# A deterministic polynomial time algorithm for non-commutative rational identity testing

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**Abstract**

In this paper we present a deterministic polynomial time algorithm for testing if a symbolic matrix in *non-commuting* variables over $\mathbb{Q}$ is invertible or not. The analogous question for commuting variables is the celebrated polynomial identity testing (PIT) for symbolic determinants. In contrast to the commutative case, which has an efficient probabilistic algorithm, the best previous algorithm for the non-commutative setting required exponential time [IQS15] (whether or not randomization is allowed). The algorithm efficiently solves the "word problem" for the free skew field, and the identity testing problem for arithmetic formulae with division over non-commuting variables, two problems which had only exponential-time algorithms prior to this work.

The main (and simple) technical contribution of this paper is an analysis of an existing algorithm due to Gurvits [Gur04] which solved the same problem in some special cases. We also extend the algorithm to compute the (non-commutative) rank of a symbolic matrix, yielding a factor 2 approximation on the commutative rank. This naturally raises a relaxation of the commutative PIT problem to achieving better deterministic approximation of the commutative rank.

Symbolic matrices in non-commuting variables, and the related structural and algorithmic questions, have a remarkable number of diverse origins and motivations. They arise independently in (commutative) invariant theory and representation theory, linear algebra, optimization, linear system theory, quantum information theory, approximation of the permanent and naturally in non-commutative algebra. We provide a detailed account of some of these sources and their interconnections. In particular we explain how some of these sources played an important role in the development of Gurvits' algorithm and in our analysis of it here.

## 1 Introduction

This introduction will be unusually long, due to the unusual number of settings in which the problems we study appear, and their interconnections and history.

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The main object of study in this paper are *symbolic* matrices whose entries are linear functions in variables $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ over a field[1] $\mathbb{F}$. Any such matrix can be expressed as a linear combination of the variables with matrix coefficients

$$L = x_1 A_1 + x_2 A_2 + \cdots + x_m A_m$$

where $A_1, A_2 \ldots, A_m$ are $n \times n$ matrices[2] over $\mathbb{F}$.

The main computational problem we will be concerned with in this paper (which we call SINGULAR) is determining whether such a symbolic matrix is invertible or not (over the field of fractions in the given variables). This problem has a dual life, depending on whether the variables commute or don't commute. In the *commutative* case this problem has an illustrious history and significance. It was first explicitly stated by Edmonds [Edm67], and shown to have a randomized polynomial time algorithm by Lovasz [Lov79]. It was shown by Valiant [Val79] to capture the celebrated Polynomial Identity Testing (PIT) problem, and so in the commutative setting we will refer to it as PIT. The derandomization of the latter probabilistic algorithm for PIT (namely, proving PIT $\in \mathcal{P}$) became all-important overnight when Kabanets and Impagliazzo [KI04] showed it would imply nontrivial arithmetic or Boolean lower bounds well beyond current reach.

On the other hand, in the *non-commutative* case even the meaning of this problem SINGULAR is unclear. It took decades to fully define and understand the related notion of a "field of fractions" for non-commutative polynomials, namely the *free skew field* over which we (attempt to) invert the matrix[3]. But as we will explain below, this non-commutative SINGULAR problem has many intuitive and clean equivalent formulations (some entirely commutative!). It captures a non-commutative version of identity testing for polynomials and rational functions, provides a possible avenue to attack the notorious commutative PIT version, and quite surprisingly, its different formulations arise naturally in diverse areas of mathematics, revealing surprising connections between them. We only note that unlike the commutative PIT, it is not even clear from its definition that the problem SINGULAR is decidable. It requires some of the very nontrivial characterizations above, and other important results in commutative algebra, to prove a deterministic *exponential time* upper bound on its complexity (see two very different proofs in [CR99, IQS15]), the best known before this work. No better bound was known even allowing randomness.

The main result of this paper is a *deterministic* polynomial time algorithm for this problem!

**Theorem 1.1.** *For non-commutative variables over* $\mathbb{Q}$, SINGULAR $\in \mathcal{P}$.
*More specifically, there is a deterministic algorithm which, given* $m$ $n \times n$ *integer matrices* $A_1, \ldots A_m$ *with entries of bit-size* $b$*, decides in time* $\mathrm{poly}(n, m, b)$ *if the matrix* $L = \sum_{i=1}^{m} x_i A_i$ *is invertible over the free skew field.*

We now formulate the problem SINGULAR for non-commuting variables in its various contexts and incarnations. We will keep throughout an analogy with the commutative case. To avoid

[1] Our main results will be for the rationals $\mathbb{Q}$ (and will hold for the Reals and Complex numbers as well) but not for finite fields. However many of the questions are interesting for any field.

[2] For all purposes we may assume that the matrices $A_i$ are linearly independent, namely span a space of matrices of dimension exactly $m$.

[3] For now, the reader may think of the elements of this "free skew field" simply as containing all expressions (formulas) built from the variables and constants using the arithmetic operations of addition, multiplication and division (we define it more formally a bit later). We note that while this is syntactically the same definition one can use for commuting variables, the skew field is vastly more complex, and in particular its elements cannot be described canonically as ratios of polynomials.

confusion, we will switch to calling commuting variables $y$, and keep $x$ to denote non-commuting variables. As is common let $\mathbb{F}[\mathbf{y}]$ denote the algebra of polynomials in commuting variables over a field $\mathbb{F}$, and $\mathbf{y} = \{y_1, y_2, \ldots, y_m\}$, and $\mathbb{F}(\mathbf{y})$ the field of rational functions over these variables (every element of this field is a ratio of two polynomials in $\mathbb{F}[\mathbf{y}]$). For commuting variables there are many simple equivalent formulations of the problem.

**Fact 1.2** (PIT)**.** *The following are equivalent for a matrix $L = \sum_{i=1}^{m} y_i A_i$ in commuting variables $\mathbf{y}$. Some of these equivalences only hold for large enough fields $\mathbb{F}$.*

*(1) $L$ is singular (namely not invertible) over $\mathbb{F}(\mathbf{y})$.*

*(2) $L$ has nontrivial factors in $\mathbb{F}(\mathbf{y})$. Namely, there exist $r < n$, matrices $K, M$ of dimensions $n \times r, r \times n$ (respectively) with entries in $\mathbb{F}(\mathbf{y})$ such that $L = KM$. The smallest $r$ with this property is called the (commutative)* rank *of $L$, and is denoted* $\mathrm{rank}(L)$.

*(3) The linear space defined by $L$ is* singular*, namely contains no non-singular matrix over $\mathbb{F}$. In other words, for every choice of constants $\beta_i \in \mathbb{F}$ the matrix $\sum_{i=1}^{m} \beta_i A_i$ is singular over $\mathbb{F}$.*

*(4) $Det(L) \equiv 0$ as a polynomial over $\mathbb{F}[\mathbf{y}]$, where Det is the determinant polynomial.*

Now consider the non-commutative situation. The algebra of polynomials over non-commutative variables $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ is familiar enough, and denoted $\mathbb{F}\langle\mathbf{x}\rangle$. Its "field of fractions"[4], which we have yet to define, is denoted $\mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$. As in the commutative case, its elements include all inverses of nonzero polynomials in $\mathbb{F}\langle\mathbf{x}\rangle$, but as we shall see, these elements may be far more complex than ratios of polynomials in $\mathbb{F}\langle\mathbf{x}\rangle$. One can formally define, as we will, the non-commutative SINGULAR problem, exactly as (1) in Fact 1.2 for the commutative case. But then most analogs of these equivalences seem to break down if we take them literally. E.g. (3) clearly doesn't make sense (as it will not distinguish commuting from non-commuting elements), and (4) is ill-defined due to the plethora of non-commutative variants of the determinant polynomial (see e.g. [GGRW02]). However, the beauty of the non-commutative setting is revealed in many more equivalences, which we collect in the following theorem (which in particular has interesting analogs of (2),(3),(4)). These equivalences illustrate some of the connections to optimization, commutative algebra, (commutative) invariant theory, non-commutative algebra and quantum information theory. All of these equivalences will be discussed in this introduction, and elaborated on more formally later. We note that some of them are highly nontrivial theorems we shall later reference individually. To compare between the commutative and non-commutative worlds, and appreciate better the equivalences, it is instructive to have in mind the following example.

**Example 1.3.** *The following $3 \times 3$ skew symmetric matrix is singular over $\mathbb{F}(x, y)$ but not singular over $\mathbb{F}\langle\langle x, y\rangle\rangle$.*

$$\begin{bmatrix} 0 & x & y \\ -x & 0 & 1 \\ -y & -1 & 0 \end{bmatrix}$$

**Theorem 1.4** (SINGULAR)**.** *The following are equivalent for a matrix $L = \sum_{i=1}^{m} x_i A_i$ in non-commuting variables $\mathbf{x}$. Again some of these equivalences only hold for large enough fields $\mathbb{F}$.*

*1. $L$ is singular (namely not invertible) over $\mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$.*

[4] Actually there are many, but only one "universal field of fractions"

2. *$L$ has nontrivial* polynomial *factors in $\mathbb{F}\langle \mathbf{x} \rangle$. Namely, there exist $r < n$, matrices $K, M$ of dimensions $n \times r, r \times n$ (respectively) with entries in $\mathbb{F}\langle \mathbf{x} \rangle$[5] such that $L = KM$. The smallest $r$ with this property is called the* non-commutative rank *(and sometimes* inner rank*) of $L$, and is denoted* nc-rank$(L)$.

3. *The linear space defined by $L$* with matrix coefficients *is* singular. *Namely, for every integer $k$ and every tuple of $k \times k$ matrices $(B_1, B_2, \ldots, B_m)$ over $\mathbb{F}$, the matrix $\sum_{i=1}^{m} B_i \otimes A_i$ is singular over $\mathbb{F}$.*

4. *For every integer $k$ and tuples of $k \times k$ matrices $(B_1, B_2, \ldots, B_m)$ over $\mathbb{F}$, we have Det$(\sum B_i \otimes A_i) = 0$ (where Det is still the commutative determinant polynomial, acting here on $kn \times kn$ matrices.*

5. *$L$ has a* shrunk subspace. *Namely, there are subspaces $U, W$ of $\mathbb{F}^n$, with $\dim W < \dim U$, such that for all $i$ $A_i U \subseteq W$.*

6. *$L$ has a* Hall-blocker *(in analogy with the Hall theorem for perfect matchings). Namely, there exist nonsingular matrices $B, C$ over $\mathbb{F}$ such that the symbolic matrix $BLC$ has an all-zeros minor of size $i \times j$, with $i + j > n$.*

7. *The* completely positive *quantum operator (or map) associated with $L$ is rank-decreasing. Namely, there exist a positive semi-definite matrix $P$ such that rank$(\sum_{i=1}^{m} A_i P A_i^\dagger) <$ rank$(P)$.*

8. *The tuple of matrices $(A_1, A_2, \ldots, A_m)$ is in the* null-cone *of the Left-Right action of $(SL_n(\mathbb{F}))^2$ on $m$ $n \times n$ matrices, namely they evaluate to zero on every non-constant **invariant** polynomial under this action*[6]*.*

In the rest of this introduction we will try to explain how and why the problem SINGULAR arises in these different contexts, how are they related. We will discuss the algorithm that solves SINGULAR (which already appears in [Gur04]), its complexity analysis (which is the main result of this paper), and how they arise from these (and other!) sources. We will also discuss the extension of this algorithm to non-commutative rank computation, and the implications of these algorithms to the various settings. We also highlight the recurring analogs and connections between the commutative and non-commutative worlds. There are probably many ways to weave this meandering story due to the multiple connections between the topics; we chose one (some accounts of subsets of these connections appear e.g. in [Gur04, HW14, IQS15]). Note that most descriptions will be informal or semi-formal; formal definitions for everything that is actually needed will be given in the technical sections, and for the rest we provide references.

