Almost intersecting families

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Abstract

Let $n > k > 1$ be integers, $[n] = \{1, \dots, n\}$. Let $\mathcal{F}$ be a family of k-subsets of $[n]$. The family $\mathcal{F}$ is called *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. It is called *almost intersecting* if it is *not* intersecting but to every $F \in \mathcal{F}$ there is at most one $F' \in \mathcal{F}$ satisfying $F \cap F' = \emptyset$. Gerbner et al. [GLPPS] proved that if $n \geq 2k + 2$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ holds for almost intersecting families. The main result (Theorem 1.6) implies the considerably stronger and best possible bound $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$ for $n > (2 + o(1))k$.

1 Introduction

Let $[n] = \{1, \dots, n\}$ be the standard n-element set, $2^{[n]}$ its power set and $\binom{[n]}{k}$ the collection of all its k-subsets. Subsets of $2^{[n]}$ are called *families*.

A family $\mathcal{F}$ is called *intersecting* if $F \cap G \neq \emptyset$ for all $F, G \in \mathcal{F}$. One of the fundamental results in extremal set theory is the Erdős–Ko–Rado Theorem:

Theorem 1.1 ([EKR]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, $n \geq 2k > 0$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{1.1}$$

Gerbner et al. [GLPPS] proved an interesting generalisation of (1.1). To state it we need a definition.

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Definition 1.2. A family $\mathcal{F} \subset 2^{[n]}$ is called *almost intersecting* if it is *not* intersecting, but to every $F \in \mathcal{F}$ there is at most one $G \in \mathcal{F}$ satisfying $F \cap G = \emptyset$.

Theorem 1.3 ([GLPPS])**.** *Suppose that $n \geq 2k+2$, $k \geq 1$, $\mathcal{F} \subset \binom{[n]}{k}$. If $\mathcal{F}$ is intersecting or almost intersecting then* (1.1) *holds.*

A natural example of almost intersecting families is $\binom{[2k]}{k}$. For $n = 2k$ and $2k+1$ the best possible bound $|\mathcal{F}| \leq \binom{2k}{k}$ is proven in [GLPPS].

To present another example let us first define some k-uniform intersecting families. For integers $1 \leq a \leq b \leq n$ set $[a, b] = \{a, a+1, \ldots, b\}$. For a fixed $x \in [n]$ let $\mathcal{S} = \mathcal{S}(n, k, x)$ be the full star with center in x, i.e., $\mathcal{S} = \left\{S \in \binom{[n]}{k} : x \in S\right\}$. Every non-empty family $\mathcal{F} \subset \mathcal{S}$ for some x is called a star.

For $3 \leq r \leq k+1$ let us define

$$\mathcal{B}_r = \mathcal{B}_r(n,k) = \left\{B \in \binom{[n]}{k} : 1 \in B,\ B \cap [2,r] \neq \emptyset\right\} \cup \cup \left\{B \in \binom{[n]}{k} : 1 \notin B, [2,r] \subset B\right\}.$$

Obviously, $|B_r| = \binom{n-1}{k-1} - \binom{n-r}{k-1} + \binom{n-r}{k-r+1}$. In particular, $|\mathcal{B}_3| = |\mathcal{B}_4|$. For $n > 2k$ one has

$$|\mathcal{B}_4| < |\mathcal{B}_5| < \ldots < |\mathcal{B}_{k+1}|.$$

The family $\mathcal{B}_{k+1}$ is called the Hilton–Milner family. It has a single set, namely $[2, k+1]$, which does not contain 1.

For $x, y \in [n]$ let us recall the standard notation:

$$\mathcal{F}(x) = \{F \setminus \{x\} : x \in F \in \mathcal{F}\}, \mathcal{F}(\bar{x}) = \{F \in \mathcal{F} : x \notin F\},$$
$$\mathcal{F}(x, \bar{y}) = \mathcal{F}(\bar{y}, x) = \{F \setminus \{x\} : x \in F \in \mathcal{F}, y \notin F\}.$$

The *maximum degree* $\Delta(\mathcal{F})$ of a family $\mathcal{F} \subset 2^{[n]}$ is $\max\{|\mathcal{F}(x)| : x \in [n]\}$. For $3 \leq r \leq k+1$,

$$\Delta(\mathcal{B}_r) = \binom{n-1}{k-1} - \binom{n-r}{k-1} = \binom{n-2}{k-2} + \ldots + \binom{n-r}{k-2} = |\mathcal{B}_r(1)|.$$

Hilton and Milner [HM] proved the following stability result for intersecting families.

Theorem 1.4 ([HM]). *Suppose that $n > 2k \geq 4$, $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, but $\mathcal{F}$ is not a star (not contained in a full star). Then*

$$|\mathcal{F}| \leq |\mathcal{B}_{k+1}|, \tag{1.2}$$

moreover, equality holds only if $\mathcal{F}$ is isomorphic to $\mathcal{B}_{k+1}$ or $k = 3$ and $\mathcal{F}$ is isomorphic to $\mathcal{B}_3$.

Example 1.5. Let $B \subset \binom{[n]}{k}$ be an arbitrary set satisfying $1 \in B$, $B \cap [2, k+1] = \emptyset$. Set $\mathcal{B}^+ = \mathcal{B}_{k+1} \cup \{B\}$. Then $|\mathcal{B}^+| = |\mathcal{B}_{k+1}| + 1$ and $\mathcal{B}^+$ is almost intersecting.

Our main result is the following.

