# MAXIMAL DEGREES IN SUBGRAPHS OF KNESER GRAPHS 

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#### Abstract

In this paper, we study the maximum degree in non-empty induced subgraphs of the Kneser graph $K G(n, k)$. One of the main results asserts that, for $k>k_{0}$ and $n>64 k^{2}$, whenever a non-empty subgraph has $m \geq k\binom{n-2}{k-2}$ vertices, its maximum degree is at least $\frac{1}{2}\left(1-\frac{k^{2}}{n}\right) m-\binom{n-2}{k-2} \geq$ 0.49 m . This bound is essentially best possible. One of the intermediate steps is to obtain structural results on non-empty subgraphs with small maximum degree.


## 1. Introduction

Let $n \geq 2 k>1$ be positive integers and let $[n]$ denote the standard $n$-element set $\{1, \ldots, n\}$. Set $\binom{[n]}{k}:=\{F \subset[n]:|F|=k\}$. Let $K G(n, k)$ denote the famous Kneser graph. Its vertex set is $\binom{[n]}{k}$, and two vertices $F, G \in\binom{[n]}{k}$ form an edge iff $F \cap G=\emptyset$.

The interest in Kneser graphs goes back to 1955 when Kneser 9 formulated the conjecture that the chromatic number $\chi(K G(n, k))$ equals $n-2 k+2$. This conjecture was settled in an influential paper of Lovász [10] some twenty years later.

In the meantime, Erdős, Ko and Rado [4] determined the independence number $\alpha(K G(n, k))$. Note that a family $\mathcal{F} \subset\binom{[n]}{k}$ is an independent set in $K G(n, k)$ iff $F \cap G \neq \emptyset$ for all $F, G \in\binom{[n]}{k}$. In extremal set theory such an $\mathcal{F}$ is called intersecting.
Theorem 1 ([4). For $n \geq 2 k>0$,

$$
\begin{equation*}
\alpha(K G(n, k))=\binom{n-1}{k-1} . \tag{1}
\end{equation*}
$$

Equality in (11) is attained for the star $\mathcal{S}_{x}:=\left\{S \in\binom{[n]}{k}: x \in S\right\}$. Hilton and Milner [6] proved that for $n>2 k$ no other independent set (intersecting family) attains the size $\binom{n-1}{k-1}$. More specifically, they showed that if no element is contained in all sets of an intersecting family, then it is at most as large as $\mathcal{H}(x, F)$, where for $x \notin F \in\binom{[n]}{k}$ we define

$$
\mathcal{H}(x, F):=\{F\} \cup\left\{A \in\binom{[n]}{k}: x \in A, A \cap F \neq \emptyset\right\} .
$$

Definition 2. For a family $\mathcal{F} \subset\binom{[n]}{k}$, let $K G(\mathcal{F})$ denote the induced subgraph of $K G(n, k)$ on the vertex set $\mathcal{F}$. Let $e(\mathcal{F})(d(\mathcal{F}))$ denote the number of edges (maximum degree) of $K G(\mathcal{F})$.

In view of (1), for $|\mathcal{F}|>\binom{n-1}{k-1}$ both $e(\mathcal{F})$ and $d(\mathcal{F})$ are positive. Defining $\mathcal{S}^{+}:=\mathcal{S}_{x} \cup\{T\}$, where $x \notin T \in\binom{[n]}{k}$, one easily verifies

$$
e\left(\mathcal{S}^{+}\right)=d\left(\mathcal{S}^{+}\right)=\binom{n-k-1}{k-1} .
$$

Katona, Katona, and Katona [8] proved

$$
\begin{equation*}
e(\mathcal{F}) \geq\binom{ n-k-1}{k-1} \tag{2}
\end{equation*}
$$

for all $\mathcal{F} \subset\binom{[n]}{k}$ with $|\mathcal{F}|=\binom{n-1}{k-1}+1$. This result was extended to the case $|\mathcal{F}| \leq\binom{ n-1}{k-1}+$ $\frac{n-2 k}{n}\binom{n-k-1}{k-1}$ by Balogh et. al. [2] (see Theorem 4 below).

Very recently, Friedgut (personal communication) raised the problem of determining the minimum of $d(\mathcal{F})$ for $|\mathcal{F}|=\binom{n-1}{k-1}+1$. One of the motivations for this question is a recent breakthrough result of Huang [7], who showed that in any subset of the hypercube $\{0,1\}^{n}$ of size $2^{n-1}+1$ the maximum degree of a vertex (in the standard hypercube graph) is at least $\sqrt{n}$. This settled an old problem from Theoretical Computer Science, known as the Sensitivity Conjecture.

Getting back to Kneser graphs, let us state this problem in a more general form.
Problem 1. Define

$$
d(m, n, k):=\min \left\{d(\mathcal{F}): \mathcal{F} \subset\binom{[n]}{k},|\mathcal{F}|=m \text { and } \mathcal{F} \text { is not intersecting }\right\} .
$$

Determine or estimate $d(m, n, k)$.
We should note that $K G(2 k, k)$ is a perfect matching. Thus $d(m, 2 k, k)=1$ identically for $2 \leq m \leq\binom{ 2 k}{k}$.

Example. Consider the family $\mathcal{D}:=\mathcal{H}(x, F) \cup\left\{F^{\prime}\right\}$, where $F^{\prime}$ contains $x$ and is disjoint from $F$. Obviously, $|\mathcal{D}|=\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+2, e(\mathcal{D})=d(\mathcal{D})=1$. This example shows that the problem of estimating $d(m, n, k)$ is only interesting for $m>\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+2$. Note that, for $k \geq 4$, we have $|\mathcal{D}| \geq k\binom{n-k}{k-2}$.
Definition 3. The lexicographic order $A<_{L} B$ is defined for distinct $A, B \in\binom{[n]}{k}$ by $A<_{L} B$ iff $\min \{x: x \in A \backslash B\}<\min \{y: y \in B \backslash A\}$. let $\mathcal{L}(m):=\mathcal{L}(m,[n], k)$ denote the family of the first $m$ members in $\binom{[n]}{k}$ in the lexicographic order. Note that if $1 \leq \ell \leq n-k$ is an integer then $\mathcal{L}\left(\binom{n}{k}-\binom{n-\ell}{k}\right)=\left\{A \in\binom{[n]}{k}: A \cap[\ell] \neq \emptyset\right\}$.

For a wide range of the values of the parameters Das, Gan, and Sudakov [3] proved that $e(\mathcal{F}) \geq$ $e(\mathcal{L}(|\mathcal{F}|)$. Later, their results were extended to another range by Balogh et. al. [2].