### 1.1 The free skew field

Polynomials, both in commuting and non-commuting variables, can always be uniquely defined by their coefficients. And so, while their computational complexity is of course interesting, it is not

[5] Moreover, the polynomial entries of $K, M$ in such a minimal decomposition can actually be taken to be polynomials of *degree 1*, namely *affine* combinations of the variables.

[6] The Left-Right action and its invariant polynomials are defined as follows. Consider $mn^2$ commuting variables $(Y_1, Y_2, \ldots Y_m)$, each a matrix, and consider polynomials in these variables. Every pair $B, C$ of determinant 1 matrices over $\mathbb{F}$ defines a linear map of these variables by sending this tuple to $(BY_1C, BY_2C, \ldots BY_mC)$. A polynomial in these variables is *invariant* if it remains unchanged by this action for every such pair $B, C$.

part of their description. For rational functions this persists in the commutative setting, as they are simply ratios of polynomials. It is perhaps amusing that in the non-commutative setting, the first definition of the *free skew field* of rational functions was computational, namely one whose objects are *described* by their computation via a sequence of arithmetic operations. Both this description, and the discovery of a syntactic representation play an important role in our story.

Books on history and construction of the skew field include [Coh95, Row80], and a thorough discussion from the computational perspective is given in [HW14]. Here we will be relatively brief, highlighting the key roles played by matrices and symbolic matrices in these constructions of the free skew field.

Fix a field $\mathbb{F}$ and a set of non-commuting variables $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$. As before, let $\mathbb{F}\langle\mathbf{x}\rangle$ denote the the (free) algebra of non-commutative polynomials. A major objective is to construct a (minimal) "skew field of fractions", which contains this algebra and is a division ring, namely every non-zero element is invertible. As it happens, there can be many non-isomorphic ways of doing that, but only one that is *universal*[7], the *free skew field* $\mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$, first defined by Amitsur [Ami66] as follows (see a more detailed and precise description in [KVV10]).

A *rational expression* $r(\mathbf{x})$ is simply a formula whose inputs are variables and constants from $\mathbb{F}$, over the standard arithmetic operations of addition, multiplication and division, e.g. $1 + x_1^{-1} + x_2^{-1}$ and $(x_1 + x_2 x + 3^{-1}x_4)^{-1}$. Essentially, the elements of the field will consist of these expressions, modulo an equivalence relation that we now define. Substituting $d \times d$ matrices for the variables, $r$ becomes a (partial) function[8]. Two expressions $r, s$ are *equivalent* if they agree (output the same matrix) for *all* inputs for which they are defined, namely on all tuples of matrices from $M_d(\mathbb{F})$ for all finite $d$.

**Theorem 1.5** ([Ami66])**.** *The rational expressions under the above equivalence relation comprise the (universal) free skew field* $\mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$*.*

One important complication (or perhaps a fountain of interesting problems) in the non-commutative setting is that a non-zero rational expression (or even a polynomial expression, namely one with no inverses) can be an identically zero function for some finite $M_d(\mathbb{F})$. The simple example $x_1x_2 - x_2x_1$ is a *polynomial identity* for $d = 1$, and more generally the *standard polynomial* $\sum_{\pi \in S_{2d}} (-1)^{\mathrm{sgn}(\pi)} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(2d)}$ is a polynomial identity for $M_d(\mathbb{F})$. By the celebrated Amitsur-Levitsky theorem, this example is tight in terms of the relation between the degree of the polynomial identity and dimension of matrices.

**Theorem 1.6** ([AL50])**.** *A degree $k$ polynomial cannot vanish on all inputs in $M_d(\mathbb{F})$ for $d > \lfloor k/2 \rfloor$.*

This theorem immediately implies a polynomial identity test in probabilistic polynomial time (just like in the commutative case) mentioned in the next subsection - simply plug random matrices of appropriate size for the variables. However, for rational expressions, which is our main interest here, such upper bounds on the sufficient dimension to exclude *rational identities* are much harder to prove, and what is known is much weaker. The best bounds (and only for some fields) are exponential, which seem to only provide probabilistic exponential time rational identity tests. The last subsection of this introduction shows how these exponential dimension bounds arise and obtained from invariant theory, and how we use them to nonetheless derive a *deterministic, polynomial* time identity test promised in our main theorem.

[7] This is a technical term which we will not define here.

[8] when the expression attempts to invert a singular matrix it is undefined

A second construction of the free skew field which is somehow more concrete was developed by Cohn. In the notation we have already established of symbolic matrices Cohn proved:

**Theorem 1.7** ([Coh71])**.** *Every element of the free skew field $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$ is an entry of the inverse of some (invertible) symbolic matrix whose entries are polynomials in $\mathbb{F}\langle \mathbf{x} \rangle$.*

For this definition not to be self-referential (and other reasons) Cohn proved some of the important characterizations of invertible matrices, including items (2) and (5) in Theorem 1.4. It is clear that (each entry of) the inverse of such a matrix can be given by a rational expression, and the question of whether a matrix is invertible again arises naturally, and it turns out to capture rational identity testing.

## 1.2 Word Problems and Identity Tests

Word problems and their complexity are central throughout mathematics, arising whenever mathematical objects in a certain class have several representations. In such cases, a basic problem is whether two such representations describe the same object. Often, indeed, some problems of this form have served as early examples of decidable and undecidable problems[9].

For us, the objects in question are polynomials and rational functions in commuting and non-commuting variables. Their standard representations will be arithmetic formulas and circuits, which take the variables (and constants in $\mathbb{F}$) as inputs, and use plus, times and *division* gates. An excellent exposition of arithmetic complexity models and the state-of-art in the subject is [SY10] (which discusses at length the polynomial identity testing problem[10]). We stress that we will specifically allow division gates in our computational models, both because we want to consider rational functions, and because unlike for computing polynomials, in the non-commutative case there is no known analog of Strassen's theorem [Str73] that divisions can be efficiently eliminated. This issue and partial results, many relevant to this paper, are discussed in [HW14]. The word problem in our arithmetic context is equivalent to an Identity Test, checking if a given representation of a polynomial or rational function describes the trivial element, the zero polynomial.

As is well known, the main reason why PIT is so important in the commutative setting is that it captures the polynomial and rational function identity test for formulas, as proved by Valiant [Val79]. Combining it with the equivalences of Fact 1.2, we have

**Theorem 1.8** ([Val79])**.** *There is an efficient algorithm which converts every arithmetic formula $\phi(\mathbf{y})$ in* commuting *variables $\mathbf{y}$ of size $s$ to a symbolic matrix $L_\phi$ of size* poly$(s)$*, such that the rational expression computed by $\phi$ is identically zero if and only if $L_\phi \in$ PIT (i.e., $L_\phi$ is singular).*

Theorem [Coh71] of the previous subsection, showing that in the non-commutative setting as well SINGULAR captures polynomial and rational function identity test, is the analog Valiant's completeness theorem. It was proved even earlier by Cohn (see also Malcolmson's version [Mal78]), using similar linear algebraic ideas. Here is a more precise statement which gives the analogy.

[9] E.g. deciding if two knots diagrams describe the same knot was proved decidable by Haken [Hak61], and deciding if two presentations with generators and relations describe the same group was proved undecidable by Rabin [Rab58]

[10] while this paper focuses on identity testing, we note that our interest is partly motivated by the more basic problem of proving lower bounds for non-commutative circuits. We refer the reader to the papers [Nis91, HWY10, HWY11, LMS15] and their references for existing lower bounds on weaker models, some completeness results, and possible approaches to proving stronger lower bounds

**Theorem 1.9** ([Coh71] )**.** *There is an efficient algorithm which converts every arithmetic formula* $\phi(\mathbf{x})$ *in* non-commuting *variables* $\mathbf{x}$ *of size* $s$ *to a symbolic matrix* $L_\phi$ *of size* $\mathrm{poly}(s)$*, such that the rational expression computed by* $\phi$ *is identically zero if and only if* $L_\phi \in$ SINGULAR*.*

As mentioned, the structure of the free skew field is so complex that unlike the commutative case, even decidability of SINGULAR (and rational identity testing) is far from obvious. The first to prove decidability was Cohn in [Coh73, Coh75]. The first explicit bound on time was given by Cohn and Reutenauer in [CR99], reducing it to a system of commutative polynomial equations using characterization (5) of SINGULAR, proved earlier by Cohn, which puts it in $\mathcal{PSPACE}$. The best upper bound before this work was singly exponential time, obtained by [IQS15], and of course yields the same bound for rational identity testing.

From our Theorem 1.1 we conclude a *deterministic, polynomial* identity test for non-commuting rational expressions.

**Corollary 1.10.** *The non-commutative rational identity testing problem is in* $\mathcal{P}$*. Namely, there is an algorithm which for any non-commutative formula over* $\mathbb{Q}$ *of size* $s$ *and bit complexity* $b$ *determines in* $\mathrm{poly}(s, b)$ *steps if it is identically zero.*

It is important to stress that unlike the commutative case, where symbolic matrix inversion can be simulated by quasi-polynomial sized formulas, in the non-commutative case symbolic matrix inversion is exponentially more powerful than formulas (and so Theorem 1.1 is far more powerful than its corollary above). We state these two results formally below for contrast. Note that when saying "computing the inverse" we mean computing any entry in the inverse matrix.

**Theorem 1.11** ([Hya79, Ber84])**.** *The inverse of a commutative* $n \times n$ *symbolic matrix in* $\mathbb{F}(\mathbf{y})$ *can be computed by formulas of size* $n^{O(\log n)}$*.*

**Theorem 1.12** ([HW14])**.** *The inverse of a non-commutative* $n \times n$ *symbolic matrix in* $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$ *requires formulas of size* $2^{\Omega(n)}$*.*

Our algorithm of Theorem 1.1 and corollary 1.10 is a "white-box" identity test (namely one which uses the description of the given matrix or formula), in contrast to a "black-box" algorithm which only has input-output access to the function computed by them. The best-known "black-box" identity test for these formulas, even if randomization is allowed(!), requires exponential time. It is a very interesting question to find a faster black-box algorithm.

When division is *not* allowed, efficient deterministic identity tests of non-commutative *polynomials* were known in several models. The strongest is for arithmetic branching programs (ABPs). As this model is easily stimulable by matrix inversion (see Theorem 6.5 in [HW14], our algorithm provides an alternative (and very different) proof of the following theorem of Raz and Shpilka [RS05].

**Theorem 1.13** ([RS05])**.** *There is a deterministic polynomial time white-box identity testing algorithm for non-commutative ABPs.*

For certain classes computing only polynomials even efficient black-box algorithms are known, and we mention two below; the first deterministic, for read-once branching programs, and the second probabilistic (but for circuits).

**Theorem 1.14** ([FS13b])**.** *There is a deterministic quasi-polynomial time black-box identity testing algorithm for non-commutative* read-once *ABPs.*

**Theorem 1.15** ([BW05, AL50])**.** *There is a probabilistic polynomial time black-box identity testing algorithm for non-commutative circuits.*

We reiterate the challenge of finding a sub-exponential black-box identity test, even probabilistic, and even for only formulas, that include division. Recall that the best known elimination of division for the non-commutative formulae (discussed in [HW14]) incurs exponential size blow-up.