Theorem 1.6. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is almost intersecting, $k \geq 3$. Then*

$$|\mathcal{F}| \leq |\mathcal{B}^+| \quad \textit{holds in the following cases:} \tag{1.3}$$

(i) $k = 3$, $n \geq 13$,

(ii) $k \geq 4$, $n \geq 3k + 3$,

(iii) $k \geq 10$, $n > 2k + 2\sqrt{k} + 4$.

The case $k = 2$ is easy. Suppose that $\mathcal{G} \subset \binom{[n]}{2}$ is almost intersecting and let $F, G \in \mathcal{G}$ be pairwise disjoint. Set $X = F \cup G$ and note $|X| = 4$.

Claim 1.7. $\mathcal{G} \subset \binom{X}{2}$.

Proof. If $\mathcal{G} = \{F, G\}$ then we have nothing to prove. On the other hand, for any further edge $H \in \mathcal{G}$, both $F \cap H$ and $G \cap H$ must be non-empty. Since $|H| = 2$, $H \subset X$ follows. □

Let us make two simple but important observations.

Proposition 8. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be almost intersecting. Then there is a unique partition $\mathcal{F} = \mathcal{F}_0 \sqcup \mathcal{P}_1 \sqcup \ldots \sqcup \mathcal{P}_\ell$ where $\mathcal{F}_0$ is intersecting ($\mathcal{F}_0 = \emptyset$ is allowed) and for $1 \leq i \leq \ell$, $\mathcal{P}_i = \{P_i, Q_i\}$ with $P_i \cap Q_i = \emptyset$.*

The above partition of $\mathcal{F}$ is called the *canonical* partition. The function $\ell(\mathcal{F}) = \ell$ is an important parameter of $\mathcal{F}$.

Definition 1.9. A family $\mathcal{T} = \{T_1, \dots, T_\ell\}$ satisfying $T_i \in \mathcal{P}_i$, is called a full tail (of $\mathcal{F}$).

Proposition 10. *There are 2^ℓ full tails $\mathcal{T}$ and for each of them $\mathcal{F}_0 \cup \mathcal{T}$ is intersecting.*

Let us close this section by a short proof of (1.3) for the special case $\ell(\mathcal{F}) = 1$.

There are two cases to consider according whether the families $\mathcal{F}_0 \cup \{P_1\}$, $\mathcal{F}_0 \cup \{Q_1\}$ are stars or not. Suppose first that one of them, say $\mathcal{F}_0 \cup \{P_1\}$ is not a star. By Theorem 1.4, $\left|\mathcal{F}_0 \cup \{P_1\}\right| = |\mathcal{F}| - 1 \le \left|\mathcal{B}_{k+1}\right|$, implying (1.3). For $k \ge 4$ uniqueness in the Hilton–Milner Theorem implies uniqueness in Theorem 1.6 as well. In the case $k = 3$, one has the extra possibility $\mathcal{F}_0 \cup \{P_1\} = \mathcal{B}_3$. However, it is easy to check that adding a new 3-set to $\mathcal{B}_3$ will *never* produce an almost intersecting family.

The second case is even easier. If both $\mathcal{F}_0 \cup \{P_1\}$ and $\mathcal{F}_0 \cup \{Q_1\}$ are stars then $P_1 \cap Q_1 = \emptyset$ implies that there are two distinct elements (the centres of the stars) x, y such that $\{x, y\} \subset F$ for all $F \in \mathcal{F}_0$. Consequently,

$$|\mathcal{F}| = |\mathcal{F}_0| + 2 \le \binom{n-2}{k-2} + 2 \le \binom{n-2}{k-2} + 2\binom{n-3}{k-2} = |\mathcal{B}_3| \le |\mathcal{B}_{k+1}| < |\mathcal{B}^+|.$$

2 Preliminaries

Let us first prove an inequality on the size $\ell = \ell(\mathcal{F})$ of full tails.

Proposition 11.

$$\ell(\mathcal{F}) \le \binom{2k-1}{k-1}. \tag{2.1}$$

The proof of (2.1) depends on a classical result of Bollobás [B].

Theorem 2.2 ([B], cf. also [JP] and [Ka1])**.** *Suppose that a, b are positive integers, $\mathcal{A} = \{A_1, \dots, A_m\}$, $\mathcal{B} = \{B_1, \dots, B_m\}$ are families satisfying $|A_i| = a$, $|B_i| = b$, $A_i \cap B_i = \emptyset$ for $1 \le i \le m$ and also*

$$A_i \cap B_j \ne \emptyset \quad \text{for all} \quad 1 \le i \ne j \le m. \tag{2.2}$$

Then

$$m \le \binom{a+b}{a}. \tag{2.3}$$

Proof of Proposition 11. Define $A_i = P_i$ for $1 \le i \le \ell$, $A_i = Q_{i-\ell}$ for $\ell+1 \le i \le 2\ell$ and similarly $B_i = Q_i$ for $1 \le i \le \ell$, $B_i = P_{i-\ell}$ for $\ell+1 \le i \le 2\ell$. Then $\mathcal{A} = \{A_1, \dots, A_{2\ell}\}$ and $\mathcal{B} = \{B_1, \dots, B_{2\ell}\}$ satisfy the conditions of Theorem 2.2 with $a = b = k$. Thus $2\ell \le \binom{2k}{k}$ and thereby (2.1) follows. □

If $\mathcal{F}_0 \ne \emptyset$, then one can use an extension (cf. [F1]) of (2.3) to show that (2.1) is strict.