Theorem 4 (3], [2]). Suppose that $m, n, k, \ell$ are positive integers, $m \leq\binom{ n}{k}-\binom{m-\ell}{k}$ and $n \geq$ $108 \ell k^{2}\left(k+\ell\right.$ ) ([3]) or $n \geq C k^{2} \ell^{3}$ with some absolute $C$ ([2]). Then

$$
\begin{equation*}
e(\mathcal{F}) \geq e(\mathcal{L}(m)) \quad \text { for all } \mathcal{F} \subset\binom{[n]}{k} \text { satisfying }|\mathcal{F}|=m \tag{3}
\end{equation*}
$$

The same holds for any $n>2 k$ and $|\mathcal{F}| \leq\binom{ n-1}{k-1}+\frac{n-2 k}{n}\binom{n-k-1}{k-1}([2])$.
More generally, the number of pairs of sets with intersection at most $t$ was studied in [5].
Note the obvious relationship

$$
d(\mathcal{F}) \geq 2 e(\mathcal{F}) /|\mathcal{F}|
$$

For $1 \leq i<k$ and a set $P \in\binom{[n]}{i}$, we use the standard notation $\mathcal{F}(P):=\{F \backslash P: P \subset F \in \mathcal{F}\}$ and $\mathcal{F}(\bar{P}):=\{F \in \mathcal{F}: F \cap P=\emptyset\}$.

With this notation, $d(\mathcal{F})=\max \{|\mathcal{F}(\bar{F})|: F \in \mathcal{F}\}$. Recall that, for a family $\mathcal{F}$, its covering number $\tau(\mathcal{F})$ is the minimum size of $S \subset[n]$ such that $\mathcal{F}(\bar{S})=\emptyset$.

The structure of the extremal examples that minimize $d(\mathcal{F})$, although related to the families $\mathcal{L}(m)$, appears to be significantly more complicated. Our first two results are structural and apply in a more general setting.

Theorem 5. Fix an integer $t \geq 2$ and let $k_{0}$ be sufficiently large. If $k \geq k_{0}$ and $n \geq 16 t^{2} k^{2}$ then
 exists a set $S$ of size at most $t$ such that $|\mathcal{F}(\bar{S})| \leq\binom{ n-4}{k-4}$. Moreover, for each $x$ in $S$, we have $|\mathcal{F}(x)| \geq k^{-1 / 2}|\mathcal{F}|$.

Remark. It is sufficient to take $k_{0}$ such that $\left(10 t \log k_{0}\right)^{7} \leq k_{0}$. Here and in what follows, all log's are base 2 .

For somewhat large $\mathcal{F}$, we can get rid of $\mathcal{F}(\bar{S})$ completely.
Theorem 6. Fix and integer $t \geq 2$ and let $k_{0}$ be sufficiently large. Let further $k \geq k_{0}$ and $n \geq 16 t^{2} k^{2}$. If $\mathcal{F} \subset\binom{[n]}{k}$ and $|\mathcal{F}| \geq 2 t^{2} k\binom{n-2}{k-2}$, then $d(\mathcal{F}) \geq\left(1-\frac{1}{t}\right)|\mathcal{F}|$ unless $\mathcal{F}(\bar{S})=\emptyset$, where $S$ is as in Theorem 5. In particular, if $d(\mathcal{F})<\left(1-\frac{1}{t}\right)|\mathcal{F}|$ then $\tau(\mathcal{F}) \leq t$.

Proof. Assume that $d(\mathcal{F})<\left(1-\frac{1}{t}\right)|\mathcal{F}|$. In view of Theorem 5, we can choose a set $S,|S| \leq t$, satisfying $|\mathcal{F}(\bar{S})| \leq\binom{ n-4}{k-4}$. If $\mathcal{F}(\bar{S})=\emptyset$ then we are done. Otherwise fixe some $U \in \mathcal{F}(\bar{S})$. Consider an arbitrary set $F \in \mathcal{F} \backslash \mathcal{F}(\bar{U})$. Then either $F \in \mathcal{F}(\bar{S})$, or $F \cap U \neq \emptyset$ and $F \cap S \neq \emptyset$ hold simultaneously. Thus $|\mathcal{F}|-d(\mathcal{F}) \leq|\mathcal{F} \backslash \mathcal{F}(\bar{U})| \leq\binom{ n-4}{k-4}+t k\binom{n-2}{k-2}<\frac{1}{t}|\mathcal{F}|$, a contradiction. Therefore, $|\mathcal{F}(\bar{S})|=\emptyset$ must hold.

The next theorem focuses on the case $t=2$ and the application to the maximum degree question for non-intersecting families. It gives a more precise structural information depending on the size of $\mathcal{F}$. The situation turns out to be quite complicated for small values of $|\mathcal{F}|$. Consider the following examples:

$$
\mathcal{E}_{i}:=\left\{F \in\binom{[n]}{k}: 1 \in F, F \cap[2, k+i+1] \neq \emptyset\right\} \cup\binom{[2, k+i+1]}{k} .
$$

Note that, for $i<k$ and $n>k^{2} \geq 100$,

$$
\begin{equation*}
d\left(\mathcal{E}_{i}\right)=\left|\mathcal{E}_{i}([2, k+1])\right|=\binom{n-k-1}{k-1}-\binom{n-k-i-1}{k-1} . \tag{4}
\end{equation*}
$$

Theorem 7. Suppose that $k \geq k_{0}$ and $n \geq 64 k^{2}$. Let $\mathcal{F} \subset\binom{[n]}{k}$ satisfy $d(\mathcal{F})=d(|\mathcal{F}|, n, k)$. Then there exists $S$ as in Theorem 5 such that the following holds.
(1) If, for some integer $1 \leq i \leq \frac{k}{2}$,

$$
\begin{equation*}
|\mathcal{D}| \leq|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-k-i-1}{k-1}+\binom{k+i}{k}, \tag{5}
\end{equation*}
$$

then $S=\{x\}$ for some $x \in[n]$ and $\mathcal{F}(\bar{x}) \subset\binom{Y}{k}$ for some set $Y$ of size $k+i, x \notin Y$. In particular, $|\mathcal{F}(\bar{x})| \leq\binom{ k+i}{k}$. Moreover, if equality holds in (5) then $\mathcal{F}$ is isomorphic to $\mathcal{E}_{i}$.
(2) If $|\mathcal{F}| \geq 2|\mathcal{D}|$ then $|S|=2$.
(3) If $|\mathcal{F}| \geq 4|\mathcal{D}|$ then $\mathcal{F}(\bar{S})=\emptyset$.