## 1.3 Commutative and non-commutative rank of symbolic matrices

There are many sources, motivations and results regarding invertibility, and more generally rank, of *commutative* symbolic matrices, which are much older than the complexity theory interest in it. These mathematical papers, like the ones in the computational complexity literature, regard it as a difficult problem, and attempt to understand a variety of special cases (of a different nature than restrictions of the computational power of the model). Some of the many references to this body of work can be found in the papers [FR04, GM02]. Often, the *non-commutative* rank (explicitly or implicitly) is used to give upper bound on the commutative rank, and their relationship becomes of interest. We focus on this connection here, and explain how our main result implies a deterministic *approximation algorithm* to the commutative rank.

In this section we assume that the underlying field $\mathbb{F}$ is infinite. We will use the same notation $L$ for both a symbolic matrix, as well as the subspace of matrices spanned by it (when fixing the variables to constants in the field). We repeat the definitions and elaborate on what is known regarding the commutative and non-commutative ranks, from now on denoted by $\text{rank}(L)$ and $\text{nc-rank}(L)$. Note that the characterizations allow thinking about both without reference to the respective fields of fractions (over which they are most naturally defined), but only to polynomials.

**Fact 1.16.** *The following are equivalent for a commutative matrix $L(\mathbf{y})$.*

- $\text{rank}(L(\mathbf{y})) = r$ *over* $\mathbb{F}(\mathbf{y})$
- $r$ *is the* maximal *rank of any matrix in the subspace $L$ over $\mathbb{F}$.*

While the characterization above is simple, the one below is very substantial, mostly developed by Cohn for his construction of the free skew field. The first characterization is due to him [Coh95] and the second is due to Fortin and Reutenauer [FR04] who heavily use Cohn's techniques.

**Theorem 1.17.** *The following are equivalent for a non-commutative matrix $L(\mathbf{x})$.*

- $\text{nc-rank}(L(\mathbf{x})) = r$ *over* $\mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$
- *The* inner rank *of $L$ over $\mathbb{F}\langle\mathbf{x}\rangle$ is $r$. Namely, $r$ is the* minimal *number such there exists matrices $K, M$ of dimensions $n \times r, r \times n$ (respectively) with* polynomial *entries in $\mathbb{F}\langle\mathbf{x}\rangle$ such that $L = KM$. Moreover, in this decomposition the factors $K, M$ can be assumed to have* affine linear *entries.*
- *The space $L$ is $r$-*decomposable*. Namely, $r$ is the* minimal *number such that there exists invertible matrices $B, C$ over $\mathbb{F}$ such that $BLC$ has a minor of zeros of size $i \times j$ with $i + j = 2n - r$.*

We can extend our main Theorem 1.1 from testing singularity to efficiently computing the non-commutative rank over $\mathbb{Q}$, this is done in Theorem 4.4.

**Theorem 1.18.** *There is a deterministic algorithm which, given* $m$ $n \times n$ *integer matrices* $A_1, \ldots A_m$ *with entries of bit-size* $b$*, computes* nc-rank$(L)$ *in time* poly$(n, m, b)$ *(where* $L = \sum_{i=1}^{m} x_i A_i$*).*

**Remark 1.19.** *We note here that in some formulations of the problem,* $L$ *is represented in* affine form*, namely where an additional constant matrix* $A_0$ *is added to* $L$ *(this may be viewed as a non-commutative analog of the rank-completion problem). However, the algorithm above works in this case as well , since nc-rank*$(L)$ *remains unchanged if an additional variable* $x_0$ *was added as well. So, we will stick with the linear formulation.*

It is not hard to see from the definitions that for every $L$ we have $\text{rank}(L) \leq \text{nc-rank}(L)$. We have already seen that they can be different in Example 1.3. Taking many copies of that $3 \times 3$ matrix we see that there can be a factor $3/2$ gap between the two: for any $r$ there are matrices $L$ with $\text{rank}(L) = 2r$ and $\text{nc-rank}(L) = 3r$. However, Fortin and Reutenauer [FR04] proved that this gap is never more than a factor of 2.

**Theorem 1.20** ([FR04])**.** *For every* $L$ *we have* nc-rank$(L) \leq 2\text{rank}(L)$*.*

An immediate corollary of our main result is thus a factor 2 approximation of commutative rank.

**Corollary 1.21.** *There is a polynomial time algorithm which for every symbolic matrix in* commuting *variables over* $\mathbb{Q}$ *approximates* rank$(L)$ *to within a factor of 2.*

We find the question of obtaining a better approximation ratio efficiently a very interesting problem, a different relaxation of the commutative PIT problem that as far as we are aware was not studied till now.

## 1.4 Compression spaces, optimization and Gurvits' algorithm G

An important class of spaces of matrices $L$, which is studied in many of the papers mentioned above, is the class of *compression spaces* (this notation was apparently introduced by Eisenbud and Harris [EH88]). They are defined as those spaces $L$ for which $L$ is rank$(L)$-decomposable. A simpler definition follows the characterization of [FR04] above, namely a space $L$ is a compression space iff $\text{rank}(L) = \text{nc-rank}(L)$. The importance of compression spaces and their many origins will be surveyed below.

A deterministic polynomial time *commutative* PIT for compression spaces (over $\mathbb{Q}$) was discovered by Gurvits [Gur04], a paper which serves as the starting point for this work.[11] Indeed Gurvits' algorithm, which we will denote by Algorithm $G$, is *the same* one we use here for our main Theorem 1.1; our main contribution here is the *analysis* of its performance for any input $L$, not necessarily a compression space. We will return to the algorithm and its analysis, but first lets discuss its extension from testing singularity to rank computation. Before that, we state Gurvits' theorem, which actually proves more about Algorithm $G$.

**Theorem 1.22** ([Gur04])**.** *There is a deterministic polynomial time algorithm, Algorithm* $G$*, which for every* $n$ *and every* $n \times n$ *matrix* $L$ *given by a set of integer matrices* $(A_1, A_2, \ldots, A_m)$*, outputs*

[11] It is interesting to note that most recent progress on deterministic PIT algorithms (e.g. [KS01, DS06, KS07, SV11, FS13b] among many others) are for polynomials computed by a variety of restricted classes of arithmetic circuits. Algorithm $G$ seems to differ from all of them in solving PIT for a very different class of polynomials, which we do not know how to classify in arithmetic complexity terms.

*"singular" or "invertible", and its output is guaranteed to be correct*[12] *when either* $\text{rank}(L) = n$ *or* $\text{nc-rank}(L) < n$.

In particular, the algorithm always gives the correct answer for compression spaces. Our result on efficient computation of the non-commutative rank implies that for compression spaces we can determine the *commutative rank*.

Many natural spaces are compression spaces, and these arise from a variety of motivations and sources in linear algebra, geometry and optimization. One such source is the attempt to characterize singular spaces (namely when $\text{rank}(L) < n$) by relating the rank to the dimension of the space matrix $L$ defines, denoted $\dim(L)$. Perhaps the earliest result of this type is due to Dieudonné [Die49] who proved that if $L$ is singular and $\dim(L) = n^2 - n$, then for some nonsingular matrices $B, C$ over $\mathbb{F}$ the matrix $BLC$ has an all-zero row or all-zero column. In other words, it has a Hall-blocker as in condition (6) in Theorem 1.4, and so is in particular decomposable. Many other similar results appear in e.g. [Atk80, AL80, Bea87, EH88, GM02], which prove decomposability under more general conditions on the dimension, and study a variety of other properties of singular and rank-bounded spaces.

Yet another source, elaborated on in [Gur04], is geometric duality theorems, such as the ones for matroid intersection and matroid parity problems. These sometimes give rise to compression spaces, with one prototypical example being spaces where the generating matrices $A_i$ all have rank-1; this follows from the Edmonds-Rado [Edm69, Rad42] duality theorem for matroid intersection[13]. Another, simpler example are spaces generated by upper-triangular matrices. Other examples of compression spaces are given in [Gur04]. Following his paper, we note that our rank algorithm above can solve such optimization problems *in the dark*, namely when the given subspace has such *structured* spanning generators, but the generators $A_i$ actually given to the algorithm may be *arbitrary*. This is quite surprising, and in general cannot work by uncovering the original structured generators from the given ones: an efficient algorithm is not known for the problem of deciding if a given space of matrices is spanned by rank-1 matrices and we believe it is hard. It would be interesting to explore which other optimization problems can be encoded as the rank of a compression space, and whether the fact that it can be computed "in the dark" has any applications.

In recent related work, [IKQS15] give a different "in the dark" algorithm for computing the rank of certain compression spaces (including the ones spanned by rank-1 matrices) using so-called second Wong sequences, a linear algebraic analog of "augmenting paths for matchings". Its main advantage is that it works not only over subfields of $\mathbb{C}$ as our algorithm, but also over large enough finite fields. This resolves an open problem in [Gur04].

### 1.5 Permanents, Quantum Operators, Origins and Nature of Algorithm $G$

We will describe the algorithm formally in Section 2.2. Here we give an informal description, which makes it easy to explain its origins and nature. We find these aspects particularly interesting. One (which has very few parallels) is that while the problem SINGULAR solved by Algorithm $G$ in Theorem 1.1 is purely *algebraic*, the algorithm itself is purely *analytic*; it generates from the input a sequence of complex-valued matrices and attempts to discover if it is convergent. Another is that

[12] For both the commutative and non-commutative definitions.

[13] It is an interesting question whether a compression space naturally arises from matroid parity duality of Lovasz [Lov80, Lov89].

the algorithm arises as a quantum analog of another algorithm with very different motivation that we now discuss.

To give more insight to the working of algorithm $G$, let us describe another algorithm (for a different problem) which inspired it, which we call Algorithm $S$. This *matrix scaling* algorithm was developed by Sinkhorn [Sin64] for applications in numerical analysis, and has since found many other applications (see survey and references in [LSW98], who used it as a basis for their deterministic algorithm for approximating the permanent of non-negative matrices). Two different analyses of Sinkhorn's algorithm $S$, one of [LSW98] and the other in the unpublished [GY98] inspire the analysis of Algorithm $G$ in [Gur04].

We describe the [LSW98] analysis for Algorithm $S$. We need a few definitions. For a non-negative matrix $A$, let $R(A)$ denote the diagonal matrix whose $(i,i)$-entry is the inverse of the $L_1$ norm of row $i$ (which here is simply the sum of its entries as $A$ is non-negative). Similarly $C(A)$ is defined for the columns[14].

Algorithm $S$ gets as input a non-negative integer matrix $A$. For a fixed polynomial (in the input size) number of iterations it repeats the following two steps

- Normalize rows: $A \leftarrow R(A) \cdot A$
- Normalize columns: $A \leftarrow A \cdot C(A)$

What does this algorithm do? It is clear that in alternate steps either $R(A) = I$ or $C(A) = I$, where $I$ is the identity matrix. Thus $A$ itself alternates being row-stochastic and column-stochastic. The question is whether both converge to $I$ together, namely, if this process converts $A$ to a *doubly stochastic* matrix. In [LSW98] it is proved that this happens if and only if $\mathrm{Per}(A) > 0$, where Per is the permanent polynomial. Moreover, convergence is easy to detect after a few iterations! If we define $\mathrm{ds}(A) = ||R(A) - I||^2 + ||C(A) - I||^2$ as a notion of distance between $A$ and the doubly stochastic matrices, then the convergence test is simply whether $\mathrm{ds}(A) < 1/n$. If it is that small at the end of the algorithm then $\mathrm{Per}(A) > 0$, otherwise $\mathrm{Per}(A) = 0$.