Another ingredient of the proof of Theorem 1.6 is the following

Theorem 2.3 ([F2])**.** *Suppose that* $\mathcal{A} \subset \binom{[n]}{k}$*,* $n > 2k \ge 6$*. Let* r *be an integer,* $4 \le r \le k+1$*. If* $\mathcal{A}$ *is intersecting and* $\Delta(\mathcal{A}) \le \Delta(\mathcal{B}_r)$ *then*

$$|\mathcal{A}| \le |\mathcal{B}_r|. \tag{2.4}$$

Let us note that if $\mathcal{A}$ is not a star then for all $x \in [n]$ there exists $A(x) \in \mathcal{A}$ with $x \notin A(x)$. There are only $\binom{n-1}{k-1} - \binom{n-k-1}{k-1}$ sets $A \in \binom{[n]}{k}$ satisfying $x \in A$, $A \cap A(x) \ne \emptyset$. Thus $|\mathcal{A}(x)| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} = |\mathcal{B}_{k+1}(1)|$. This shows that Theorem 2.3 extends the Hilton–Milner Theorem.

The last ingredient of the proof is the Kruskal–Katona Theorem ([Kr], [Ka2]). We use it in a form proposed by Hilton [H].

For fixed n and k let us define the *lexicographic order* $<_L$ on $\binom{[n]}{k}$ by setting

$$A <_L B \quad \text{iff} \quad \min\{x \in A \setminus B\} < \min\{x \in B \setminus A\}.$$

For an integer $1 \le m \le \binom{n}{k}$ let $\mathcal{L}(m) = \mathcal{L}(m, n, k)$ denote the family of the first m subsets $A \in \binom{[n]}{k}$ in the lexicographic order.

Let a, b be positive integers, $a + b \le n$. Two families $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ are called *cross-intersecting* if $A \cap B \ne \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem 2.4 ([Kr], [Ka2], [H])**.** *Let* $X \subset [n]$ *and* $|X| \ge a + b$*. If* $\mathcal{A} \subset \binom{X}{a}$ *and* $\mathcal{B} \subset \binom{X}{b}$ *are cross-intersecting then* $\mathcal{L}(|\mathcal{A}|, X, a)$ *and* $\mathcal{L}(|\mathcal{B}|, X, b)$ *are cross-intersecting as well.*

Note that if $\mathcal{G} \subset \binom{[n]}{k}$ is intersecting then the two families $\mathcal{G}(1) \subset \binom{[2,n]}{k-1}$ and $\mathcal{G}(\bar{1}) \subset \binom{[2,n]}{k}$ are cross-intersecting. Usually we apply Theorem 2.4 to these families (with $X = [2, n]$).

In our situation with $\mathcal{F} \subset \binom{[n]}{k}$ being almost intersecting and $\mathcal{F}_0 \subset \mathcal{F}$ defined by Proposition 8, $\mathcal{F}_0(1)$ and $\mathcal{F}(\bar{1})$ are cross-intersecting.

Using Theorem 2.4 one easily deduces the following.

Corollary 2.5. *Let $r \geq 3$ be an integer. Suppose that $\mathcal{A} \subset \binom{[2,n]}{k-1}$ and $\mathcal{B} \subset \binom{[2,n]}{k}$ are cross-intersecting, $n > 2k$, $k \geq r$. If*

$$|\mathcal{A}| \geq \binom{n-1}{k-1} - \binom{n-r}{k-1}. \tag{2.5}$$

Then

$$|\mathcal{B}| \leq \binom{n-r}{k-r+1}. \tag{2.6}$$

Proof. Note that $\mathcal{L}\left(\binom{n-1}{k-1} - \binom{n-r}{k-1}, [2,n], k-1\right) = \left\{L \in \binom{[2,n]}{k-1} : L \cap [2,r] \neq \emptyset\right\}$. Since $n > 2k$, $[2,r] \subset B$ must hold for every $B \in \binom{[2,n]}{k}$ which intersects *each* member of $\mathcal{L}\left(\binom{n-1}{k-1} - \binom{n-r}{k-1}, [2,n], k-1\right)$. Via Theorem 2.4 this implies (2.6). □

Corollary 2.6. *Suppose that $\mathcal{A} \subset \binom{[2,n]}{k-1}$, $\mathcal{B} \subset \binom{[2,n]}{k}$ are cross-intersecting, $n > 2k > 2$,*

$$|\mathcal{B}| \geq k. \tag{2.7}$$

Then

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k}{k-1}. \tag{2.8}$$

Proof. Just note that $\mathcal{L}(k, [2,n], k) = \left\{[2,k] \cup \{j\}, k+1 \leq j \leq 2k\right\}$ and the only $(k-1)$-sets intersecting each of these k-sets are those which intersect $[2,k]$. □

3 Some inequalities concerning binomial coefficients

In this section we present some inequalities that we use in Section 5. The proofs are via standard manipulations, the reader might just glance through them briefly.

Lemma 3.1.

$$\binom{2k}{k-2} \geq \binom{2k-1}{k-1} \quad \textit{for} \quad k \geq 6, \tag{3.1}$$

$$\binom{2k+1}{k-2} \geq \binom{2k-1}{k-1} \quad \textit{for} \quad k \geq 4. \tag{3.2}$$

Proof. $\binom{2k}{k-2} \big/ \binom{2k-1}{k-1} = \frac{2k\cdot(k-1)}{(k+1)(k+2)}$ which is a monotone increasing function of k. Since for $k = 6$, $2 \times 6 \times 5 = 60 > 56 = 7 \times 8$, (3.1) is proved. To prove (3.2) just note $\binom{2k+1}{k-2} > \binom{2k}{k-2}$ and check it for $k = 4$ and 5. □

