin{aligned}
\sum_{i=r^{\frac{1}{k-1}}}^{nd^{-\frac{1}{k-1}}} \binom{n}{i} \binom{\binom{i}{k}}{ri} \left(\frac{d}{\binom{n}{k-1}}\right)^{ri} &\leq \sum_{i=r^{\frac{1}{k-1}}}^{nd^{-\frac{1}{k-1}}} \left[\frac{\mathrm{e}n}{i}\left(\frac{\mathrm{e}i^{k-1}}{r}\right)^r \left(\frac{d}{\binom{n}{k-1}}\right)^r\right]^i \qquad (43)\\
&\leq \sum_{i=r^{\frac{1}{k-1}}}^{nd^{-\frac{1}{k-1}}} \left[\mathrm{e}^{1+\frac{1}{k-1}}(k-1)\left(\frac{d}{r}\right)^{\frac{1}{k-1}} \left(\frac{\mathrm{e}i^{k-1}d}{r\binom{n}{k-1}}\right)^{r-\frac{1}{k-1}}\right]^i = o(1),
\end{aligned}$$

for sufficiently large d. Note that in the lefthand side of (43) we used the fact that any subset of vertices of size $s < r^{\frac{1}{k-1}}$ cannot violate the assertion of the lemma, since it can span at most $s^k < rs$ hyperedges. In deriving the final inequality we used that for any pair of integers α, β, we have that $\binom{\alpha}{\beta} \geq \left(\frac{\alpha}{\beta}\right)^{\beta}$.

∎

Next we show that, as far as the number of vertices of $H(k, n, d/\binom{n}{k-1})$ that have a constant degree c is concerned, the degree of each vertex of H is essentially a Poisson random variable with mean d.

Lemma 5.2. *For constants $c \geq 1$, $k \geq 2$ and d, let X_c denote the number of vertices of degree c in $H(k, n, d/\binom{n}{k-1})$. Then, for $c = O(1)$, with high probability,*

$$X_c \leq \frac{d^c \mathrm{e}^{-d}}{c!} n \left(1 + O\left(\frac{\log n}{\sqrt{n}}\right)\right).$$

Proof. The lemma follows from standard ideas for estimation of the degree distribution of random graphs (see for example the proof of Theorem 3.3 in [19] for the case $k = 2$). In particular, assume that the vertices of $H(k,n,d/\binom{n}{k-1})$ are labeled $1,2,\dots,n$. Then,

$$\begin{aligned}
\mathbb{E}[X_c] &= n\Pr[\deg(1)=c] \\
&= n\binom{\binom{n-1}{k-1}}{c}\left(\frac{d}{\binom{n}{k-1}}\right)^c\left(1-\frac{d}{\binom{n}{k-1}}\right)^{\binom{n-1}{k-1}-c} \\
&\le n\frac{\left(\binom{n-1}{k-1}\right)^c}{c!}\left(1+O\left(\frac{c^2}{\binom{n-1}{k-1}}\right)\right)\left(\frac{d}{\binom{n}{k-1}}\right)^c\exp\left(-\left(\binom{n-1}{k-1}-c\right)\frac{d}{\binom{n}{k-1}}\right) \\
&\le n\frac{d^c\mathrm{e}^{-d}}{c!}\left(1+O\left(\frac{1}{n^{k-1}}\right)\right).
\end{aligned}$$

To show concentration of X_c around its expectation, we will use Chebyshev's inequality. In order to do so, we need to estimate $\Pr[\deg(1)=\deg(2)=c]$. For $\ell\in\{0,\dots,c\}$, let $E^{\ell}_{1,2}$ denote the event that there exist exactly ℓ hyperedges that contain both vertices 1 and 2. Then, letting $p=\frac{d}{\binom{n}{k-1}}$, we see that

$$\begin{aligned}
\Pr[\deg(1)=\deg(2)=c] &\le \sum_{\ell=0}^{c}\Pr\left[E^{\ell}_{1,2}\right]\left(\binom{\binom{n-1}{k-1}}{c-\ell}p^c(1-p)^{\binom{n-1}{k-1}-c}\right)^2 \\
&= \sum_{\ell=0}^{c}\binom{\binom{n-2}{k-2}}{\ell}p^{\ell}\,(1-p)^{\binom{n-2}{k-2}-\ell}\left(\binom{\binom{n-1}{k-1}}{c-\ell}p^c(1-p)^{\binom{n-1}{k-1}-c}\right)^2 \\
&= \Pr[\deg(1)=c]\cdot\Pr[\deg(2)=c]\left(1+O\left(\frac{1}{n^{k-1}}\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}[X_c] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Pr[\deg(i)=c,\deg(j)=c]-\Pr[\deg(1)=c]\Pr[\deg(2)=c]\right) \\
&\le \sum_{i\neq j=1}O\left(\frac{1}{n^{k-1}}\right)+\mathbb{E}[X_c]=An\ ,
\end{aligned}$$

for some constant $A=A(c,d)$.