The analysis of convergence of Algorithm $S$ in [LSW98] is extremely simple, using the permanent itself as a progress measure. It has three parts:

1. The input size provides an exponential lower bound on the starting value of $\mathrm{Per}(A)$,
2. The arithmetic-geometric mean inequality guarantees that it grows by a factor of $1 + 1/n$ at every iteration, and
3. The permanent of any stochastic matrix is upper bounded by 1.

We shall return to this analysis of Algorithm $S$ soon.

As it happens, Algorithm $G$ is a *quantum* analog of Algorithm $S$! In quantum analogs of classical situations two things typically happen, diagonal matrices (which commute) become general matrices (which do not), and the $L_1$ norm is replaced by $L_2$. This happens here as well, and we do so almost syntactically, referring the reader to [Gur04] for their quantum information theoretic intuition and meaning of all notions we mention.

The input to algorithm $G$ is a symbolic matrix $L = \sum_i x_i A_i$, given by the $n \times n$ integer matrices $(A_1, A_2, \ldots, A_m)$. Briefly, $L$ is viewed as a *completely positive (quantum) operator*, or map, on psd matrices, mapping such a (complex valued) matrix $P$ to $L(P) = \sum_i A_i P A_i^\dagger$ ($P$ is

[14] A "non-triviality" assumption is that no row or column in $A$ is all zero.

typically a "density matrix" describing a quantum state, namely a psd matrix with unit trace, and the operator $L$ will typically preserve trace or at least not increase it). The dual operator $L^*$ acts (as you'd expect) by $L^*(P) = \sum_i A_i^\dagger P A_i$. The analog "normalizing factors" for $L$, named $R(L)$ and $C(L)$ are defined[15] by $R(L) = (\sum_i A_i A_i^\dagger)^{-\frac{1}{2}}$, and $C(L) = (\sum_i A_i^\dagger A_i)^{-\frac{1}{2}}$. Note that $R(L) = L(I)^{-\frac{1}{2}}$ and $C(L) = L^*(I)^{-\frac{1}{2}}$.

On input $(A_1, A_2, \ldots, A_m)$ Algorithm $G$ repeats, for a fixed polynomial (in the input size) number of iterations, the following analogous two steps

- Normalize rows: $L \leftarrow R(L) \cdot L$
- Normalize columns: $L \leftarrow L \cdot C(L)$

So again, row and column operations are performed alternately, simultaneously on all matrices $A_i$. It is clear, as above, that after each step either $R(L) = I$ or $C(L) = I$. It is natural to define the case when both occur as "doubly stochastic", and wonder under what conditions does this sequence converges, namely both $R(L)$ and $C(L)$ simultaneously approach $I$, and alternatively the limiting $L$ simultaneously fixes the identity matrix $I$. A natural guess would be that it has to do with a "quantum permanent". Indeed, Gurvits [Gur04] defines such a polynomial (in the entries of the $A_i$) $\mathrm{QPer}(L)$, and proves several properties and characterizations (for example, it is always non-negative like the permanent, and moreover degenerates to the permanent when the operator $L$ is actually a "classical" operator described by a single non-negative matrix $A$).

One can similarly define in an analogous way to the classical setting a "distance from double stochastic" by $\mathrm{ds}(L) = ||R(L) - I||^2 + ||C(L) - I||^2$, and and test (after polynomially many iterations) if $\mathrm{ds} < 1/n$. This is precisely what Algorithm $G$ does. It is not hard to see that if $L$ (with non-commuting variables) is singular, then there is no convergence, the test above fails (and indeed $\mathrm{QPer}(L) = 0$). However, in contrast to Algorithm $S$, the analysis in [Gur04] falls short of proving that, otherwise we have convergence (and $\mathrm{QPer}(L) > 0$). It does so *only* under the strong (commutative) assumption $\mathrm{Det}(L) \neq 0$, namely when the symbolic matrix, viewed with *commuting* variables, is nonsingular. This result was stated above as Theorem 1.22. The 3-step complexity analysis proceeds exactly as in [LSW98] for the classical Algorithm $S$ described above, using the quantum permanent as a progress measure. However, if $\mathrm{Det}(L) = 0$ then $\mathrm{QPer}(L) = 0$, and lower bounding the initial quantum permanent from below in part (1) of that analysis fails.

To prove our main theorem, showing that convergence happens *if and only if* $L$ is non-singular over the non-commutative skew field, we use another ingredient from Gurvits' paper. He proposes another progress measure, called *capacity* and denoted $\mathrm{cap}(L)$. Capacity (defined in the next section) is a quantum analog of another classical analysis of Algorithm $S$ provided in [GY98], based on KL-divergence. The advantage of capacity, as pointed out in [Gur04] is that it does not necessarily vanish when $\mathrm{Det}(L) = 0$. However Gurvits could only bound it sufficiently well from below as a function of the input size if $\mathrm{Det}(L) \neq 0$. Our main contribution is proving such a lower bound for any $L$ nonsingular over the skew field. The source of the proof turns out to be commutative algebra and invariant theory, as we discuss next.

### 1.6 Invariant Theory, Left-Right Action and the Analysis of Algorithm $G$

Invariant theory is a vast field. The books [CLO07, DK02, KP96] provide expositions on the general theory. More focused discussions towards our applications appear in the appendix of [HW14] and

[15] Again using a "non-triviality" assumption these matrices are invertible.

Section 1.2 of [FS13a]. We will exposit here only what is necessary for this paper.

Invariant theory deals with understanding the symmetries of mathematical objects, namely transformations of the underlying space which leave an object unchanged or *invariant*. Such a set of transformations always form a group (and every group arises this way). One major question in this field is, given a group acting on a space, characterize the objects left invariant under all elements of the group. Here we will only discuss very specific space and actions: polynomials (with commuting variables!) that are left invariant under certain linear transformations of the variables. The invariant polynomials under such action clearly form a ring (called the *invariant ring*), and *null-cone* of the action is simply all assignments to variables which vanish on *all* non-constant invariant polynomials.

Here are two motivating examples surely familiar to the reader. The first example is given by polynomials in $n$ variables $y_1, \ldots, y_n$ invariant under the full group of permutations $S_n$; here the invariant polynomials are (naturally) all *symmetric* polynomials. While this is an infinite set, containing polynomials of arbitrarily high degrees, note that this set of polynomials is generated (as an algebra) by the $n+1$ *elementary symmetric polynomials*, with the largest degree being $n$. This finite generation is no coincidence! A general, important theorem of Hilbert [Hil93] assures us that for such linear actions there is always a finite generating set (and hence a finite upper bound on their maximum degree). Obtaining upper bounds on the degree of generating sets, finding descriptions of minimal generating sets for natural actions are the classical goals of this area, and a more modern one[16] is obtaining succinct descriptions and efficient computation of these invariants (e.g. see [Mul12, FS13a]). The case of the action of the symmetric group is an excellent example of a perfect understanding of the invariant polynomial ring. Note that for this action the null-cone is simply the all-zero vector.

The second example, much closer to our interest here, is the "Left-Right" action of the group $(SL_n(\mathbb{F}))^2$ acting on $n^2$ variables $y_{ij}$ arranged in an $n \times n$ matrix $Y$. Specifically, two $n \times n$ matrices (of determinant 1) $(B, C)$ take $Y$ to $BYC$, changing the basis on the left and right. It is not hard to see that all invariant polynomials under this action are generated by one - the determinant $\mathrm{Det}(Y)$ (which again has degree $n$). Note that the null-cone of this action is all singular matrices over the field.

What we consider here is the left-right action on $m$ $n \times n$ matrices, where a pair $(B, C)$ as above takes $(Y_1, Y_2, \ldots, Y_m)$ to $(BY_1C, BY_2C, \ldots, BY_mC)$. The study of the invariant ring of polynomials (in the $mn^2$ variables sitting in the entries of these matrices) for this action was achieved[17] by [DW00, DZ01, SdB01, ANS10], and will look very familiar from condition (4) in Theorem 1.4, as well as from Amitsur's definition of the free skew field[18]

**Theorem 1.23** ([DW00, DZ01, SdB01, ANS10])**.** *Over algebraically closed fields, the invariant ring of polynomials of the left-right action above is generated by all polynomials of the form* $Det(\sum_i D_i \otimes Y_i)$*, for all* $d$ *and all* $d \times d$ *matrices* $D_i$*.*

It is probably worthwhile to stress the connection forged between the commutative and non-commutative worlds by this theorem when combined with Amitsur's and Cohn's constructions of the skew field. A set of matrices $(A_1, A_2, \ldots, A_m)$ is in the null-cone of the left-right action if and

[16] Arising in particular in the GCT program of Mulmuley and Sohoni

[17] We note that this is part of the larger project of understanding *quiver representations*, started by the works of Procesi, Razmysolov, and Formanek [Pro76, Raz74, For86].

[18] Note though that the roles of which matrices in the tensor product are variable, and which are constant, has switched!

only if the symbolic matrix $L = \sum_i x_1 A_i$ is not invertible in the free skew field! In other words, the non-commutative SINGULAR problem (and thus rational identity testing, and the word problem in the skew field) arises completely naturally in commutative algebra. Of course, invertibility itself is invariant under the left-right action (indeed, even by any invertible matrices $B, C$, not necessarily of determinant 1), so one expects a connection in one direction. At any rate, one may hope now that commutative algebraic geometric tools will aid in solving these non-commutative problems, and they do!

The generating set of the invariant ring given in the theorem is still an infinite set, and as mentioned above, Hilbert proved that some finite subset of it is already generating. A natural definition to make now is $\beta(n, m)$, the smallest integer such that taking only matrices with $d \leq \beta(n, m)$ generates the invariant ring. Hilbert did not give an explicit bound. The first to do so was Popov [Pop82], and this bound was significantly improved by Derksen [Der01]. A related quantity $\sigma(n, m)$ is the smallest integer such that taking only matrices with $d \leq \sigma(n, m)$ suffice for testing membership in the null-cone. Derksen [Der01] proved that $\sigma(n, m)$ and $\beta(n, m)$ are polynomially related (over algebraically closed fields of characteristic zero) and that (for algebraically closed fields of characteristic zero) $\sigma(n, m) \leq n^2 4^{n^2}$. This bound was further improved very recently by [IQS15], and extended to finite fields, but remains exponential.

**Theorem 1.24** ([IQS15])**.** $\sigma(n, m) \leq (n + 1)!$

This exponential degree bound already gives a probabilistic exponential time algorithm for SINGULAR. It is not hard to see (by the usual Schwartz-Zippel lemma [Sch80, Zip79, DL78]) that if matrices $D_i$ exist for which $\mathrm{Det}(\sum_i D_i \otimes A_i) \neq 0$, then random ones (indeed, even with entries of poly$(n)$ number of bits!) will do so with high probability (lets call such a choice of $D_i$ *good* for future reference). The randomness in this procedure was eliminated by [IQS15]; they showed that one can use Algorithm $G$ applied to a new (exponentially larger) set of (exponentially larger) matrices $A'_1, A'_2, \ldots, A'_{m'}$ to test singularity of the original set $A_i$. This is quite simple: the subspace spanned by all $D_i \otimes A_i$ for all $D_i$ of dimension $d$ is actually spanned by poly$(d, n)$ matrices of dimension $dn$. Importantly, this subspace is guaranteed to contain a non-singular matrix iff $L$ is invertible in the skew field. Thus in the new subspace of matrices Gurvits' condition on the non-vanishing of the determinant is met, and Algorithm $G$ delivers the correct answer by Theorem 1.22.