Lemma 3.2. *Suppose that* $k \geq 10$ *and* $3k + 2 \geq m \geq 2k - 4$. *Then*

$$2 \geq \binom{m}{k-2} \bigg/ \binom{m-1}{k-2} \geq 4/3. \tag{3.3}$$

Moreover, if $m - s \geq 2k - 4$ *then*

$$\sum_{0 \leq i \leq s} \binom{m-i}{k-2} \geq \left(2 - \frac{1}{2^s}\right) \binom{m}{k-2}. \tag{3.4}$$

Proof. $\binom{m}{k-2} \big/ \binom{m-1}{k-2} = \frac{m}{m-k+2}$. Now (3.3) is equivalent to

$$2m - 2k + 4 \geq m \geq \frac{4}{3}m - \frac{4}{3}k + \frac{8}{3}.$$

The first part is equivalent to $m \geq 2k - 4$, the second to $4k - 8 \geq m$. As for $k \geq 10$, $4k - 8 \geq 3k + 2$, we are done. The inequality (3.4) is a direct application of (3.3). □

Lemma 3.3. *Suppose that* $n \geq 2(k + \sqrt{k} + 2)$, $k \geq 9$, $r \geq \sqrt{k} + 5$. *Then*

$$\binom{n-r+1}{k-r+2} < \binom{n-r-1}{k-2}. \tag{3.5}$$

Proof. Let us first show that for n, k fixed the function $f(r) = \binom{n-r+1}{k-r+2} \big/ \binom{n-r-1}{k-2}$ is monotone decreasing in r. Indeed, $f(r+1)/f(r) = \frac{n-r-1}{n-r+1} \cdot \frac{k-r+2}{n-k-r+1} < 1$ as both factors are less than 1 for $n > 2k + 1$.

Consequently it is sufficient to check (3.5) in the case $r = t + 1$ where $t = \left\lfloor \sqrt{k} \right\rfloor + 4$. Fixing k and thereby r, t, define

$$g(n) = \binom{n-t}{k-t+1} \bigg/ \binom{n-t-2}{k-2}.$$

Claim 3.4. *For $n \geq 2k$, $g(n)$ is a monotone decreasing function of n. Indeed,*

$$g(n+1)/g(n) = \frac{n-t+1}{n-t-1} \cdot \frac{n-k-t+1}{n-k} \leq \frac{(n-t+1)(n-k-2)}{(n-t-1)(n-k)} < 1$$

where we used $t \geq 3$ and $ab > (a-2)(b+2)$ for $a > b+2 > 0$.

In view of the claim it is sufficient to prove (3.5) for the case $n = 2k + 2\sqrt{k} + 4$.

$$\frac{\binom{n-t}{k-t+1}}{\binom{n-t-2}{k-2}} = \frac{(n-t)(n-t-1)}{(n-k-t+2)(n-k-t+1)} \cdot \prod_{0 \leq j \leq t-4} \frac{k-2-j}{n-k-1-j}. \tag{3.6}$$

To estimate the RHS, note that the first part is at most $2 \times 2 = 4$. As to the product part, we can use the inequality $\frac{(a-i)(a+i)}{(b-i)(b+i)} < \left(\frac{a}{b}\right)^2$, valid for all $b > a > i > 0$ to get the upper bound

$$\left(\frac{k-\frac{t}{2}}{n-k+1-\frac{t}{2}}\right)^{t-3} = \left(1 - \frac{n+1-2k}{n-k+1-\frac{t}{2}}\right)^{t-3}.$$

To prove (3.5) we need to show that this quantity is at most $1/4$. We show the stronger upper bound $e^{-\frac{3}{2}}$. Using the inequality $1-x < e^{-x}$, it is sufficient to show

$$\frac{n+1-2k}{n+1-k-\frac{t}{2}} > \frac{3}{2(t-3)}.$$

Plugging in $n = 2k + 2\sqrt{k} + 4$, $t = \sqrt{k} + 4$ the above inequality is equivalent to

$$2(\sqrt{k}+1)\big(2\sqrt{k}+5\big) > 3k + \frac{9}{2}\sqrt{k} + 9, \quad \text{or}$$

$k + 5.5\sqrt{k} > 9$ which is true for $k \geq 2$. □

Lemma 3.5. *Suppose that $n \geq 3k+3$, $k \geq 4$ then*

$$\binom{n-4}{k-3} + \binom{2k-1}{k-1} \leq \binom{n-5}{k-2} + \binom{n-5}{k-4}. \tag{3.7}$$

Proof. Let us first prove (3.7) in the case $n = 3k+3$,

$$\binom{3k-1}{k-3} + \binom{2k-1}{k-1} \leq \binom{3k-2}{k-2} + \binom{3k-2}{k-4}. \tag{3.8}$$

The cases $k=4,5,6$ can be checked directly. Let $k\ge 7$. Note that

$$\binom{3k-1}{k-3}\Big/\binom{3k-2}{k-2}=\frac{(3k-1)(k-2)}{(2k+1)(2k+2)}=\frac{3k^2-7k+2}{4k^2+6k+2}<\frac{3}{4}.$$

Thus it is sufficient to show

$$\binom{2k-1}{k-1}\Big/\binom{3k-2}{k-2}\le\frac{1}{4}. \tag{3.9}$$

In view of $k\ge 7$, $\binom{2k-1}{k-1}\big/\binom{2k}{k-2}$ is less than 1. Thus (3.9) will follow from

$$\binom{2k}{k-2}\Big/\binom{2k+4}{k-2}=\frac{(k+6)(k+5)(k+4)(k+3)}{(2k+4)(2k+3)(2k+2)(2k+1)}<\frac{1}{4}. \tag{3.10}$$

Since $\frac{k+i+2}{2k+i}=\frac{1}{2}+\frac{\frac{i}{2}+2}{2k+i}$ is a decreasing function of k, it is sufficient to check (3.10) for $k=7$. Plugging in $k=7$ we obtain $\frac{143}{612}<\frac{1}{4}$, as desired.