Case 2 of the theorem is necessary since, while the example for the tightness of case 3 is expected to be of the following type:

$$
\mathcal{W}_{\ell}:=\left\{F \in\binom{[n]}{k}:\{1,2\} \subset F\right\} \cup\left\{F \in\binom{[n]}{k}:|F \cap[1,2]|=1, F \cap[3, \ell+2] \neq \emptyset\right\}
$$

(or, more generally, $\mathcal{L}(m)$-type on $[3, n]$ ), for small $\ell$ these examples may be 'improved' by adding the following family:

$$
\mathcal{W}_{\ell^{\prime}}^{\prime}:=\left\{F \in\binom{[n]}{k}:|F \cap[3, \ell+2]| \geq \ell^{\prime}\right\} .
$$

Indeed, if $\ell^{\prime}$, say, satisfies $\ell^{\prime}>\frac{2}{3} \ell>\frac{3 k}{4}$, then $\mathcal{W}_{\ell} \cup \mathcal{W}_{\ell^{\prime}}^{\prime}$ beats families with $|S|=1$ and has the same maximum degree as $\mathcal{W}_{\ell}$ since the elements from $\mathcal{W}_{\ell^{\prime}}^{\prime}$ have low degree.

Our next result gives a concrete numerical implication of Theorems 5 ,
Theorem 8. In the conditions of Theorem 7 (3), we have

$$
\begin{equation*}
d(\mathcal{F}) \geq \frac{\frac{|\mathcal{F}|-\binom{n-2}{k-2}}{2}\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\binom{n-k-2}{k-2} \geq \frac{1}{2}\left(1-\frac{k^{2}}{n}\right)|\mathcal{F}|-\frac{3}{2}\binom{n-k-2}{k-2} . \tag{6}
\end{equation*}
$$

Tightness. Although the bound (6) may appear somewhat arbitrary, it is actually optimal up to some lower order terms. See Section 6 for details.

## 2. Simple bounds

In this section, we present proofs of several weaker bounds which are, however, less technical. The aim is to convey a feeling of the problem to the reader. Moreover, the methods used here are different and may be interesting in their own right.

Throughout this section, we fix a family $\mathcal{F} \subset\binom{[n]}{k},|\mathcal{F}|>\binom{n-1}{k-1}$. Note that, due to the Erdős-Ko-Rado theorem, $\mathcal{F}$ is not intersecting. Our goal is to present several relatively simple cases in which $d(\mathcal{F})>0.49|\mathcal{F}|$ and approaches to prove such a bound.

For $1 \leq i<k$, define $c(i)=c(i, \mathcal{F})=\binom{n-i}{k-i}^{-1} \max \left\{|\mathcal{F}(P)|: P \in\binom{[n]}{i}\right\}$.
Lemma 9. Define $\gamma:=|\mathcal{F}| /\binom{n-1}{k-1}$. Then

$$
\begin{equation*}
d(\mathcal{F}) \geq \frac{1}{2}\left(1-\frac{c(2) k^{3}}{\gamma n}\right)|\mathcal{F}| . \tag{7}
\end{equation*}
$$

Proof. Since $\mathcal{F}$ is not intersecting, we may choose $G, H \in \mathcal{F}, G \cap H=\emptyset$. If $F \in \mathcal{F}$ intersects both $G$ and $H$ then it contains at least one of the $k^{2}$ pairs $(x, y)$ with $x \in G, y \in H$. This allows at most $k^{2} c(2)\binom{n-2}{k-2}$ such sets. Note that

$$
\binom{n-2}{k-2}<\frac{k}{n}\binom{n-1}{k-1}=\frac{k}{\gamma n}|\mathcal{F}| .
$$

Now (77) follows from the fact that each of the remaining $F \in \mathcal{F}$ are disjoint to either $G$ or $H$ (or both).

This result immediately implies the following corollary.
Corollary 10. If $n \geq 50 k^{3}$ then $d(\mathcal{F})>0.49|\mathcal{F}|$.
Proof. Then we can apply Lemma 9 and use $\gamma>1, c(2) \leq 1$.
For $k \geq 50$ we can prove something stronger.
Theorem 11. If $n \geq 4 k^{3}$ and $k \geq 50$ then $d(\mathcal{F})>0.49|\mathcal{F}|$.

Proof. For two families $\mathcal{G} \subset\binom{[n]}{g}, \mathcal{H} \subset\binom{[n]}{h}$, let $e(\mathcal{G}, \mathcal{H})$ denote the number of disjoint pairs $G, H$, where $G \in \mathcal{G}, H \in \mathcal{H}$. In the case $\mathcal{H}=\{H\}$ consists of one member, we define $e(\mathcal{G}, H):=e(\mathcal{G},\{H\})$. The following simple lemma will be essential for the proof.

Lemma 12. Suppose that $1 \leq i<k, P \in\binom{[n]}{i}$ and $|\mathcal{F}(P)|=c(i)\binom{n-i}{k-i}$. then

$$
\begin{align*}
e(\mathcal{F}(P), H) & \geq|\mathcal{F}(P)|-c(i+1) k\binom{n-i-1}{k-i-1} \quad \text { for all } H \in \mathcal{F}(\bar{P}),  \tag{8}\\
e(\mathcal{F}(P), \mathcal{F}(\bar{P})) & \geq\left(1-\frac{c(i+1) k^{2}}{c(i) n}\right)|\mathcal{F}(P)||\mathcal{F}(\bar{P})| . \tag{9}
\end{align*}
$$

Proof. Fixing $H \in \mathcal{F}(\bar{P}), G \cap H \neq \emptyset$ for some $G \in \mathcal{F}(P)$ is equivalent to $G \cup P \in \mathcal{F}$ containing at least one of the $k$ sets $P \cup\{x\}, x \in H$. This allows for less than $c(i+1) k\binom{n-i-1}{k-i-1}$ such sets, implying (8).

To deduce (9), simply sum (8) over all $H \in \mathcal{F}(\bar{P})$ and use $|\mathcal{F}(P)|=c(i)\binom{n-i}{k-i}$ together with $\binom{n-i-1}{k-i-1}=\frac{k-i}{n-i}\binom{n-i}{k-i}<\frac{k}{n}\binom{n-i}{k-i}$.