So, randomness eliminated, we need only reduce the exponential complexity to polynomial. The key idea is that even if Algorithm $G$ is applied to the original matrices $A_i$, instead of the exponentially larger matrices $A'_i$, its behavior can still be controlled by the the fact that the space spanned by the $A'_i$ is non-singular (even if the space spanned by the $A_i$ is singular). Indeed, there is no need to span the large space (which had exponential cost) - instead the mere existence of *good* set of $m$ matrices $D_i$ suffice. This magic works since the existence of a non-singular matrix in the large space suffices for a large enough lower bound in step (1) of the complexity analysis. For this, we use as our progress measure the *capacity* parameter mentioned in the previous section, which we now define: $\mathrm{cap}(L) = \min \mathrm{Det}(L(P))$, where the minimum is taken over all psd matrices $P$ with $\mathrm{Det}(P) = 1$ [19]. This measure has the extremely nice "composition" property under tensor product. To explain it we return to the quantum language, and besides the completely positive operator $L$ defined by the $A_i$'s (acting on $n \times n$ matrices) consider another one, $M$, defined by the $D_i$ (acting on $d \times d$ matrices). Their tensor product $L \otimes M$ naturally acts on $nd \times nd$ matrices. The main result which enables the analysis is the following, which is proven in Proposition 2.20 of Section 2.

[19] It is worth mentioning that this notion of capacity seems to have nothing to do with the usual capacity of a quantum channel

**Theorem 1.25.** *For any two completely positive operators $L, M$ (of dimensions $n, d$ respectively), their capacity satisfies the following inequality:*

$$cap(L \otimes M) \leq (cap(L))^d (cap(M))^n$$

The proof is extremely simple and follows almost directly from the definition. It allows lower bounding cap$(L)$, as required, by a $d^{th}$ root of cap$(L \otimes M)$, which is positive since this space has a non-singular matrix. Taking the $d^{th}$ root allows us to lower bound (1) by a quantity that is only exponential in $n$, rather than $nd$. This is true for any $d$. The fact that $d$ itself is only exponential (Theorem 1.24) enables us to bound the base of that exponent by a function that is only exponential in the size of the original input $(A_1, A_2, \ldots, A_n)$, via the existence of *good* $D_i$'s. With that lower bound on (1), the 3-step complexity analysis works just like the one in [LSW98] described above, and concludes the proof of our main theorem.

### 1.7 Organization

In Section 2, we describe Algorithm G for testing if quantum operators are rank-decreasing. We explain the notion of capacity of quantum operators, its properties, and how it is used in the analysis to prove that Algorithm G runs in polynomial time. In Section 3, we give (after the necessary formal definitions) the many reincarnations of the SINGULAR problem solved by Algorithm $G$. In Section 4, we show how to compute the non-commutative rank. The reader interested in the computation of the non-commutative rank can skip to Section 4 after reading Sections 3.1.1 and 3.2.2. We conclude in Section 5 with a short discussion and open problems.

## 2 Quantum Operators and Analysis of Algorithm G

This section is devoted to Algorithm G and its analysis. We start with preliminaries about quantum operators, their properties, and basic quantities associated with them. We then formally describe Algorithm G, and proceed to give a full analysis of its running time, proving the main theorem of this paper.

### 2.1 Quantum Operators and Capacity

Given a complex matrix $A$, we will use $A^\dagger$ to denote the conjugate-transpose of $A$. For matrices with real entries this will just be $A^T$. For a matrix $A$, Im$(A)$ will denote the image of $A$ i.e. $\{v \in \mathbb{C}^n | v = Au \text{ for some } u \in \mathbb{C}^n\}$.

**Definition 2.1** (Quantum Operators)**.** An operator (or map) $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called completely positive if there are $n \times n$ complex matrices $A_1, \ldots, A_m$ s.t. $T(X) = \sum_{i=1}^m A_i X A_i^\dagger$. The matrices $A_1, \ldots, A_m$ are called Kraus operators of $T$ (and they are not unique). $T$ is called completely positive trace preserving (cptp) if $T^*(I) = \sum_{i=1}^m A_i^\dagger A_i = I$.

**Remark 2.2.** *The above is actually not the usual definition of completely positive operators. Positive operators are those that map psd matrices into psd matrices. Completely positive matrices map psd matrices to psd matrices even if they act on a small part of the whole matrix. Choi [Cho75] proved that an operator is completely positive iff it is of the form stated above.*

**Remark 2.3.** *We will slightly abuse the name quantum operators and use it to refer to completely positive operators. Usually the name quantum operators is reserved for completely positive trace preserving operators. Also worth noting that $X$ above is just a $n \times n$ complex matrix.*

**Definition 2.4.** If $T(X) = \sum_{i=1}^{m} A_i X A_i^\dagger$ is a quantum operator, we define its dual $T^*$ by $T^*(X) = \sum_{i=1}^{m} A_i^\dagger X A_i$. If both $T$ and $T^*$ are trace preserving, namely $T(I) = T^*(I) = I$ then we call $T$ (and $T^*$) *doubly stochastic.*

**Definition 2.5** (Rank Decreasing Operators)**.** A quantum operator $T$ is said to be rank-decreasing if there exists an $X \succeq 0$ s.t. $\text{rank}(T(X)) < \text{rank}(X)$. $T$ is said to be $c$-rank-decreasing if there exists an $X \succeq 0$ s.t. $\text{rank}(T(X)) \leq \text{rank}(X) - c$.

Now that we defined quantum operators, we define their *capacity*, which is a very important complexity measure of such operators suggested in [Gur04]. Its evolution will be central to the complexity analysis of our algorithm.

**Definition 2.6** (Capacity)**.** [Gur04] The capacity of a quantum operator $T$, denoted by $\text{cap}(T)$, is defined as

$$\text{cap}(T) = \inf\{\text{Det}(T(X)) : X \succ 0,\ \text{Det}(X) = 1\}$$

The next proposition shows how capacity changes when linear transformations are applied (as in the algorithm) to the quantum operator.

**Proposition 2.7** ([Gur04])**.** *Let $T$ be the operator defined by $A_1, \ldots, A_m$ and let $T_{B,C}$ be the operator defined by $BA_1C, \ldots, BA_mC$, where $B, C$ are invertible matrices. Then*

$$cap(T_{B,C}) = |Det(B)|^2 |Det(C)|^2 cap(T)$$

*Proof.*

$$\begin{aligned}
\text{cap}(T_{B,C}) &= \inf\left\{\text{Det}\left(\sum_{i=1}^{m} BA_iCXC^\dagger A_i^\dagger B^\dagger\right) : X \succ 0,\ \text{Det}(X) = 1\right\} \\
&= |\text{Det}(B)|^2 \cdot \inf\left\{\text{Det}\left(\sum_{i=1}^{m} A_iCXC^\dagger A_i^\dagger\right) : X \succ 0,\ \text{Det}(X) = 1\right\} \\
&= |\text{Det}(B)|^2 \cdot \inf\left\{\text{Det}\left(\sum_{i=1}^{m} A_iXA_i^\dagger\right) : X \succ 0,\ \text{Det}(X) = |\text{Det}(C)|^2\right\} \\
&= |\text{Det}(B)|^2 \cdot |\text{Det}(C)|^2 \cdot \text{cap}(T)
\end{aligned}$$

□

The next proposition gives a useful upper bound on the capacity of trace preserving quantum operators; this will be used for the convergence analysis of the algorithm.

**Proposition 2.8** (Capacity Upper Bound)**.** *Let $T$ be a completely positive operator with Kraus operators $A_1, \ldots, A_m$ (which are $n \times n$ complex matrices). Also suppose that either $\sum_{i=1}^{m} A_i^\dagger A_i = I$ or $\sum_{i=1}^{m} A_i A_i^\dagger = I$. Then $cap(T) \leq 1$.*

*Proof.* Note that $\mathrm{tr}(T(I)) = n$ in either case.

$$\begin{aligned}\mathrm{cap}(T) &\leq \mathrm{Det}(T(I)) \leq (\mathrm{tr}(T(I))/n)^n \\ &= 1\end{aligned}$$

The second inequality follows from the AM-GM inequality. □

## 2.2 Algorithm G

We now describe Algorithm G to test if a quantum operator $T$ (given in terms of Kraus operators $A_1, \ldots, A_m$) is rank non-decreasing (equivalently $A_1, \ldots, A_m$ admit no shrunk subspace).

Since the property of $A_1, \ldots, A_m$ having a shrunk subspace or not remains invariant if we replace $A_1, \ldots, A_m$ by a basis spanning the subspace spanned by $A_1, \ldots, A_m$, we can always assume wlog that $m \leq n^2$. Suppose the maximum bit size of the entries of $A_1, \ldots, A_m$ is $b$. Since scaling the matrices $A_1, \ldots, A_m$ doesn't change the problem, we can assume that $A_1, \ldots, A_m$ have integer entries of magnitude at most $M = 2^{O(b)}$.
Algorithm $G$ below is essentially Gurvits' algorithm [Gur04] for Edmonds' problem for subspaces of matrices having Edmonds-Rado property. It is a non-commutative generalization of the Sinkhorn scaling procedure to scale non-negative matrices to doubly stochastic matrices (see for example [LSW98] and references therein). The algorithm alternates applying a "normalizing" basis change from the left and right to the given matrices, so as to alternately make the operator or its dual trace preserving. The idea is that this process will converge to a doubly stochastic operator iff it is not rank-decreasing, and furthermore, the we can bound the number $t$ of iterations by $\mathrm{poly}(N, \log(M))$. We will use the following to measure of our operator to being doubly stochastic.

**Definition 2.9.**

$$\mathrm{ds}(T) = \mathrm{tr}\left[(T(I) - I)^2\right] + \mathrm{tr}\left[(T^*(I) - I)^2\right]$$

It is not hard to see that if a quantum operator $T$ is doubly stochastic, or ever close to being one in this measure, then it is rank non-decreasing.

**Theorem 2.10** ([Gur04])**.** *If $T$ is a quantum operator such that $ds(T) \leq 1/3n$, then $T$ is rank non-decreasing.*

**Definition 2.11.** Given a quantum operator $T$, define its *right normalization* $T_R$ as follows:

$$T_R(X) = T\left(T^*(I)^{-1/2} X T^*(I)^{-1/2}\right)$$

Note that $T_R^*(I) = I$.

**Definition 2.12.** Given a quantum operator $T$, define its *left normalization* $T_L$ as follows:

$$T_L(X) = T(I)^{-1/2} T(X) T(I)^{-1/2}$$

Note that $T_L(I) = I$. These operations are referred to as row and column operations in [Gur04]. We are now ready to present the algorithm.

**Remark 2.13.** *In algorithm G, the operators are maintained in terms of the Kraus operators and it is easy to see the effect of right and left normalizations on the Kraus operators. In fact, they are named such since for a right (left) normalization, the Kraus operators multiplied on the right (left) by $T^*(I)^{-1/2}\left(T(I)^{-1/2}\right)$.*

**Input**: Quantum operator $T$ given in terms of Kraus operators $A_1, \ldots, A_m \in \mathbb{Z}^{n\times n}$. Each entry of $A_i$ has absolute value at most $M$.
**Output**: Is $T$ rank non-decreasing?