To prove (3.7) for $n>3k+3$, we show that passing from n to $n+1$ the RHS increases more than the LHS. More exactly we show:

$$\binom{n-4}{k-4}<\binom{n-5}{k-3}. \tag{3.11}$$

We have

$$\binom{n-4}{k-4}\Big/\binom{n-5}{k-3}=\frac{(n-4)(k-3)}{(n-k)(n-k-1)}.$$

Using $n>3k$, $\frac{n-4}{n-k}<2$ and $\frac{k-3}{n-k-1}<\frac{1}{2}$, we get (3.11). □

4 The case $k=3$, $n\ge 13$

Let $\mathcal{F}=\mathcal{F}_0\cup\mathcal{P}_1\cup\ldots\cup\mathcal{P}_\ell$ be the canonical partition of the almost intersecting family $\mathcal{F}\subset\binom{[n]}{3}$. For $n=6$, $|\mathcal{F}|\le\binom{6}{3}=20$ is obvious. In [GLPPS] the same upper bound is established for $n=7$ as well. On the other hand for $n\ge 9$ one has $3n-7\ge 20$. Let us make the indirect assumption

$$|\mathcal{F}|\ge 3n-6. \tag{4.1}$$

In view of (2.1) and $\binom{6}{3}=20$ one has $\mathcal{F}_0\ne\emptyset$. Also because of the proof in Section 1, $\ell(\mathcal{F})\ge 2$.

For notational convenience we set $(a, b, c) = \{a, b, c\}$. By symmetry we assume $\mathcal{P}_1 = \{(1, 2, 3), (4, 5, 6)\}$. Note that for $F \in (\mathcal{F} \setminus \mathcal{P}_1)$, $F \cap (1, 2, 3) \neq \emptyset$ and $F \cap (4, 5, 6) \neq \emptyset$ imply

$$|F \setminus [6]| \leq 1 \tag{4.2}$$

and

$$\{a, b\} \subset F \quad \text{for at least one of the 9 choices } 1 \leq a \leq 3,\ 4 \leq b \leq 6. \tag{4.3}$$

For $\{a, b\}$, $1 \leq a \leq 3$, $4 \leq b \leq 6$ define $D(a, b) = \{c \in [7, n], (a, b, c) \in \mathcal{F}\}$. Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be some permutations of $(1, 2, 3)$ and $(4, 5, 6)$, respectively.

Lemma 4.1. (i) *If $D(a_i, b_i) \neq \emptyset$ for $i = 1, 2, 3$ then $D(a_i, b_i)$ is the same 1-element set for $1 \leq i \leq 3$.*

(ii) *If $|D(a_1, b_1)| \geq 3$ then $D(a_i, b_i) = \emptyset$ for $i = 2, 3$.*

(iii) *If $|D(a_1, b_1)| = |D(a_2, b_2)| = 2$ then $D(a_1, b_1) = D(a_2, b_2)$.*

Proof. Suppose by symmetry $|D(a_1, b_1)| \geq 2$ and let $x, y \in D(a_1, b_1)$. The almost intersecting property implies $(a_i, b_i, z) \notin \mathcal{F}$ for $i = 2, 3$ and $z \notin \{x, y\}$. This already proves (ii). To continue with the proof of (i) choose $x_2, x_3 \in \{x, y\}$, not necessarily distinct elements so that $(a_i, b_i, x_i) \in \mathcal{F}$ for $i = 2, 3$.

There are two simple cases to consider. Either $x_2 = x_3$ or $x_2 \neq x_3$. By symmetry assume $x_3 = y$. In the first case (a_1, b_1, x) is disjoint to both (a_2, b_2, y) and (a_3, b_3, y). While in the latter case (a_3, b_3, y) is disjoint to both (a_1, b_1, x) and (a_2, b_2, x). These contradict the almost intersecting property. Now (iii) follows in the same way. □

Lemma 4.2. *If $|D(a, b)| \geq 3$ for some $1 \leq a \leq 3$, $4 \leq b \leq 6$, then $\{a, b\} \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

Proof. Suppose by symmetry $(a, b) = (1, 4)$ and $(1, 4, c) \in \mathcal{F}$ for $c = 7, 8, 9$. Let indirectly $F \in \mathcal{F}$ satisfy $F \cap \{1, 4\} = \emptyset$. By (4.2), $|F \cap (7, 8, 9)| \leq 1$. Thus F is disjoint to at least two of the three triples $(1, 4, c)$, $7 \leq c \leq 9$, the desired contradiction. □

How many choices of (a, b), $1 \leq a \leq 3$, $4 \leq b \leq 6$ can be that satisfy $|D(a, b)| \geq 3$? In view of Lemma 4.1 (ii), $\{a, b\} \cap \{a', b'\} \neq \emptyset$ must hold for distinct choices. Recall the easy fact that every bipartite graph without two disjoint edges is a star. Consequently, by symmetry, we may assume that $|D(a,b)| \geq 3$ implies $a = 1$. Let us distinguish *four* cases.

(a) $$|D(1,j)| \geq 3 \quad \text{for} \quad j = 4,5,6.$$

We claim that $\mathcal{F}(\bar{1}) = \{(4,5,6)\}$. Let us prove it. Suppose that $F \in \mathcal{F}$, $1 \notin F$ and by symmetry $4 \notin F$. Choose $(x,y,z) \subset [7,n]$ such that $(1,4,x), (1,4,y), (1,4,z) \in \mathcal{F}$. In view of (4.2) at least two of them are disjoint to F, a contradiction.