## Corollary 13.

$$
\begin{equation*}
d(\mathcal{F}) \geq \max \left\{\frac{1}{2}, 1-c(1)\right\} \cdot\left(1-\frac{c(2) k^{2}}{c(1) n}\right)|\mathcal{F}| . \tag{10}
\end{equation*}
$$

Proof. Note $|\mathcal{F}|=|\mathcal{F}(x)|+|\mathcal{F}(\bar{x})|$. This implies max $\{|\mathcal{F}(x)|, \mid \mathcal{F}(\bar{x} \mid)\} \geq \frac{1}{2}|\mathcal{F}|$. Moreover, $|\mathcal{F}(\bar{x})| \geq$ $|\mathcal{F}|-c(1)\binom{n-1}{k-1}>(1-c(1))|\mathcal{F}|$. Applying (19) with $i=1$ and averaging yields (10).

Lemma 14. If $d(\mathcal{F})<\frac{1}{2}|\mathcal{F}|$ then we have

$$
\begin{equation*}
c(1)>\frac{|\mathcal{F}|}{2 k\binom{n-1}{k-1}}>\frac{1}{2 k} . \tag{11}
\end{equation*}
$$

Proof. Let $F \in \mathcal{F}$ be arbitrary. In view of our assumption, $|\mathcal{F}(\bar{F})|<\frac{1}{2}|\mathcal{F}|$. Equivalently, $\mid \mathcal{F} \backslash$ $\left.\mathcal{F}(\bar{F})\left|>\frac{1}{2}\right| \mathcal{F} \right\rvert\,$. On the other hand, $|\mathcal{F} \backslash \mathcal{F}(\bar{F})| \leq \sum_{x \in F}|\mathcal{F}(x)| \leq k c(1)\binom{n-1}{k-1}$. Comparing these two inequalities, the claim follows.

Let us continue with the proof of Theorem [11. In view of the lemma above, we may assume that $c(1)>\frac{1}{2 k}$. If $c(1) \geq \frac{1}{2}$ then (10) combined with $n \geq 100 k^{2}$ implies $d(\mathcal{F}) \geq 0.49|\mathcal{F}|$. If $\frac{1}{2 k} \leq c(1)<\frac{1}{2}$ then the right hand side of (10) is at least $(1-c(1))\left(1-\frac{1}{4 c(1) k}\right)|\mathcal{F}|$ due to $n \geq 4 k^{3}$. It is an easy calculation that the minimum of this expression is attained for $c(1)=\frac{1}{2 k}$ or $\frac{1}{2}$ and the expression is at least $0.49|\mathcal{F}|$ in both cases.

Next, we add one more idea and prove the same statement in a yet wider range.
Theorem 15. If $n \geq 100 k^{2}$ then $d(\mathcal{F}) \geq 0.49|\mathcal{F}|$.
Proof. Assume that $c(2) / c(1) \leq 5$. If $c(1) \leq \frac{1}{2}$ then (10) gives $d(\mathcal{F}) \geq(1-c(1))\left(1-\frac{c(2) k^{2}}{c(1) n}\right)|\mathcal{F}| \geq$ $(1-c(1))\left(1-\frac{c(2)}{100 c(1)}\right)|\mathcal{F}|$. Given the assumption, it is not difficult to see that this expression is minimized for $c(1)=\frac{1}{2}$, in which case we have $d(\mathcal{F}) \geq \frac{1}{2}\left(1-\frac{c(2)}{50}\right)|\mathcal{F}| \geq 0.49|\mathcal{F}|$. If $c(1) \geq \frac{1}{2}$ then $\frac{c(2)}{c(1)} \leq 2$ and (10) implies $d(\mathcal{F}) \geq \frac{1}{2}\left(1-\frac{2 k^{2}}{n}\right)|\mathcal{F}| \geq 0.49|\mathcal{F}|$.

Assume that $c(2) / c(1) \geq 5$. Define $q$ to be the smallest integer satisfying $2^{q} \geq \frac{1}{c(2)}$. Since $c(i) \geq 1$ for all $1 \leq i \leq k$ we have

$$
\frac{c(3)}{c(2)} \cdot \frac{c(4)}{c(3)} \cdot \ldots \cdot \frac{c(q+2)}{c(q+1)} \leq \frac{1}{c(2)} \leq 2^{q} .
$$

Consequently we can fix $2 \leq i \leq q+1$ such that

$$
\frac{c(i+1)}{c(i)} \leq 2 .
$$

By definition, $c(2)<2^{-(q-1)}$. Noting that $2^{q} \geq q+1$ for all $q \geq 0, i c(2) \leq(q+1) c(2)<$ $(q+1) 2^{-(q-1)} \leq 2$ follows.

Take $P,|P|=i$, such that $|\mathcal{F}(P)|=c(i)\binom{n-i}{k-i}$. Note that $i \leq j=1+\left\lceil\log _{2} \frac{1}{c(2)}\right\rceil \leq \frac{2}{c(2)}$, and so we have $|\mathcal{F}(\bar{P})| \geq|\mathcal{F}|-i c(1)\binom{n-1}{k-1} \geq|\mathcal{F}|-\frac{i c(2)}{5}\binom{n-1}{k-1} \geq|\mathcal{F}|-\frac{2}{5}\binom{n-1}{k-1} \geq 0.6|\mathcal{F}|$. Using (9) and averaging, we get that

$$
d(\mathcal{F}) \geq\left(1-\frac{2 k^{2}}{n}\right)|\mathcal{F}(\bar{P})| \geq 0.98 \cdot 0.6|\mathcal{F}|>\frac{1}{2}|\mathcal{F}| .
$$

## 3. Proof of Theorem 5

The proof is a combination of Lemmas 16, 19, and 22,
Lemma 16. For any $c \in \mathbb{N}$ there exists $k_{0}$ such that for any $k \geq k_{0}, n \geq k^{2}+k$ and a family $\mathcal{G} \subset\binom{[n]}{k}$ of size at least $\binom{n-c}{k-c}$, we can find a collection $\mathcal{G}^{\prime} \subset \mathcal{G}$ of $k^{c}$ sets such that $\left|G_{1} \cap G_{2}\right| \leq \log k$ for any $G_{1}, G_{2} \in \mathcal{G}^{\prime}$.

Proof. The proof is a simple probabilistic argument. Form a family $\mathcal{G}^{\prime} \subset \mathcal{G}$ by including each set from $\mathcal{G}$ into $\mathcal{G}^{\prime}$ independently with probability $p:=\frac{2 k^{c}}{|\mathcal{G}|}$. It is an easy calculation to see that the number of sets in $\mathcal{G}^{\prime}$ is at least $\frac{k^{c}}{|\mathcal{G}|}$ with probability at least $1 / 2$.