1. Check if $T(I)$ and $T^*(I)$ are singular. If any one of them is singular, then output that the operator is rank decreasing, otherwise proceed to step 2.
2. Perform right and left normalizations on $T$ alternatively for $t$ steps. Let $T_j$ be the operator after $j$ steps. Also let $\epsilon_j = \mathrm{ds}(T_j)$.
3. Check if $\min\{\epsilon_j : 1 \le j \le t\} \le 1/12n$. If yes, then output that the operator is rank non-decreasing otherwise output rank decreasing.

**Algorithm 1:** Algorithm G

The following theorem of [Gur04] gives the time analysis of the algorithm *assuming* an initial lower bound on the capacity of the input quantum operator. In the next subsection we will prove our main result, an appropriate lower bound, which shows that the algorithm terminates in polynomial time.

**Theorem 2.14** ([Gur04])**.** *Let $T$ be a completely positive operator that is rank non-decreasing, whose Kraus operators are given by $A_1, \ldots, A_m$, which are $n \times n$ integer matrices such that each entry of $A_i$ has absolute value at most $M$. Suppose $T_R$ is the right normalization of $T$. If $cap(T_R) \ge f(n, M)$, then Algorithm G when applied for $t = 2 + 36n \cdot \log(1/f(n, M))$ steps is correct.*

For completeness sake, we provide a full proof of this theorem. Again, this analysis follows similar ones for the classical Sinkhorn iterations, e.g. as in [LSW98, GY98]. Basically, capacity increases by a factor roughly $1 + 1/36n$ per iteration as long as it is not too close to 1.

*Proof.* It is easy to see that if either of $T(I)$ or $T^*(I)$ is singular, then $T$ decreases the rank of $I$. When $T(I)$ is singular, this is what it means. When $T^*(I)$ is singular, one way to see it is that $A_1, \ldots, A_m$ shrink the full space $\mathbb{C}^n$ since $\mathrm{Im}(A_i) \subseteq \mathrm{Im}\,(T^*(I))$ for all $i$ and hence $T(I)$ is singular by the argument in Proposition 3.25. If $T(I)$ and $T^*(I)$ are both non-singular, it is easy to verify that $T_j(I)$ and $T_j^*(I)$ will remain non-singular for all $j$ and hence step 2 is well defined. Also using Theorem 2.10 and the fact that right and left normalizations don't change the property of being rank decreasing, Algorithm $G$ will always output rank decreasing if $T$ is rank-decreasing.

So what is left to prove is if $T$ is rank non-decreasing, then $\min\{\epsilon_j : 1 \le j \le t\} \le 1/12n$. Assume on the contrary. Denote by $\mathrm{cap}_j$ to be the capacity of the operator $T_j$.

**Claim 2.15.** *If $\epsilon_j > 1/12n$, then $cap_{j+1} \ge exp(1/36n) \cdot cap_j$.*

*Proof.* Let us define

$$S_j = \begin{cases} T_j(I) & j \text{ odd} \\ T_j^*(I) & j \text{ even} \end{cases}$$

It follows from Proposition 2.7 that

$$\mathrm{cap}_{j+1} = \mathrm{Det}(S_j)^{-1} \cdot \mathrm{cap}_j$$

Also $\mathrm{tr}(S_j) = n$ for all $j$ and $\mathrm{tr}\left[(S_j - I)^2\right] > 1/12n$. From here it follows that $\mathrm{Det}(S_j)^{-1} \geq \exp(1/36n)$ (see Lemma 3.10 in [LSW98]) from which the claim follows. □

From the assumption of the theorem, we know that $\mathrm{cap}_1 \geq f(n, M)$. Also it is easy to see that $\mathrm{cap}_j \leq 1$ for all $j$ (Proposition 2.8). However by Claim 2.15 and the assumption that $\min\{\epsilon_j : 1 \leq j \leq t\} > 1/12n$, we get that

$$1 \geq \mathrm{cap}_t \geq \exp\left(\frac{t-1}{36n}\right) \cdot \mathrm{cap}_1 = \exp\left(\frac{t-1}{36n}\right) \cdot f(n, M)$$

Plugging in $t = t = 2 + 36n \cdot \log(1/f(n, M))$ gives us the required contradiction.

□

In the main Theorem 2.21 below we will prove that the quantity $f(n, M)$ used in the statement of Theorem 2.14 is $\geq \exp(-\mathrm{poly}(n, \log(M)))$, which will prove that the number of *iterations* needed in Algorithm $G$ is $\mathrm{poly}(n, \log(M))$. But this alone doesn't guarantee that the algorithm is actually polynomial time, since the bit sizes of numbers involved might get exponential. As it happens, simple truncation suffices to overcome this problem, as shown in [Gur04], and we reproduce this analysis here for completeness after the analysis of capacity.

## 2.3 Analysis of Capacity

We will need the following characterization, due to Choi, of when two sets of matrices define the same quantum operator. This will be useful for making one of the matrices defining the operator have full rank (if the space of matrices contains one).

**Proposition 2.16** ([Cho75])**.** *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be two sets of $n \times n$ complex matrices s.t. there exists a unitary $m \times m$ matrix $U$ s.t.*

$$B_j = \sum_{i=1}^{m} U_{j,i} A_i$$

*Also let $T_A$ be the operator defined by $A_1, \ldots, A_m$ and $T_B$ be the one defined by $B_1, \ldots, B_m$. Then $T_A$ and $T_B$ are the same.*

*Proof.*

$$\begin{aligned}
T_B(X) &= \sum_{j=1}^{m} B_j X A_j^\dagger \\
&= \sum_{j=1}^{m} \left( \sum_{i=1}^{m} U_{j,i} A_i \right) X \left( \sum_{i'=1}^{m} U_{j,i'} A_{i'} \right)^\dagger \\
&= \sum_{j=1}^{m} \sum_{i,i'=1}^{m} U_{j,i} \overline{U_{j,i'}} A_i X A_{i'}^\dagger \\
&= \sum_{i,i'=1}^{m} A_i X A_{i'}^\dagger \left( \sum_{j=1}^{m} U_{j,i} \overline{U_{j,i'}} \right) \\
&= \sum_{i=1}^{m} A_i X A_i^\dagger \\
&= T_A(X)
\end{aligned}$$

□

We will also need the following simple inequality.

**Proposition 2.17.** *Let $A, B$ be two (hermitian) positive semidefinite matrices. Then*

$$Det(A+B) \geq Det(A)$$

*Proof.* Since $v^\dagger(A+B)v \geq v^\dagger Av$ for all vectors $v$, by the variational characterization of eigenvalues, $\lambda_i(A+B) \geq \lambda_i(Y)$ for all $i$ and hence

$$\mathrm{Det}(A+B) = \Pi_{i=1}^n \lambda_i(A+B) \geq \Pi_{i=1}^n \lambda_i(A) = \mathrm{Det}(A)$$

□

The main contribution of this paper is a good lower bound on the capacity of cptp maps which are rank non-decreasing. This will be proved in Theorem 2.21 below. But before proving this theorem, let us look at a very important special case. Suppose we are given a cptp map $T$ which is rank non-decreasing and given by the Kraus operators $A_1, \ldots, A_m$ such that each $A_i \in M_n(\mathbb{Z})$ and moreover, we know that the space of matrices spanned by the $A_i$'s contains a non-singular matrix. In this case, the lemma below uses the existence of this singular matrix to give a very good lower bound on the capacity of $T$.

**Lemma 2.18.** *Suppose $T$ is a cptp map which is rank non-decreasing and is obtained by right normalization of an operator with Kraus operators $A_1, \ldots, A_m$, which are $n \times n$ integer matrices such that each entry of $A_i$ has absolute value at most $M$. Also suppose the subspace of matrices spanned by $A_1, \ldots, A_m$ has a non-singular matrix. Then*

$$cap(T) \geq \frac{1}{n!(M^2 m^2 n^3)^n}.$$

*Proof.* Let $T'$ be the operator defined by $A_1, \ldots, A_m$. Since $\text{Det}\left(\sum_{i=1}^m x_i A_i\right) \neq 0$, by Schwartz-Zippel lemma, there exist integers $z_1, \ldots, z_n$ such that $|z_i| \leq n$ for all $i$ and $\text{Det}\left(\sum_{i=1}^m z_i A_i\right) \neq 0$. Let $u_1 \in \mathbb{R}^m$ s.t.

$$u_1(i) = \frac{z_i}{\sqrt{\sum_{j=1}^m z_i^2}}$$

Extend $u_1$ to get an orthonormal basis $u_1, u_2, \ldots, u_m$ of $\mathbb{R}^m$. Let

$$B_j = \sum_{i=1}^m u_j(i) A_i$$

Let $T_B$ be the operator defined by $B_1, \ldots, B_m$. Then by Proposition 2.16, $T'$ and $T_B$ are the same. Hence

$$\begin{aligned}
\text{cap}(T') = \text{cap}(T_B) &= \inf\left\{\text{Det}\left(\sum_{i=1}^m B_i X B_i^\dagger\right) : X \succ 0,\ \text{Det}(X) = 1\right\} \\
&\geq \inf\left\{\text{Det}\left(B_1 X B_1^\dagger\right) : X \succ 0,\ \text{Det}(X) = 1\right\} \\
&= \text{Det}(B_1)^2
\end{aligned}$$

The inequality follows from Proposition 2.17. Now

$$B_1 = \frac{1}{\sqrt{\sum_{j=1}^m z_i^2}} \cdot \sum_{i=1}^m z_i A_i$$

Since $\sum_{i=1}^m z_i A_i$ is an integer matrix and non-singular,

$$|\text{Det}\left(\sum_{i=1}^m z_i A_i\right)| \geq 1.$$

Thus

$$\begin{aligned}
|\text{Det}(B_1)| &= \frac{|\text{Det}\left(\sum_{i=1}^m z_i A_i\right)|}{\left(\sum_{j=1}^m z_i^2\right)^{n/2}} \\
&\geq \frac{1}{\left(\sum_{j=1}^m z_i^2\right)^{n/2}} \\
&\geq \frac{1}{(mn^2)^{n/2}}
\end{aligned}$$

and hence $\text{cap}(T') \geq \frac{1}{(mn^2)^n}$. Now by Proposition 2.7,

$$\text{cap}(T) = \left(\text{Det}\left(T'^*(I)\right)\right)^{-1} \cdot \text{cap}(T')$$

Since each entry of $T'^*(I)$ is of magnitude at most $M^2 mn$,

$$\text{Det}\left(T'^*(I)\right) \leq n!(M^2 mn)^n$$

which gives us that

$$\text{cap}(T) \geq \frac{\text{cap}(T')}{n!(M^2mn)^n} \geq \frac{1}{n!(M^2m^2n^3)^n}$$

□

**Remark 2.19.** *A similar bound to the lemma above is implicit in [Gur04] using connections between capacity and quantum permanents. But here we gave a much simpler proof, which does not need to use the quantum permanent and its properties.*

If it were true that any rank non-decreasing cptp map had Kraus operators whose linear span has a non-singular matrix, we would be done, as the bound above is enough to give us a polynomial number of iterations! However, this claim is not true in general (as example 1.3 shows). And here is where the connection between quantum operators and invariant theory comes into play. Given two quantum operators $T_1$ and $T_2$, we can define their tensor $T_1 \otimes T_2$, which is also a quantum operator. The proposition below shows that the capacity of the tensor operator $T_1 \otimes T_2$ can be used to lower bound the capacities of $T_1$ or $T_2$.

What do we gain from taking the tensor product of two operators? As it turns out, Theorem 3.7, which answers a question in invariant theory, provides us with the utility of taking tensor products. Theorem 3.7 tells us that for any rank non-decreasing operator $T_1$, there exists an operator $T_2$ with small enough dimension (at least for our purposes) s.t. the operator $T_1 \otimes T_2$ has Kraus operators whose span contains a non-singular matrix. And this is exactly what we need to be able to use Lemma 2.18 in the general case!