Since $(1,2,3)$ is the only member of $\mathcal{F}$ disjoint to $(4,5,6)$, now $\mathcal{F} \subset \big\{(1,u,v) : \{u,v\} \cap (4,5,6) \neq \emptyset\big\} \cup \big\{(1,2,3),(4,5,6)\big\}$ follows.

(b) $$|D(1,j)| \geq 3 \quad \text{for} \quad j = 4,5, \quad \text{but} \quad |D(1,6)| \leq 2.$$

In view of Lemma 4.1 (ii), $D(a,b) = \emptyset$ for $a = 2,3$. Using (4.2) as well we infer

(4.4) $$\left| \mathcal{F} \setminus \binom{[6]}{3} \right| \leq 2(n-6) + |D(1,6)|.$$

To estimate $\left|\mathcal{F} \cap \binom{[6]}{3}\right|$ we need another simple lemma.

Lemma 4.3. *If $|D(a,b)| \geq 2$ for some $1 \leq a \leq 3$, $4 \leq b \leq 6$ then $[6] \setminus \{a,b\}$ contains no member of $\mathcal{F}$.*

Proof. If $E \in \binom{[6]\setminus\{a,b\}}{3}$, then $E \cap (a,b,c) = \emptyset$ for all $c \in D(a,b)$. Thus almost intersection implies $E \notin \mathcal{F}$. □

Applying the lemma to both $(a,b) = (1,4)$ and $(1,5)$ yields $\left|\mathcal{F} \cap \binom{[6]}{3}\right| \leq 20 - 7 = 13$.

In case $|D(1,6)| = 2$, $(3,4,5) \notin \mathcal{F}$ follows as well. Thus (4.4) yields

$$|\mathcal{F}| \leq 2n - 12 + 14 = 2n + 2 < 3n - 7 \quad \text{for} \quad n \geq 10.$$

For $n = 9$ as well we obtain the inequality $|\mathcal{F}| \leq 3n - 7$. However, to obtain uniqueness would require some extra case analysis.

(c) $$|D(1,4)| \geq 3 > |D(a,b)| \quad \text{for} \quad (a,b) \neq (1,4), \quad 1 \leq a \leq 3, \quad 4 \leq 6 \leq 6.$$

In view of Lemma 4.1 (ii), $D(a,b) = \emptyset$ is guaranteed if $(a,b) \cap (1,4) = \emptyset$. This leads to

$$\left|\mathcal{F} \setminus \binom{[6]}{3}\right| \le n - 6 + 4 \times 2 = n + 2. \tag{4.5}$$

On the other hand Lemma 4.3 yields

$$\left|\mathcal{F} \cap \binom{[6]}{3}\right| \le 20 - 4 = 16.$$

Together with (4.5) this implies

$$|\mathcal{F}| \le n + 18 < 3n - 7 \quad \text{for} \quad n \ge 13.$$

(d) $\qquad |D(a,b)| \le 2 \quad \text{for all} \quad (a,b), \;\; 1 \le a \le 3, \;\; 4 \le b \le 6.$

Applying Lemma 4.1 gives that

$$\big|D(a_1,b_1)\big| + \big|D(a_2,b_2)\big| + \big|D(a_3,b_3)\big| \le 4.$$

Using this for three disjoint matchings yields

$$\left|\mathcal{F} \setminus \binom{[6]}{3}\right| \le 12.$$

Thus

$$|\mathcal{F}| \le 32 \le 3n - 7 \quad \text{for} \quad n \ge 13.$$

In case of equality, $\binom{[6]}{3} \subset \mathcal{F}$. However, that would immediately imply $\mathcal{F} = \binom{[6]}{3}$. Thus the proof of the case $k = 3$, $n \ge 13$ is complete.

5 The proof of (1.3) for $k \ge 4$

We are going to distinguish three cases according to $\Delta(\mathcal{F}_0)$.

(a) $\qquad \Delta(\mathcal{F}_0) \le \binom{n-2}{k-2} + \binom{n-3}{k-2} = \binom{n-1}{k-1} - \binom{n-3}{k-1}.$

Let us suppose $n \ge 2k + 5$. In view of (3.2),

$$\binom{n-4}{k-2} > \binom{2k-1}{k-1}.$$

Consequently, for any choice of a full tail $\mathcal{T}$,

$$\Delta(\mathcal{F}_0 \cup \mathcal{T}) \leq \Delta(\mathcal{F}_0) + \ell \leq \binom{n-2}{k-2} + \binom{n-3}{k-2} + \binom{n-4}{k-2} = \binom{n-1}{k-1} - \binom{n-4}{k-1}.$$

Thus we may apply (2.4) with $r = 4$:

$$|\mathcal{F}_0 \cup \mathcal{T}| \leq \binom{n-1}{k-1} - \binom{n-4}{k-1} + \binom{n-4}{k-3}. \tag{5.1}$$

From (5.1) and $\ell \leq \binom{2k-1}{k-1}$ we infer

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-4}{k-1} + \binom{n-4}{k-3} + \binom{2k-1}{k-1}. \tag{5.2}$$

Using $\left|\mathcal{B}^+\right| > \left|\mathcal{B}_{k+1}\right| \geq \left|\mathcal{B}_5\right|$, it is sufficient to show that the RHS is not larger than $|B_5|$. Equivalently

$$\binom{n-4}{k-3} + \binom{2k-1}{k-1} \leq \binom{n-5}{k-2} + \binom{n-5}{k-4}. \tag{5.3}$$

Since (5.3) is the same as (3.7), for $n \geq 3k+3$ we are done.