Finally, applying the Chebyshev's inequality, we obtain that, for any $t>0$,

$$\Pr\left[|X_c-\mathbb{E}[X_c]|\ge t\sqrt{n}\right]\le\frac{A}{t^2},$$

and, thus, the proof is concluded by choosing $t=\log n$.

□

Lemma 5.2 implies the following useful corollary.

Corollary 5.3. *For any constants $\delta\in(0,1),k\ge 2,d>0$, let $X=X(\delta,k,d)$ denote the random variable equal to the number of vertices in $H(k,n,\ d/\binom{n}{k-1})$ whose degree is in $[(1+\delta)d,3(k-1)^{k-1}d]$. There exists a constant $d_\delta>0$ such that if $d\ge d_\delta$ then, almost surely, $X\le\frac{n}{d^2}$.*

Proof. Let X_r denote the number of vertices of degree r in $H(k,n,d/\binom{n}{k-1})$. Since k,d are constants, using Lemma 5.2 and Stirling's approximation we see that, almost surely,

$$\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d}X_r\le n\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)\right)\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d}\frac{d^r\mathrm{e}^{-d}}{r!}\le n(1+\delta)\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d}\frac{d^r\mathrm{e}^{-d}}{\sqrt{2\pi r}\left(\frac{r}{\mathrm{e}}\right)^r}\le\frac{n}{d^2},$$

for sufficiently large d and n. □

Using Lemma 5.1 and Corollary 5.2 we show that, almost surely, only a small fraction of vertices of $H(k, n, d/\binom{n}{k-1})$ have degree that significantly exceeds its average degree.

Lemma 5.4. *For every constants $k \geq 2$ and $\delta \in (0,1)$, there exists $d_{k,\delta} > 0$ such that for any constant $d \geq d_{k,\delta}$, all but at most $\frac{2n}{d^2}$ vertices of the random hypergraph $H(k, n, d/\binom{n}{k-1})$ have degree at most $(1+\delta)d$, almost surely.*

Proof. Corollary 5.3 implies that the number of vertices with degree in the interval $[(1+\delta)d, 3(k-1)^{k-1}d]$ is at most $\frac{n}{d^2}$, for sufficiently large d.

Suppose now there are more than $\frac{n}{d^2}$ vertices with degree at least $3(k-1)^{k-1}d$. Denote by S a set containing exactly $\frac{n}{d^2}$ such vertices. According to Lemma 5.1, almost surely, the induced subhypergraph $H[S]$ has at most

$$e(H[S]) \leq \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}} |S| = \frac{n}{d^{2-\frac{1}{k-1}}(\ln d)^{\frac{2}{k-1}}}$$

hyperedges. Therefore, the number of hyperedges between the sets of vertices S and $V \setminus S$ is at least

$$3(k-1)^{k-1}d|S| - ke(H[S]) \geq \frac{2.9(k-1)^{k-1}n}{d} =: N.$$

However, the probability that $H(k, n, d/\binom{n}{k-1})$ contains such a subhypergraph is at most

$$\binom{n}{\frac{n}{d^2}}\binom{\frac{n^k}{d^2}}{N}\left(\frac{d}{\binom{n}{k-1}}\right)^N \leq \left(\mathrm{e}d^2\right)^{\frac{n}{d^2}}\left(\frac{n^k\mathrm{e}}{d^2N}\cdot\frac{d}{\binom{n}{k-1}}\right)^N = o(1),$$

for sufficiently large d. Note that in deriving the final equality we used that for any pair of integers α, β, we have that $\binom{\alpha}{\beta} \geq \left(\frac{\alpha}{\beta}\right)^{\beta}$. Therefore, almost surely there are at most $\frac{n}{d^2}$ vertices in G with degree greater than $3(k-1)^{k-1}d$, concluding the proof. $\square$

Finally, we show that the neighborhood of a typical vertex of $H(k, n, d/\binom{n}{k-1})$ is locally tree-like.