Therefore, our strategy will be to lower bound the capacity of out input operator $T_1$ by the capacity of a "nicer" operator $T_1 \otimes T_2$, which fits the conditions of Lemma 2.18. This is what we do next.

**Proposition 2.20.** *Let $T_1$ and $T_2$ be two cptp maps and $T_1$ acts on matrices of dimension $d_1$ and $T_2$ acts on matrices of dimension $d_2$. Then $cap(T_1) \geq cap(T_1 \otimes T_2)^{1/d_2}$.*

*Proof.* We begin by proving that $\text{cap}(T) \geq \text{cap}(T \otimes I_d)^{1/d}$. Suppose $X$ be any matrix s.t. $X \succ 0$ and $\text{Det}(X) = 1$. Then $\text{Det}(X \otimes I_d) = 1$ and $\text{Det}((T \otimes I_d)(X \otimes I_d)) = \text{Det}(T(X) \otimes I_d) = \text{Det}(T(X))^d$. So for any $X$ s.t. $X \succ 0$ and $\text{Det}(X) = 1$ it holds that

$$\text{Det}(T(X)) = \text{Det}((T \otimes I_d)(X \otimes I_d))^{1/d} \geq \text{cap}(T \otimes I_d)^{1/d}$$

Hence $\text{cap}(T) \geq \text{cap}(T \otimes I_d)^{1/d}$. Therefore, we have that

$$\text{cap}(T_1) \geq \text{cap}(T_1 \otimes I_{d_2})^{1/d_2} \geq (\text{cap}(T_1 \otimes I_{d_2}) \times \text{cap}(I_{d_1} \otimes T_2))^{1/d_2} = \text{cap}(T_1 \otimes T_2)^{1/d_2}.$$

Here we have used the fact that for any cptp map $T$, $\text{cap}(T) \leq 1$ (Proposition 2.8) and that $\text{cap}(T_1 \otimes I_{d_2}) \times \text{cap}(I_{d_1} \otimes T_2) = \text{cap}(T_1 \otimes T_2)$. □

Now that we have the lower bound on the capacity of $T_1$ based on the capacity of $T_1 \otimes T_2$, we are ready to state and prove our main theorem.

**Theorem 2.21.** *Suppose $T$ is a cptp map which is rank non-decreasing and is obtained by right normalization of an operator with Kraus operators $A_1, \ldots, A_m$, which are $n \times n$ integer matrices such that each entry of $A_i$ has absolute value at most $M$. Then $cap(T) \geq exp(-poly(n, \log(M)))$.*

*Proof.* Since $T$ is rank non-decreasing, $A_1, \ldots, A_m$ do not admit a shrunk subspace. Hence by Theorem 3.8, there exist (complex) matrices $B_1, \ldots, B_m$ of dimension $(n+1)!$ (let us denote this by $d$) s.t. $\mathrm{Det}(A_1 \otimes B_1 + \cdots A_m \otimes B_m) \neq 0$. If we think of the entries of $B_1, \ldots, B_m$ as formal variables, $\mathrm{Det}(A_1 \otimes B_1 + \cdots A_m \otimes B_m)$ is a nonzero polynomial of degree $\leq nd$ in the variables of $B_1, \ldots, B_m$. Hence by Schwartz-Zippel lemma, there exist integer matrices $B_1, \ldots, B_m$ s.t. the magnitude of each entry in $B_1, \ldots, B_m$ is $\leq nd$ and $\mathrm{Det}(A_1 \otimes B_1 + \cdots A_m \otimes B_m) \neq 0$. Let $T'$ be the cptp map obtained by right normalization of an operator with Kraus operators $\{A_i \otimes B_j\}_{i,j=1}^m$. The subspace of matrices spanned by $\{A_i \otimes B_j\}_{i,j=1}^m$ has a non-singular matrix (namely $A_1 \otimes B_1 + \cdots + A_m \otimes B_m$) and also the magnitude of each entry in $A_i \otimes B_j$ is at most $ndM$. Hence by Lemma 2.18, we have that

$$\begin{aligned}\mathrm{cap}(T') &\geq \frac{1}{(nd)!\left((ndM)^2 (m^2)^2 (nd)^3\right)^{nd}} \\ &\geq \exp(-nd\log(nd)) \cdot \left((ndM)^2 m^4 (nd)^3\right)^{-nd}\end{aligned}$$

Now $T' = T \otimes T''$, where $T''$ is the cptp map obtained by right normalization of an operator with Kraus operators $\{B_j\}_{j=1}^m$. Hence by Proposition 2.20,

$$\begin{aligned}\mathrm{cap}(T) &\geq \mathrm{cap}(T')^{1/d} \\ &\geq \exp(-n\log(nd)) \cdot \left((ndM)^2 m^4 (nd)^3\right)^{-n} \\ &= \exp(-\mathrm{poly}(n, \log(M)))\end{aligned}$$

For the last step, we used $d = (n+1)!$ and $m \leq n^2$, which we can always assume wlog. □

## 2.4 Bit complexity of Algorithm $G$

Here we repeat for completeness, and in more detail, Gurvits' [Gur04] analysis that if one runs Algorithm $G$ while truncating the numbers to an appropriate polynomial number of bits there is essentially no change in the convergence and required number of iterations. We will call the algorithm working with truncated inputs Algorithm $G'$.

Recall that we defined $T_j$ to be the operator after $j$ steps in algorithm G ($T_0 = T$). And

$$S_j = \begin{cases} T_j(I) & j \text{ odd}, j \geq 1 \\ T_j^*(I) & j \text{ even}, j \geq 1 \end{cases}$$

Then $\epsilon_j = \mathrm{tr}\left[(S_j - I)^2\right]$. Also starting with $S_0 = T_0^*(I)$, we get

$$S_j = \begin{cases} T(S_{j-1}^{-1}) & j \text{ odd}, j \geq 1 \\ T^*(S_{j-1}^{-1}) & j \text{ even}, j \geq 1 \end{cases}$$

Given a matrix $A$, let $\mathrm{Trn}(A)$ be the matrix obtained by truncating the entries of A up to $P(n, \log(M))$ bits after the decimal point. $P(n, \log(M))$ is a polynomial which we will specify later. Note that $||A - \mathrm{Trn}(A)||_\infty \leq 2^{-P(n,\log(M))}$. We now describe Algorithm $G'$, which is the variant of Algorithm $G$ with truncation.

**Input**: Quantum operator $T$ given in terms of Kraus operators $A_1, \ldots, A_m \in \mathbb{Z}^{n\times n}$. Each entry of $A_i$ has absolute value at most $M$.
**Output**: Is $T$ rank non-decreasing?

1. Check if $T(I)$ and $T^*(I)$ are singular. If any one of them is singular, then output that the operator is rank decreasing, otherwise proceed to step 2.
2. Let $U_0 = S_0$. for($j = 1$ to $t$): $U_j = \mathrm{Trn}(T(U_{j-1}^{-1}))$ if $j$ odd and $U_j = \mathrm{Trn}(T^*(U_{j-1}^{-1}))$ if $j$ even. Let $\tilde{\epsilon}_j = \mathrm{tr}\left[(U_j - I)^2\right]$
3. Check if $\min\{\tilde{\epsilon}_j : 1 \le j \le t\} \le 1/6n$. If yes, then output that the operator is rank non-decreasing otherwise output rank decreasing.

**Algorithm 2:** Algorithm $G'$ (with truncation)

$t$ will be chosen as before: $t = 2 + 36n \cdot \log(1/f(n, M))$. We now show that throughout the iterations, distances to double stochasticity is essentially the same for the original and truncated algorithms $G$ and $G'$.

**Lemma 2.22.** *For an appropriate choice of $P(n, \log(M))$, $|\epsilon_j - \tilde{\epsilon}_j| \le 1/12n$, for $1 \le j \le t = 2 + 36n \cdot \log(1/f(n, M))$.*

Let us first prove the correctness of Algorithm $G'$ assuming Lemma 2.22. As before, we can assume that $T(I)$ and $T^*(I)$ are both non-singular. If $\min\{\tilde{\epsilon}_j : 1 \le j \le t\} \le 1/6n$, then by Lemma 2.22,

$$\min\{\epsilon_j : 1 \le j \le t\} \le 1/6n + 1/12n \le 1/3n$$

Hence by Theorem 2.10, $T$ is rank non-decreasing. Now assume, on the contrary, that $T$ is rank non-decreasing. Then by the analysis in previous section $\min\{\epsilon_j : 1 \le j \le t\} \le 1/12n$. Hence by Lemma 2.22

$$\min\{\tilde{\epsilon}_j : 1 \le j \le t\} \le 1/12n + 1/12n = 1/6n$$

This proves the correctness of Algorithm $G'$.

**Proposition 2.23.** *For a real symmetric positive definite matrix $X$, let $l(X)$ denote its smallest eigenvalue and $u(X)$ its largest. Suppose $\alpha = (M^2n^2m)^{n-1}$.*

1. *Suppose $A$ is an $n \times n$ real symmetric positive definite matrix with integer entries s.t. the magnitude of any entry of $A$ is at most $M$. Then $l(A) \ge \frac{1}{(Mn)^{n-1}}$ and $u(A) \le Mn$.*
2. *Let $T$ be a completely positive operator whose Kraus operators $A_1, \ldots, A_m$ are $n \times n$ integer matrices and each entry of $A_i$ is of magnitude at most $M$. Also $T(I), T^*(I)$ are both non-singular. Then*
$$(M^2n^2m)I \succeq T(I), T^*(I) \succeq \frac{1}{(M^2n^2m)^{n-1}} I$$
3. *Let $T$ be as before. Let $X$ be a real symmetric matrix. Then $||T(X)|| \le \alpha||X||$. Also if $X$ is real symmetric positive definite, then $l(T(X)) \ge \alpha^{-1} \cdot l(X)$ and $u(T(X)) \le \alpha \cdot u(X)$. Similarly $||T^*(X)|| \le \alpha||X||$, $l(T^*(X)) \ge \alpha^{-1} \cdot l(X)$ and $u(T^*(X)) \le \alpha \cdot u(X)$*

4. *$S_j$ are real symmetric positive definite matrices for all $0 \leq j \leq t$. Also $l(S_j) \geq \alpha^{-(j+1)}$ and $u(S_j) \leq \alpha^{j+1}$ for all $0 \leq j \leq t$.*

5. *For all $1 \leq k, l \leq n$, $|S_j(k,l)| \leq \alpha^{j+1}$.*

*Proof.* 1. Let us first prove that $u(A) \leq Mn$.

$$\begin{aligned}
u(A) &= \max_{v \text{ st. } ||v||_2 = 1} ||Av||_2 \\
&= \max_{v \text{ st. } ||v||_2 = 1} \sqrt{\sum_{k=1}^{n} \left( \sum_{l=1}^{n} A_{k,l} v_l \right)^2} \\
&\leq \max_{v \text{ st. } ||v||_2 = 1} \sqrt{\sum_{k=1}^{n} \left( \sum_{l=1}^{n} M|v_l| \right)^2} \\
&= \max_{v \text{ st. } ||v||_2 = 1} \sqrt{n} \cdot M \cdot \left( \sum_{l=1}^{n} |v_l| \right) \\
&\leq nM
\end{aligned}$$

The last inequality follows from Cauchy-Schwarz. Now since $A$ is positive definite and the determinant is an integer, $\text{Det}(A) \geq 1$. Then

$$1 \leq \text{Det}(A) \leq u(A)^{n-1} l(A) \leq (Mn)^{n-1} l(A)$$

from which it follows that $l(A) \geq \frac{1}{(Mn)^{n-1}}$.