To deal with the case (iii), we cannot be so generous. We assume that $n \leq 3k+2$. Note that

$$|\mathcal{B}^+| > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} \geq \binom{n-1}{k-1} - \binom{2k+1}{k-1}.$$

Using (5.2) and the inequality above, it is sufficient for us to show that

$$\binom{n-4}{k-1} - \binom{n-4}{k-3} \geq 2\binom{2k+1}{k-1}.$$

The left hand side is $\left(1 - \frac{(k-1)(k-2)}{(n-k-1)(n-k-2)}\right)\binom{n-4}{k-1} \geq \left(1 - \frac{k^2}{(n-k)^2}\right)\binom{n-4}{k-1} \geq \Big(1 - \left(\frac{2\sqrt{k}+4}{k+2\sqrt{k}+4}\right)^2\Big)\binom{n-4}{k-1} \geq \Big(1 - \left(1 - \frac{2}{\sqrt{k}} + \frac{1}{k}\right)^2\Big)\binom{n-4}{k-1} \geq 2k^{-1/2}\binom{n-4}{k-1}$. Thus, it is sufficient for us to show that

$$\binom{n-4}{k-1} \Big/ \binom{2k+1}{k-1} \geq k^{1/2}.$$

Let us define $2p = n - 2k - 4$ and note $p > \sqrt{k}$ and thus $p \geq 4$. In view of (3.3) and $n \leq 3k + 2$ we have

$$(5.4) \qquad \binom{n-4}{k-1} \Big/ \binom{2k+1}{k-1} > (4/3)^{2p-1}.$$

Thus, putting $t := \sqrt{k}$, we are done if $t \leq (4/3)^{2t-1}$ for any $t \geq 4$. The latter is verified via an easy calculation. This concludes the proof of (1.3) in this case.

$$\text{(b)} \qquad \binom{n-1}{k-1} - \binom{n-3}{k-1} < \Delta(\mathcal{F}_0) \leq \binom{n-1}{k-1} - \binom{n-k}{k-1}.$$

Let 1 be the vertex of highest degree in $\mathcal{F}_0$.

Claim 5.1. *Let $\mathcal{G} \subset \binom{[n]}{k}$ be any intersecting family containing $\mathcal{F}_0$. Then 1 is the unique vertex of highest degree in $\mathcal{G}$.*

Proof. By assumption $|\mathcal{G}(1)| \geq |\mathcal{F}_0(1)| > \binom{n-2}{k-2} + \binom{n-3}{k-2}$.

Let $2 \leq x \leq n$ be an arbitrary vertex. In view of Corollary 2.5,

$$\big|\mathcal{G}(\bar{1}, x)\big| \leq \big|\mathcal{G}(\bar{1})\big| \leq \binom{n-3}{k-2}.$$

The inequality

$$|\mathcal{G}(1, x)| \leq \binom{n-2}{k-2}$$

is obvious. Therefore $|\mathcal{G}(x)| = |\mathcal{G}(\bar{1}, x)| + |\mathcal{G}(1, x)| \leq \binom{n-2}{k-2} + \binom{n-3}{k-2} < |\mathcal{G}(1)|$. □

Define the parameter r, $4 \leq r \leq k$ by

$$(5.5) \qquad \binom{n-1}{k-1} - \binom{n-(r-1)}{k-1} < \Delta(\mathcal{F}_0) \leq \binom{n-1}{k-1} - \binom{n-r}{k-1}.$$

Let us choose the full tail $\mathcal{T}$ so that $1 \notin T$ for all $T \in \mathcal{T}$. Applying Claim 5.1 to $\mathcal{G} = \mathcal{F}_0 \cup \mathcal{T}$ yields $\Delta(\mathcal{F}_0 \cup \mathcal{T}) = \Delta(\mathcal{F}_0)$. Thus Theorem 2.3 implies

$$(5.6) \qquad \big|\mathcal{F}_0 \cup \mathcal{T}\big| \leq \binom{n-1}{k-1} - \binom{n-r}{k-1} + \binom{n-r}{k-r+1}.$$

Let us first prove (1.3) in the case $n \geq 3k+3$. Using $|\mathcal{B}_r| \leq |\mathcal{B}_k|$ and $\ell(\mathcal{F}) \leq \binom{2k-1}{k-1}$ it is sufficient to show $\binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k}{1} + \binom{2k-1}{k-1} < \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$, or equivalently $\binom{2k-1}{k-1} < \binom{n-k-1}{k-2} - (n-k) + 2$. For $n \geq 3k+3$ the RHS is an increasing function of n. Thus it is sufficient to check the case $n = 3k+3$:

$$\binom{2k-1}{k-1} < \binom{2k+2}{k-2} - 2k - 1 = \binom{2k+1}{k-2} + \left(\binom{2k+1}{k-3} - 2k - 1\right).$$

This inequality is true by (3.2) and $k - 3 \geq 1$.

Now let us turn to the case $k \geq 10$, $3k+2 \geq n \geq 2\big(k + \sqrt{k} + 2\big)$. Recall the definition of r from (5.5).