Now let us calculate the expected number of pairs $A, B \in \mathcal{G}^{\prime}$ that intersect in at least $1+\log k$ elements. The number of such pairs in $\mathcal{G}$ is at most $|\mathcal{G}| \cdot D$, where

$$
D:=\sum_{i=1+\log k}^{k}\binom{k}{i}\binom{n-k}{k-i} \leq\binom{ k}{\log k}\binom{n-k}{k-\log k} \leq \frac{k^{\log k}}{(\log k)!} \cdot \frac{k^{\log k-c}}{(n-k)^{\log k-c}}\binom{n-c}{k-c} \leq \frac{k^{c}}{(\log k)!}\binom{n-c}{k-c} .
$$

Remark that $(\log k)!\gg k^{C}$ for any constant $C$, provided $k$ is large enough. Thus, the expected number of pairs of pairs $A, B$ as above is at most

$$
p^{2}|\mathcal{G}| D \leq \frac{4 k^{3 c}}{(\log k)!}<\frac{1}{2},
$$

provided $k$ is large enough. Thus, by Markov's inequality, with probability strictly greater than $1 / 2$, there are no such pairs in $\mathcal{G}^{\prime}$. Fix such a choice of $\mathcal{G}^{\prime}$ that satisfies both properties simultaneously.

Using a bit of algebra, we can get a stronger statement. We do not require it for the proof since it would improve the bounds on $k$ only slightly, but decided to keep it for the interested reader.
Lemma 17. For any $c \in \mathbb{N}$ there exists $k_{0}$ such that for any $k \geq k_{0}, n \geq 2 k^{2}$ and a family $\mathcal{G} \subset\binom{[n]}{k}$ of size at least $\binom{n-c}{k-c}$, we can find a collection $\mathcal{G}^{\prime} \subset \mathcal{G}$ of $k^{c}$ sets such that $\left|G_{1} \cap G_{2}\right| \leq 4 c$ for any $G_{1}, G_{2} \in \mathcal{G}^{\prime}$.

Proof. Fix integer $d>0$. Take the largest prime power $q$ that satisfies $k \leq q<\frac{n}{k}$. Consider $G F(q)$ and take some set $U \subset G F(q)$ of size $k$. Consider the graphs of all polynomials of degree at most $d$ in $U \times G F(q)$ :

$$
G(f):=\{(x, f(x)): x \in U\} \subset U \times G F(q) .
$$

For $f^{\prime} \neq f,\left|G(f) \cap G\left(f^{\prime}\right)\right| \leq d$ is obvious. Consider the family $\mathcal{G}(d)$ of all such $G(f)$. Then $|\mathcal{G}(d)|=q^{d+1}$. Next, consider a random injection $\phi: U \times G F(q) \rightarrow[n]$. This defines the family $\phi(\mathcal{G}(d)):=\{\phi(G): G \in \mathcal{G}(d)\}$. By linearity of expectation,

$$
\mathrm{E}[|\phi(\mathcal{G}(d)) \cap \mathcal{G}|]=\frac{|\mathcal{G}| q^{d+1}}{\binom{n}{k}}=: y
$$

Note that $y \sim \frac{k^{c} q^{d+1}}{n^{c}}$ and that $q \sim \frac{n}{k}$ for large $k$. Thus, clearly $y \geq k^{c}$ for $d \geq 4 c$. We get that there is a choice of $\phi$ that gives at least $k^{c}$ images of sets from $\mathcal{G}(d)$ in $\mathcal{G}$. Then $\mathcal{G}^{\prime}:=\phi(\mathcal{G}(d))$ is the desired subfamily.

We need the following easy claim for the next lemma.
Claim 18. Fix positive integers $n, k, s$ and let $n \geq k^{2}$. Assume that $F_{1}, \ldots, F_{s}$ are pairwise disjoint sets of size at most $k$. Then the proportion of $k$-element sets intersecting each of $F_{1}, \ldots, F_{s}$ is at most $\left(\frac{k^{2}}{n}\right)^{s}$.
Proof. There are at most $k^{s}$ transversal sets $T$ with $|T|=s,\left|T \cap F_{i}\right|=1$ for all $i$. Each transversal $T$ is contained in $\binom{n-s}{k-s}<\binom{n}{k} \cdot\left(\frac{k}{n}\right)^{s}$ sets from $\binom{[n]}{k}$.
Lemma 19. In the requirements of Theorem [5, assume that there exists $S$ such that $|\mathcal{F}(\bar{S})| \geq\binom{ n-4}{k-4}$ and, moreover, for any $i \in[n] \backslash S$ we have $|\mathcal{F}(i)| \leq \frac{|\mathcal{F}|}{100(t \log k)^{3}}$. Then there is a set $F \in \mathcal{F}(\bar{S})$ that is disjoint with more than $\frac{(t-1)|\mathcal{F}|}{t}$ sets from $\mathcal{F}$.
Proof. Apply Lemma 16 and get a family $\mathcal{G}^{\prime} \subset \mathcal{F}(\bar{S})$ of $10 t \log k$ sets with pairwise intersections of size at most $\log k$. Then the set $I=\bigcup_{A \neq B \in \mathcal{G}^{\prime}} A \cap B$ has size at most $50 t^{2} \log ^{3} k$. Given the condition on $|\mathcal{F}(i)|$ for each $i \in[n] \backslash S$, at most $\frac{50 t^{2} \log ^{3} k}{100(t \log k)^{3}}|\mathcal{F}|=\frac{1}{2 t}|\mathcal{F}|$ sets from $\mathcal{F}$ intersect $I$.

Let us next bound the total number of sets in $\binom{[n]}{k}$ intersecting at least $5 \log k$ sets from $\mathcal{G}^{\prime}$. For a fixed choice of $\ell \geq 5 \log k$ sets, Claim 18 asserts that the proportion of such sets in $\binom{[n]}{k}$ is at most $\left(\frac{k^{2}}{n}\right)^{\ell}$. At the same time, there are $\left({ }^{10 t}{ }_{\ell}^{\log k}\right) \leq\left(\frac{10 e t \log k}{\ell}\right)^{\ell} \leq(2 e t)^{\ell}$ possible subsets of $\mathcal{G}^{\prime}$, and so we can bound the number of all such sets in $\binom{[n]}{k}$ by

$$
\binom{n}{k} \cdot \sum_{\ell=5 \log k}^{10 t \log k}\left(\frac{2 e t k^{2}}{n}\right)^{\ell} \leq\binom{ n}{k} \cdot \begin{cases}2^{-5 \log k}=k^{-5}, & \text { if } n \leq k^{2.2}  \tag{12}\\ n^{-4} & \text { otherwise }\end{cases}
$$

Note that we used the condition $n \geq 16 t k^{2}$ in the first case. It is evident that in both cases the number of such sets is much smaller than $\binom{n-4}{k-4}$.