Lemma 5.5. *For every constants $k \geq 2, \delta \in (0,1)$, almost surely, the random hypergraph $H(k, n, d/\binom{n}{k-1})$ has a subset $U \subseteq V(H)$ of size at most $n^{1-\delta}$ such that the induced hypergraph $H[V \setminus U]$ is of girth at least 5.*

Proof. Let Y_2, Y_3, Y_4, denote the number of 2-, 3- and 4-cycles in $H(n, k, d/\binom{n}{k-1})$, respectively. A straightforward calculation reveals that for $i \in \{2, 3, 4\}$:

$$\mathbb{E}[Y_i] \leq \sum_{s=1}^{i(k-1)} \binom{n}{s}\binom{\binom{s}{k-1}}{i}\left(\frac{d}{\binom{n}{k-1}}\right)^i \leq i(k-1)n^{i(k-1)}\left(\frac{(i(k-1))^{k-1}\mathrm{e}^2(k-1)^{k-1}}{in^{k-1}}\right)^i = O(1).$$

By Markov's inequality this implies that $Y_2 + Y_3 + Y_4 \leq n^{1-\sqrt{\delta}}$ almost surely. Denote by U the union of all 2-, 3- and 4- cycles in H. Then the induced subhypergraph $H[V \setminus U]$ has girth at least 5 and, almost surely, $|U| \leq n^{1-\delta}$.

$\square$

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Our goal will be to find a subset $U \subset V$ of size $|U| \leq nd^{-\frac{1}{k-1}}$ such that the induced subgraph $H[V \setminus U]$ is of girth at least 5 and maximum degree at most $(1+\delta)d$ and, further, every vertex v in $V \setminus U$ has at most $9k^2 \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}} = o\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$ neighbors in U. Note that in this case, according to Lemma 5.1, $H[U]$ is $k\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$-degenerate, concluding the proof assuming d is sufficiently large. A similar idea has been used in [7, 8, 31].

Towards that end, let U_1 be the set of vertices of degree more than $(1+\delta)d$, and U_2 the set of vertices that are contained in a 2-,3- or a 4-cycle. Notice that U_1, U_2, can be found in polynomial time and, according to Lemmas 5.4 and 5.5, the size of $U_0 := |U_1 \cup U_2|$ is at most $\frac{3n}{d^2}$ for sufficiently large n and d.

We now start with $U := U_0$ and as long as there exists a vertex $v \in V \setminus U$ having at least $9k^2\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$ neighbors in U we do the following. Let $S_v = \{u_1, u_2, \ldots, u_N\}$ be the neighbors of v in U. We choose an arbitrary hyperedge h that contains v and u_1 and update U and S_v by defining $U := U \cup h$ and $S_v := S_v \setminus h$. We keep repeating this operation until S_v is empty.

This process terminates with $|U| < nd^{-\frac{1}{k-1}}$ because, otherwise, we would get a subset $U \subset V$ of size $|U| = nd^{-\frac{1}{k-1}}$ spanning more than

$$\frac{1}{k}\left(\frac{n}{d^{\frac{1}{k-1}}} - |U_0|\right) \times 9k^2\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}} \times \frac{1}{k} > \frac{n}{d^{\frac{1}{k-1}}} \times \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$$

hyperedges, for sufficiently large d. According to Lemma 5.1 however, H does not contain any such set almost surely.

□

6 Acknowledgements

The author is grateful to Dimitris Achlioptas and Irit Dinur for detailed comments and feedback.

References

[1] Dimitris Achlioptas and Amin Coja-Oghlan. Algorithmic barriers from phase transitions. In *49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA*, pages 793–802. IEEE Computer Society, 2008.

[2] Dimitris Achlioptas, Fotis Iliopoulos, and Alistair Sinclair. Beyond the lovász local lemma: Point to set correlations and their algorithmic applications. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 725–744. IEEE Computer Society, 2019.

[3] Dimitris Achlioptas and Michael Molloy. The analysis of a list-coloring algorithm on a random graph. In *Proceedings 38th Annual Symposium on Foundations of Computer Science*, pages 204–212. IEEE, 1997.

[4] Dimitris Achlioptas and Assaf Naor. The two possible values of the chromatic number of a random graph. In László Babai, editor, *Proceedings of the 36th Annual ACM Symposium on Theory of Computing, Chicago, IL, USA, June 13-16, 2004*, pages 587–593. ACM, 2004.

[5] Dimitris Achlioptas, Assaf Naor, and Yuval Peres. Rigorous location of phase transitions in hard optimization problems. *Nature*, 435(7043):759–764, 2005.

[6] Miklós Ajtai, János Komlós, Janos Pintz, Joel Spencer, and Endre Szemerédi. Extremal uncrowded hypergraphs. *Journal of Combinatorial Theory, Series A*, 32(3):321–335, 1982.

[7] Noga Alon and Michael Krivelevich. The concentration of the chromatic number of random graphs. *Combinatorica*, 17(3):303–313, 1997.

[8] Noga Alon, Michael Krivelevich, and Benny Sudakov. List coloring of random and pseudo-random graphs. *Combinatorica*, 19(4):453–472, 1999.

[9] Peter Ayre, Amin Coja-Oghlan, and Catherine Greenhill. Hypergraph coloring up to condensation. *Random Structures & Algorithms*, 54(4):615–652, 2019.