Using (2.1) and Corollary 2.5 we have

$$\ell = \ell(\mathcal{F}) \leq \min\left\{\binom{2k-1}{k-1}, \binom{n-r+1}{k-r+2}\right\}. \tag{5.7}$$

Let us first consider the case

$$r < \sqrt{k} + 5.$$

We are going to prove (1.3) in the form

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-r}{k-1} + \binom{n-r}{k-r+1} + \binom{2k-1}{k-1} \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1},$$

or equivalently

$$\binom{n-r}{k-r+1} + \binom{2k-1}{k-1} \leq \binom{n-r-1}{k-2} + \binom{n-r-2}{k-2} + \ldots + \binom{n-k-1}{k-2}. \tag{5.8}$$

We want to apply (3.4) to the RHS. Note that $n - s \geq 2k - 4$ is satisfied if $s \leq 2\sqrt{k} + 8$. Since $r < \sqrt{k} + 5$, $\big(2 - 2^{-\sqrt{k}}\big)\binom{n-r-1}{k-2}$ is a lower bound for the RHS. As to $\binom{2k-1}{k-1}$, in view of (3.1) and (3.3) it is very small, e.g.,

$$\binom{2k-1}{k-1} < \text{RHS} \times \left(\frac{4}{3}\right)^{-\sqrt{k}}.$$

As to the main term, $\binom{n-r}{k-r+1}$, using $r \geq 4$ we have

$$\binom{n-r}{k-r+1} \leq \binom{n-r}{k-3} = \binom{n-r-1}{k-2} \frac{(n-r)(k-2)}{(n-r-k+3)(n-r-k+2)} \leq$$

$$\leq \frac{n-4}{n-4-(k-3)} \cdot \frac{k-2}{n-4-(k-2)} \binom{n-r-1}{k-2}.$$

Both factors in the coefficient of $\binom{n-r-1}{k-2}$ are decreasing functions of n. Thus the maximum is attained for $n = 2k + 2\sqrt{k} + 4$ and its value is

$$\frac{2\big(k+\sqrt{k}\big)}{\big(k+\sqrt{k}\big)+\big(\sqrt{k}+3\big)} \cdot \frac{k-2}{k-2+2\sqrt{k}+2} \overset{\text{def}}{=} h(k).$$

To prove (5.7) it is sufficient to show

$$h(k) + \left(\frac{4}{3}\right)^{-\sqrt{k}} < 2 - 2^{-\sqrt{k}}.$$

Since

$$h(k) < \frac{2}{1+\frac{1}{\sqrt{k}}} \cdot \frac{1}{1+\frac{2}{\sqrt{k}}} < 2 - \frac{2}{\sqrt{k}},$$

we are done.

Let us now suppose that $\sqrt{k} + 5 \leq r < k$. We want to establish (1.3) in the form

$$|\mathcal{F}| = \big|\mathcal{F}_0 \cup \mathcal{T}\big| + \ell(\mathcal{F}) < \big|\mathcal{B}_{r+2}\big|.$$

Using (5.6) and (5.7) one sees that the following inequality is sufficient:

$$\binom{n-r}{k-r+1} + \binom{n-r+1}{k-r+2} \leq \binom{n-r-1}{k-2} + \binom{n-r-2}{k-2}.$$

This inequality is the sum of (3.5) applied once for r and once for $r+1$.

The final subcase is $r = k$. Using (5.6) and (5.7) we obtain

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k}{1} + \binom{n-k+1}{2}.$$

To show $|\mathcal{F}| < |\mathcal{B}^+|$ it is sufficient to show

$$\binom{n-k}{1} + \binom{n-k+1}{2} \leq \binom{n-k-1}{3} < \binom{n-k-1}{k-2} + 2. \tag{5.9}$$

The second half of (5.9) is evident from $k \geq 10$ and $n > 2k+4$. To show the first half note that

$$\binom{n-k+1}{1} + \binom{n-k+1}{2} = \binom{n-k+2}{2} < 2\binom{n-k-1}{2},$$

where the last inequality is true for $n-k-1 \geq 8$.

On the other hand, for $n-k-1 \geq 8$ one has also $2\binom{n-k-1}{2} \leq \binom{n-k-1}{3}$, concluding the proof of (5.9). $\square$

$$\text{(c)} \qquad \binom{n-1}{k-1} - \binom{n-k}{k-1} < \Delta(\mathcal{F}_0).$$

In view of Corollary 2.6 we have

$$\left|\mathcal{F}_0(\bar{1})\right| + \ell(\mathcal{F}) \leq k-1. \tag{5.10}$$

On the other hand, having solved the case $\ell(\mathcal{F}) = 1$ in Section 1, we know that $\ell(\mathcal{F}) \geq 2$.

The first two k-subsets of $\binom{[2,n]}{k}$ in the lexicographic order are $[2, k+1]$ and $[2, k] \cup \{k+2\}$. Using Theorem 2.4 we infer

$$\left|\mathcal{F}_0(1)\right| \leq \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-2}{k-2}. \tag{5.11}$$

Adding (5.10), (5.11) and using $\ell(\mathcal{F}) \leq k-1$ we obtain

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-2}{k-2} + 2(k-1).$$

To prove (1.3) we need

$$\binom{n-k-2}{k-2} + 2(k-1) < \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + 2.$$

Rearranging yields

$$2(k-1) < \binom{n-k-2}{k-3} + 2.$$

For $k = 4$ this is simply

$$6 < (n-6) + 2, \qquad \text{i.e.,} \qquad n \geq 11.$$

For $k \geq 5$, $k-3 \geq 2$ and therefore

$$\binom{n-k-2}{2} > 2(k-2) \qquad \text{is sufficient.}$$

This inequality is satisfied for $n \geq 2k+2$. Indeed,

$$\binom{k}{2} = \frac{k}{2}(k-1) > 2(k-2) \qquad \text{already for } \ k \geq 3.$$

This concludes the entire proof. $\square$

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