Thus, the majority (at least ( $1-\frac{1}{2 t}$ )-proportion, excluding those intersecting $I$ ) of sets from $\mathcal{F}$ intersect at most $\frac{1}{2 t}$-proportion of the sets from $\mathcal{G}^{\prime}$. Via simple double counting, it is clear that one of the sets from $\mathcal{G}^{\prime}$ intersects at most a $\frac{1}{t}$-fraction of sets from $\mathcal{F}$.
Claim 20. In the assumptions of Theorem [5, there are fewer than $(10 t \log k)^{3}$ elements $i \in[n]$ such that $|\mathcal{F}(i)| \geq \frac{|\mathcal{F}|}{100(t \log k)^{3}}$.

Proof. Assume that there is a set $S$ of such elements of size $(10 t \log k)^{3}$. By the inclusion-exclusion principle,

$$
|\mathcal{F}| \geq \sum_{i \in S}|\mathcal{F}(i)|-\sum_{i \neq j \in S} \left\lvert\, \mathcal{F}\left(\{ i , j \} | \geq 1 0 | \mathcal { F } \left|-|S|^{2}\binom{n-2}{k-2}\right.\right.\right.
$$

Note that $|S|^{2}=(10 t \log k)^{6}$ is less than $k$, on the other hand, applying Claim 18 to $\mathcal{D}$ we infer $|\mathcal{F}|>|\mathcal{D}|>\frac{k}{2}\binom{n-2}{k-2}$. This shows that the right hand side of the displayed inequality is more than $8|\mathcal{F}|+\left(2|\mathcal{F}|-k\binom{n-2}{k-2}\right)>8|\mathcal{F}|$, giving the contradiction $|\mathcal{F}|>8|\mathcal{F}|$.

Combining Claim 20 and Lemma 19, we immediately get the following corollary.
Corollary 21. If $\mathcal{F}$ satisfies the requirements of Theorem 5 and minimizes $d(\mathcal{F})$ for fixed $|\mathcal{F}|$ then there exists a set $S$ of size at most $(10 t \log k)^{3}$ such that $|\mathcal{F}(\bar{S})| \leq\binom{ n-4}{k-4}$.

Lemma 22. If $\mathcal{F}, S$ are as in Corollary 21 then $|S| \leq t$.
Proof. Indeed, assume that $|S|>t$ and take $i \in S$ such that

$$
\begin{equation*}
\frac{3}{3 t+2}|\mathcal{F}| \geq\left|\mathcal{F}\left(\bar{S}^{\prime}\right)\right| \geq\binom{ n-2}{k-2} \tag{13}
\end{equation*}
$$

where $S^{\prime}:=S \backslash\{i\}$. It is easy to see that the right inequality is satisfied for any $i \in S$, while the left one is satisfied for $i$ such that $|\mathcal{F}(i)|$ is minimal over $i \in S$, provided $|S| \geq 3$. (Both inequalities are obtained using inclusion-exclusion principle in a similar way as used in the proof of Claim 20, )

Next, we apply an argument very similar to the one given in the proof of Lemma 19. We find a family $\mathcal{G}^{\prime} \subset \mathcal{F}\left(\bar{S}^{\prime}\right)$ of $10 t^{2} \log k$ sets with pairwise intersections of size at most $\log k$. The set $I=\bigcup_{A \neq B \in \mathcal{G}^{\prime}} A \cap B$ satisfies $|I| \leq 50 t^{4} \log ^{3} k$. The number of sets from $\mathcal{F} \backslash \mathcal{F}\left(\bar{S}^{\prime}\right)$ intersecting $I$ is at most

$$
\begin{equation*}
|I|\left|S^{\prime}\right|\binom{n-2}{k-2} \leq 50 t^{4} \log ^{3} k \cdot(10 t \log k)^{3}\binom{n-2}{k-2} \leq \frac{1}{20 t}|\mathcal{F}|, \tag{14}
\end{equation*}
$$

where the last inequality holds provided $k$ is large enough.
Next, we estimate the number of sets intersecting at least $5 \log k$ sets from $\mathcal{G}^{\prime}$. We do virtually the same calculation as in Lemma 19 and get that there are at most

$$
\binom{n}{k} \cdot \sum_{\ell=5 \log k}^{10 t^{2} \log k}\left(\frac{2 e t^{2} k^{2}}{n}\right)^{\ell} \leq\binom{ n}{k} \cdot \begin{cases}2^{-5 \log k}=k^{-5}, & \text { if } n \leq t^{2} k^{3} \\ n^{-5} & \text { otherwise }\end{cases}
$$

sets in $\binom{[n]}{k}$ with this property. This number is at most $\binom{n-2}{k-2} \leq \frac{1}{20 t}|\mathcal{F}|$ in any case.
We conclude that at least $1-\frac{3}{3 t+2}-\frac{1}{10 t} \geq 1-\frac{2}{2 t+1}$ of the sets from $\mathcal{F}$ intersect at most $\frac{1}{2 t^{2}}$-fraction of sets from $\mathcal{G}^{\prime}$. (Note that we had to exclude $\mathcal{F}(i)$ itself, which made the biggest contribution to the complement.) Since $\left(1-\frac{2}{2 t+1}\right)\left(1-\frac{1}{2 t^{2}}\right)>1-\frac{1}{t}$, we again get by simple double-counting that there is a set in $\mathcal{G}^{\prime}$ that intersects fewer than $\frac{1}{t}$-fraction of sets from $\mathcal{F}$, which is a contradiction.

Finally, it is not difficult to see that the condition $(10 t \log k)^{7} \leq k$ is sufficient for the argument to go through. (And is only used in Claim 20 and Lemma 22)

## 4. Proof of Theorem 7

Let us first analyze the two possible cases: $|S|=1$ and $|S|=2$. Assume that $S=\{1\}$ in the first case and $S=\{1,2\}$ in the second case. We assume that $\mathcal{F}$ minimizes $d(\mathcal{F})$ for fixed $|\mathcal{F}|$ in either case.