[10] Tom Bohman, Alan Frieze, and Dhruv Mubayi. Coloring h-free hypergraphs. *Random Structures & Algorithms*, 36(1):11–25, 2010.

[11] Karthekeyan Chandrasekaran, Navin Goyal, and Bernhard Haeupler. Deterministic algorithms for the Lovász local lemma. *SIAM J. Comput.*, 42(6):2132–2155, 2013.

[12] Amin Coja-Oghlan. Belief propagation guided decimation fails on random formulas. *Journal of the ACM (JACM)*, 63(6):1–55, 2017.

[13] Amin Coja-Oghlan, Amir Haqshenas, and Samuel Hetterich. Walksat stalls well below satisfiability. *SIAM Journal on Discrete Mathematics*, 31(2):1160–1173, 2017.

[14] Amin Coja-Oghlan and Dan Vilenchik. Chasing the k-colorability threshold. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 380–389. IEEE, 2013.

[15] Ewan Davies, Ross J Kang, François Pirot, and Jean-Sébastien Sereni. An algorithmic framework for coloring locally sparse graphs. *arXiv preprint arXiv:2004.07151*, 2020.

[16] Jian Ding, Allan Sly, and Nike Sun. Proof of the satisfiability conjecture for large k. In *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*, pages 59–68, 2015.

[17] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[18] Paul Erdös and Joel Spencer. Lopsided Lovász local lemma and latin transversals. *Discrete Applied Mathematics*, 30(2-3):151–154, 1991.

[19] Alan Frieze and Michał Karoński. *Introduction to random graphs*. Cambridge University Press, 2016.

[20] Alan Frieze and Dhruv Mubayi. Coloring simple hypergraphs. *Journal of Combinatorial Theory, Series B*, 103(6):767–794, 2013.

[21] Marylou Gabrié, Varsha Dani, Guilhem Semerjian, and Lenka Zdeborová. Phase transitions in the q-coloring of random hypergraphs. *Journal of Physics A: Mathematical and Theoretical*, 50(50):505002, 2017.

[22] A. Johansson. Asympotic choice number for triangle free graphs., 1996.

[23] A. Johansson. The choice number of sparse graphs. *Unpublished manuscript*, 1996.

[24] Jeff Kahn. Asymptotics of the chromatic index for multigraphs. *Journal of Combinatorial Theory, Series B*, 68(2):233–254, 1996.

[25] Jeff Kahn. Asymptotics of the list-chromatic index for multigraphs. *Random Structures & Algorithms*, 17(2):117–156, 2000.

[26] Jeong Han Kim. On Brooks' theorem for sparse graphs. *Combinatorics, Probability and Computing*, 4(2):97–132, 1995.

[27] János Komlós, János Pintz, and Endre Szemerédi. A lower bound for heilbronn's problem. *Journal of the London Mathematical Society*, 2(1):13–24, 1982.

[28] Alexandr Kostochka, Dhruv Mubayi, Vojtěch Rödl, and Prasad Tetali. On the chromatic number of set systems. *Random Structures & Algorithms*, 19(2):87–98, 2001.

[29] Florent Krzakała, Andrea Montanari, Federico Ricci-Tersenghi, Guilhem Semerjian, and Lenka Zdeborová. Gibbs states and the set of solutions of random constraint satisfaction problems. *Proceedings of the National Academy of Sciences*, 104(25):10318–10323, 2007.

[30] Hanno Lefmann. Sparse parity-check matrices over GF (q). *Combinatorics, Probability and Computing*, 14(1-2):147–169, 2005.

[31] Tomasz Łuczak. The chromatic number of random graphs. *Combinatorica*, 11(1):45–54, 1991.

[32] Marc Mezard, Marc Mezard, and Andrea Montanari. *Information, physics, and computation*. Oxford University Press, 2009.

[33] Michael Molloy. The list chromatic number of graphs with small clique number. *Journal of Combinatorial Theory, Series B*, 134:264–284, 2019.

[34] Michael Molloy and Bruce Reed. A bound on the total chromatic number. *Combinatorica*, 18(2):241–280, 1998.

[35] Michael Molloy and Bruce Reed. *Graph colouring and the probabilistic method*, volume 23 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2002.

[36] Robin A. Moser and Gábor Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2):Art. 11, 15, 2010.

[37] Michel Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques*, 81(1):73–205, 1995.

[38] Van H Vu. A general upper bound on the list chromatic number of locally sparse graphs. *Combinatorics, Probability and Computing*, 11(1):103–111, 2002.

[39] Lenka Zdeborová and Florent Krzakala. Phase transitions in the coloring of random graphs. *Physical Review E*, 76(3):031131, 2007.