Assume that $|S|=1$ and $|\mathcal{F}| \leq\binom{ n-1}{k-1}+1$. Then $\mathcal{F}(\overline{1})$ contains at least one set $U$, while $|\mathcal{F}(1)| \geq|\mathcal{F}|-\binom{n-4}{k-4}$ due to Theorem [5, At the same time, the number of sets from $\mathcal{F}(1)$ intersecting $U$ is at most $|\mathcal{H}(1, U)|$. Thus,

$$
\begin{equation*}
|\mathcal{F}|-|\mathcal{H}(1, U)|-\binom{n-4}{k-4} \leq d(\mathcal{F}) \tag{15}
\end{equation*}
$$

We can bound $d(|\mathcal{F}|, n, k)$ from above using the following construction: take $\mathcal{H}(1, U)$ together with some other sets containing 1 (so that we have $|\mathcal{F}|$ sets in total). This gives

$$
\begin{equation*}
d(\mathcal{F}) \leq|\mathcal{F}|-|\mathcal{H}(1, U)| . \tag{16}
\end{equation*}
$$

Next, assume that $|S|=2$. To analyze $d(\mathcal{F})$, we shall apply the following proposition.
Proposition 23. [1, Corollary 9.2.5] Let $G=(V, E)$ be a $D$-regular $N$-vertex graph. Let $\lambda=\lambda(G)$ be the second largest absolute value of an eigenvalue of $G$. Then for any subsets $B, C \subset V$, where $|B|=b N,|C|=c N$, we have

$$
|e(B, C)-c b D N| \leq \lambda \sqrt{b c} N
$$

Let $\mathcal{F}_{1}, \mathcal{F}_{2}$ be defined as follows: $\mathcal{F}_{i}:=\{U \backslash\{i\}: U \in \mathcal{F}, U \cap[2]=i\}$. Both $\mathcal{F}_{1}, \mathcal{F}_{2}$ can be seen as subsets of $K G(n-2, k-1)$ (on vertex set $[3, n]$ ). We note that $K G(n-2, k-1)$ is regular of degree $\binom{n-k-1}{k-1}$, and $\lambda=\lambda(K G(n-2, k-1))=\binom{n-k-2}{k-2}$. Let $\delta_{i}$ be the average degree of a vertex in $\mathcal{F}_{i}$. Applied to our situation, the proposition above implies the following.

Proposition 24. Assume that $\{i, j\}=\{1,2\}$. Then

$$
\begin{equation*}
\delta_{i} \geq \frac{\left|\mathcal{F}_{j}\right|\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\binom{n-k-2}{k-2} \sqrt{\frac{\left|\mathcal{F}_{j}\right|}{\left|\mathcal{F}_{i}\right|}} . \tag{17}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
d(\mathcal{F}) \geq \frac{\frac{\left|\mathcal{F}_{1}\right|+\left|\mathcal{F}_{2}\right|}{2}\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\binom{n-k-2}{k-2} . \tag{18}
\end{equation*}
$$

Proof. The first part is just an application of Proposition 23. Next, assume that $\left|\mathcal{F}_{1}\right| \geq\left|\mathcal{F}_{2}\right|$. It is not difficult to see that, for fixed $\left|\mathcal{F}_{1}\right|+\left|\mathcal{F}_{2}\right|$, the bound on $\delta_{2}$ is the smallest when $\left|\mathcal{F}_{1}\right|=\left|\mathcal{F}_{2}\right|$. Indeed, recall that $|\mathcal{F}| \geq|\mathcal{D}|$, and thus $\left|\mathcal{F}_{1}\right|,\left|\mathcal{F}_{2}\right| \geq \frac{1}{3}|\mathcal{D}| \geq \sqrt{k}\binom{n-k-2}{k-2}$ by Theorem 5. The first term on the right hand side of (17) is essentially $C\left|\mathcal{F}_{1}\right|$, where $C>\frac{1}{2}$, while the absolute value of the second term grows slower than $c\left|\mathcal{F}_{1}\right|$ with $c \leq\binom{ n-k-2}{k-2} /\left|\mathcal{F}_{2}\right| \leq k^{-0.5}$. Thus, we get (17).

We note that

$$
\begin{equation*}
\binom{n-2}{k-2} \cdot\binom{n-k-1}{k-1} /\binom{n-2}{k-1}=\frac{k-1}{n-k}\binom{n-k-1}{k-1}<\binom{n-k-2}{k-2} . \tag{19}
\end{equation*}
$$

Knowing that $|\mathcal{F}(S)| \leq\binom{ n-2}{k-2}$ and $|\mathcal{F}(\bar{S})| \leq\binom{ n-4}{k-4}$, from (18) and (19) we derive that

$$
\begin{align*}
& d(\mathcal{F}) \geq \frac{1}{2} \frac{(|\mathcal{F}|-|\mathcal{F}(\bar{S})|-|\mathcal{F}(S)|)\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\binom{n-k-2}{k-2} \\
& \quad \geq \frac{1}{2} \frac{|\mathcal{F}|\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\frac{3}{2}\binom{n-k-2}{k-2}-\frac{1}{2}\binom{n-4}{k-4} \geq 0.4|\mathcal{F}| . \tag{20}
\end{align*}
$$

We go on to the proof of different parts of Theorem 7 .

1. We first note that $|S|=1$. Indeed, (15) implies that $d\left(\mathcal{E}_{i}\right)<\frac{1}{3}\left|\mathcal{E}_{i}\right|$ in this range, while (20) gives $d(\mathcal{F}) \geq 0.4|\mathcal{F}|$.

Next, assume that there are at least $k+i+1$ elements $y \in[2, n]$ such that $|\mathcal{F}(\{1, y\})| \geq$ $\binom{n-2}{k-2}-\binom{n-4}{k-4}$. Then $|\mathcal{F}| \geq\binom{ n-1}{k-1}-\binom{n-k-i-2}{k-1}-(k+i+1)\binom{n-4}{k-4}>\binom{n-1}{k-1}-\binom{n-k-i-1}{k-1}+\binom{n-4}{k-4}$, a contradiction with the assumption on the size of $\mathcal{F}$. Consequently, there are at most $k+i$ such elements.

In particular, if $\left|\bigcup_{U \in \mathcal{F}(\overline{1})} U\right|>k+i$, then there is a set $U \in \mathcal{F}(\overline{1})$ such that one of its elements, say, $y$, is contained in less than $\binom{n-2}{k-2}-\binom{n-4}{k-4}$ sets containing 1. Then the degree of this set is at least $|\mathcal{F}|-\binom{n-1}{k-1}+\binom{n-k-1}{k-1}+\binom{n-4}{k-4}-|\mathcal{F}(\overline{1})| \geq|\mathcal{F}|-\binom{n-1}{k-1}+\binom{n-k-1}{k-1}$. This is more than the maximum degree in a family of size $|\mathcal{F}|$ having just one set $U$ not containing 1 and containing all sets containing 1 and passing through $U$.

Finally, assume that $|\mathcal{F}|=\left|\mathcal{E}_{i}\right|$. If $|\mathcal{F}(\overline{1})|<\binom{k+i}{k}$ then, for any $U \in \mathcal{F}(\overline{1})$, the number of sets not intersecting it is at least $\left|\mathcal{E}_{i}\right|-\binom{n-1}{k-1}+\binom{n-k-1}{k-1}-\binom{k+i}{k}+1$, which is bigger than the right hand side in (4). Thus, $|\mathcal{F}(\overline{1})|=\binom{k+i}{k}$ and, by the first part, must have the form $\binom{Y}{k}$ for some $Y \subset[2, n]$ of size $k+i$.

2, 3. In the cases 2,3 of Theorem 7, we have $|S|=2$. Indeed, if $|S|=1$ then (15) implies that $d(\mathcal{F}) \geq \frac{1}{2}|\mathcal{F}|$. Next, if we have $|\mathcal{F}|>4|\mathcal{D}|$ and $\mathcal{F}(\bar{\emptyset}) \neq \emptyset$, then, for any set $U \in \mathcal{F}(\bar{S})$, there are at most $2|\mathcal{D}|-\binom{n-3}{k-3}$ sets from $\mathcal{F} \backslash \mathcal{F}(\bar{S})$ intersecting it, and thus $d(\mathcal{F}) \geq|\mathcal{F}|-2|\mathcal{D}|+\binom{n-3}{k-3}-\binom{n-4}{k-4}>$ $\frac{1}{2}|\mathcal{F}|$.

## 5. Proof of Theorem 8

Take a family $\mathcal{F}$ satisfying the restrictions of the theorem and such that $d(\mathcal{F})=d(|\mathcal{F}|, n, k)$. The conclusion of Theorem 7 holds, i.e., we must have two elements, say, 1,2 such that $\mathcal{F}([\overline{2}])=\emptyset$. We may w.l.o.g. assume that $\mathcal{F}([2])=\binom{[3, n]}{k-2}$. Next, for $i=1,2$ consider the families $\mathcal{F}_{i}:=$ $\{F \backslash\{i\}: F \in \mathcal{F}, F \cap[2]=\{i\}\}$. Inequality (18) is valid in our situation, which, together with $\left|\mathcal{F}_{1}\right|+\left|\mathcal{F}_{2}\right|=|\mathcal{F}|-\binom{n-2}{k-2}$, concludes the proof of the first part of (6).

To get to the second form of the bound in (6), we use the same calculations as in (20) together with the fact that $|\mathcal{F}(\bar{S})|=0$, and get

$$
d(\mathcal{F}) \geq \frac{|\mathcal{F}|\binom{n-k-1}{k-1}}{2\binom{n-2}{k-1}}-\frac{3}{2}\binom{n-k-2}{k-2} .
$$

Finally, we note that $\frac{\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}} \geq\left(\frac{n-2 k}{n-k}\right)^{k-1}=\left(1-\frac{k}{n-k}\right)^{k-1} \geq\left(1-\frac{k+1}{n}\right)^{k-1} \geq 1-\frac{(k+1)(k-1)}{n} \geq 1-\frac{k^{2}}{n}$.
We have used the fact that $n \geq k^{2}+k$ in the second inequality.

## 6. Optimality of Theorem 8

Fix some $s>1000 k$ and $n>10 k^{3}$ and consider the following family $\mathcal{G}$ :

$$
\mathcal{G}:=\left\{A \in\binom{[n]}{k}:\{1,2\} \subset A\right\} \cup\left\{A \in\binom{[n]}{k}:|A \cap\{1,2\}|=1,|A \cap[3, s+2]|=1\right\} .
$$

It is not difficult to see that $|\mathcal{G}|=\binom{n-2}{k-2}+2 s\binom{n-s-2}{k-2}$. Substituting this into the bound (6) , we get that

$$
\begin{align*}
d(\mathcal{G}) & \geq \frac{s\binom{n-s-2}{k-2}\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\binom{n-k-2}{k-2}=\frac{(s-1)\binom{n-s-2}{k-2}\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\Theta\left(s\binom{n-3}{k-3}\right) \\
& \geq(s-1)\binom{n-s-k}{k-2}-\Theta\left(s\binom{n-3}{k-3}\right) \tag{21}
\end{align*}
$$

where $\Theta$ stands for a constant, independent of $s, k, n$. In the last transition, we have used the result of the following calculation:

$$
\begin{aligned}
& \frac{\binom{n-s-2}{k-2}\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}=\prod_{i=1}^{k-1} \frac{n-k-i}{n-1-i}\binom{n-s-2}{k-2}=\prod_{i=1}^{k-1} \frac{n-k-i}{n-1-i} \prod_{j=0}^{k-3} \frac{n-s-2-j}{n-s-k-j}\binom{n-s-k}{k-2} \\
\geq & \prod_{i=1}^{k-1} \frac{n-k-i}{n-1-i} \prod_{j=0}^{k-4} \frac{n-s-2-j}{n-s-1-k-j}\binom{n-s-k}{k-2} \geq \prod_{i=1}^{2} \frac{n-k-i}{n-1-i}\binom{n-s-k}{k-2} \geq\left(1-\frac{2 k}{n}\right)\binom{n-s-k}{k-2},
\end{aligned}
$$

which implies

$$
\frac{\binom{n-s-2}{k-2}\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}}-\binom{n-s-k}{k-2} \geq-\frac{2 k}{n}\binom{n-s-k}{k-2} \geq-4\binom{n-3}{k-3},
$$

for $k \geq 4$.
At the same time, it is easy to see that each set from $\mathcal{G}$ that intersects [2] in 1 element has the same degree, namely $(s-1)\binom{n-s-k}{k-2}$. This is the maximum degree of $K G(\mathcal{G})$. Compare this with (211). Note that $s\binom{n-3}{k-3} \ll\binom{n-s-k}{k-2}$ for $s \ll n / k$.

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