2. Follows from previous point by noting that each entry of $T(I), T^*(I)$ is an integer of magnitude at most $M^2 n^2 m$ and $T(I), T^*(I)$ are both symmetric matrices.

3. We know that

$$\alpha I \succeq T(I), T^*(I) \succeq \alpha^{-1} I$$

Suppose $||X|| = \beta$. Then

$$\beta I \succeq X \succeq -\beta I$$

Then

$$T(X + \beta I) \succeq 0$$

since $T$ is completely positive and hence it maps psd matrices to psd matrices. Thus we get

$$T(X) + \beta T(I) = T(X + \beta I) \succeq 0$$

This follows from linearity of $T$. Then

$$T(X) \succeq -\beta T(I) \succeq -\beta\alpha I$$

Similarly one can prove that $\beta\alpha I \succeq T(X)$ which would imply $||T(X)|| \leq \alpha\beta = \alpha||X||$. Other statements can be proven in a similar manner.

4. We will prove this via induction on $j$. It is true for $S_0$ by point 2 in the proposition. Suppose the statement holds for $S_{j-1}$. Then $S_j = T(S_{j-1}^{-1})$ or $S_j = T^*(S_{j-1}^{-1})$. It does not really matter which case it is for the purpose of this proof. Lets assume the former. Then

$$\begin{aligned} l(S_j) &= l(T(S_{j-1}^{-1})) \\ &\geq \alpha^{-1} \cdot l(S_{j-1}^{-1}) \\ &= \alpha^{-1} \cdot u(S_{j-1})^{-1} \\ &\geq \alpha^{-1} \cdot \alpha^{-j} \\ &= \alpha^{-(j+1)} \end{aligned}$$

The first inequality is by point 3 in the proposition. Second inequality is by induction hypothesis. Similarly we can also prove that $u(S_j) \leq \alpha^{j+1}$, which would complete the induction step.

5.

$$\begin{aligned} |S_j(k,l)| &= |e_k^T S_j e_l| \\ &\leq ||e_k||_2 ||S_j e_l||_2 \\ &\leq \alpha^{j+1} \end{aligned}$$

Here $e_k$ and $e_l$ are the standard basis vectors.

□

Note that

$$U_j = \begin{cases} T(U_{j-1}^{-1}) + \delta_j & j \text{ odd}, j \geq 1 \\ T^*(U_{j-1}^{-1}) + \delta_j & j \text{ even}, j \geq 1 \end{cases}$$

where $||\delta_j||_\infty \leq 2^{-P(n,\log(M))}$ for all $1 \leq j \leq t$.

**Lemma 2.24.** *Suppose $\alpha = (M^2n^2m)^{n-1}$. If $||\delta_j|| \leq \frac{1}{2^j \cdot \alpha^{j+1}}$ for all $1 \leq j \leq t$, then $l(U_j) \geq \frac{1}{2^j \cdot \alpha^{j+1}}$ and $u(U_j) \leq 2^j \cdot \alpha^{j+1}$, for all $0 \leq j \leq t$.*

*Proof.* We will prove this using induction on $j$. It is true for $j = 0$ since $U_0 = S_0$. Assume that $l(U_j) \geq \frac{1}{2^j \cdot \alpha^{j+1}}$ and $u(U_j) \leq 2^j \cdot \alpha^{j+1}$. Then $U_{j+1} = T(U_j^{-1}) + \delta_{j+1}$ or $U_{j+1} = T^*(U_j^{-1}) + \delta_{j+1}$. Suppose it is $T(U_j^{-1}) + \delta_{j+1}$. The other case is similar.

$$\begin{aligned} l(U_{j+1}) &\geq l(T(U_j^{-1})) - ||\delta_{j+1}|| \\ &\geq \alpha^{-1} \cdot l(U_j^{-1}) - ||\delta_{j+1}|| \\ &= \alpha^{-1} \cdot u(U_j)^{-1} - ||\delta_{j+1}|| \\ &= \alpha^{-1} \cdot \frac{1}{2^j \alpha^{j+1}} - \frac{1}{2^{j+1}\alpha^{j+2}} \\ &= \frac{1}{2^{j+1}\alpha^{j+2}} \end{aligned}$$

Also

$$\begin{aligned} u(U_{j+1}) &\leq u(T(U_j^{-1})) + ||\delta_{j+1}|| \\ &\leq \alpha \cdot u(U_j^{-1}) + ||\delta_{j+1}|| \\ &= \alpha \cdot l(U_j)^{-1} + ||\delta_{j+1}|| \\ &\leq \alpha \cdot 2^j \alpha^{j+1} + \frac{1}{2^{j+1}\alpha^{j+2}} \\ &\leq 2^{j+1} \cdot \alpha^{j+2} \end{aligned}$$

Note that there is a lot of slack in this part, but it wouldn't matter for our purposes. □

**Lemma 2.25.** *Suppose* $\alpha = (M^2n^2m)^{n-1}$ *and* $||\delta_j|| \leq \delta$ *for all* $1 \leq j \leq t$*, where* $\delta \leq \frac{1}{(2\alpha)^{t+1}}$*. Then*

$$||S_j - U_j|| \leq (2\alpha)^{(2t+1)\cdot(t+1)}\delta$$

*for all* $0 \leq j \leq t$*.*

*Proof.*

$$\begin{aligned} ||S_j - U_j|| &= ||T(S_{j-1}^{-1}) - T(U_{j-1}^{-1}) - \delta_j|| \\ &\leq ||T(S_{j-1}^{-1} - U_{j-1}^{-1})|| + \delta \\ &\leq \alpha \cdot ||S_{j-1}^{-1} - U_{j-1}^{-1}|| + \delta \\ &= \alpha \cdot ||S_{j-1}^{-1}(U_{j-1} - S_{j-1})U_{j-1}^{-1}|| + \delta \\ &\leq \alpha \cdot ||S_{j-1}^{-1}|| \cdot ||S_{j-1} - U_{j-1}|| \cdot ||U_{j-1}^{-1}|| + \delta \\ &\leq 2^{j-1}\alpha^{2j+1} \cdot ||S_{j-1} - U_{j-1}|| + \delta \\ &\leq (2\alpha)^{2t+1} \cdot ||S_{j-1} - U_{j-1}|| + \delta \end{aligned}$$

For the fourth inequality we used Proposition 2.23 and Lemma 2.24. Note that value of $\delta$ is small enough so that conditions of Lemma 2.24 are satisfied. □

*Proof.* (Of Lemma 2.22) $\epsilon_j = \text{tr}[(S_j - I)^2]$ and $\tilde{\epsilon_j} = \text{tr}[(U_j - I)^2]$. So

$$\begin{aligned} |\epsilon_j - \tilde{\epsilon_j}| &\leq |\text{tr}(S_j^2 - U_j^2)| + 2|\text{tr}(S_j - U_j)| \\ &= |\text{tr}[(S_j + U_j)(S_j - U_j)]| + 2|\text{tr}(S_j - U_j)| \\ &\leq 2\alpha^{j+1} \sum_{k,l=1}^{n} |(S_j - U_j)(k,l)| + 2n||S_j - U_j|| \\ &\leq 2\alpha^{j+1} \cdot n \sqrt{\sum_{k,l=1}^{n} |(S_j - U_j)(k,l)|^2} + 2n||S_j - U_j|| \\ &\leq 2\alpha^{j+1} \cdot n \cdot \sqrt{n} \cdot ||S_j - U_j|| + 2n||S_j - U_j|| \\ &\leq 4\alpha^{t+1} \cdot n^{3/2} \cdot (2\alpha)^{(2t+1)\cdot(t+1)}\delta \end{aligned}$$

In the first equality, we used the fact that $\text{tr}(S_jU_j) = \text{tr}(U_jS_j)$. In the second inequality, we used the fact that the maximum magnitude of any entry in $S_j + U_j$ is bounded by $2\alpha^{j+1}$ and that

$|\mathrm{tr}(S_j - U_j)|$ is upper bounded by $n||S_j - U_j||$. The third inequality is just Cauchy-Schwarz and the fourth is the fact that Frobenius norm of a matrix is upper bounded by $n||S_j - U_j||^2$. The last inequality is by application of Lemma 2.25.

Now $\alpha = (M^2n^2m)^{n-1} = \exp(\Theta(n\log(n)\log(M)))$ (since $m \leq n^2$). Hence

$$\begin{aligned} 4\alpha^{t+1} \cdot n^{3/2} \cdot (2\alpha)^{(2t+1)\cdot(t+1)} &= \exp(\Theta(n\log(n)\log(M) \cdot t^2)) \\ &= \exp\left(\Theta\left(n^3\log(n)\log(M)\log^2(1/f(n,M))\right)\right) \end{aligned}$$

Since $||\delta_j||_\infty \leq 2^{-P(n,\log(M))}$, $||\delta_j|| \leq n2^{-P(n,\log(M))}$. Hence choosing

$$P(n,\log(M)) = n^4\log(M)\log^2(1/f(n,M))$$

suffices to get

$$|\epsilon_j - \tilde{\epsilon_j}| \leq 1/12n$$

We proved in Theorem 2.21 that $\log(1/f(n,M))$ is $\mathrm{poly}(n,\log(M))$, so $P(n,\log(M))$ is also a polynomial in $n$ and $\log(M)$. □

# 3 Problems solved by Algorithm $G$

In this section we define more formally the various reincarnations, in different mathematical areas, of the problem SINGULAR solved by algorithm $G$. We start in Subsection 3.1 with more formal definitions of the necessary concepts from these various areas (many given semi-formally in the introduction), proceed in Subsection 3.2 with formal definitions of the problem in each area, and conclude with a formal statement of what Algorithm $G$ achieves for all.

## 3.1 Preliminary definitions

### 3.1.1 Free Skew Field

Let $\mathbf{x} = \{x_1, \ldots, x_m\}$ be a set of non-commuting variables. We will denote by $\mathbb{F}\langle\mathbf{x}\rangle$, the ring of polynomials in non-commuting variables $x_1, \ldots, x_m$ over the (commuting) field $\mathbb{F}$. We will denote by $\mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$, the free skew (non-commutative) field of rational functions in non-commuting variables $x_1, \ldots, x_m$ over the (commuting) field $\mathbb{F}$. We will describe below the original construction of Amitsur [Ami66] to describe this skew field.

A non-commutative circuit $\Phi$ (with inversions) over a (commutative) field $\mathbb{F}$ and non-commuting variables $x_1, \ldots, x_m$ is a directed acyclic graph. Nodes of in-degree zero are labelled by a variable or a field element from $\mathbb{F}$. Nodes of in-degree one are labelled by $^{-1}$ and nodes of in-degree two are labelled by a $+$ or $\times$. The two edges going into a node labelled by $\times$ are labelled *left* and *right* to specify the order of multiplication. The nodes are called input gates, inverse gates and sum and product gates respectively. The nodes of out-degree zero are called output gates. The size of a circuit $\Phi$ is the number of gates in it. A formula is a circuit all of whose nodes/gates have out-degree at most one.

A node $v \in \Phi$ is supposed to compute a non-commutative rational function $\hat{v} \in \mathbb{F}\langle\langle\mathbf{x}\rangle\rangle$. However $\Phi$ may contain division by zero, in which case $\hat{v}$ would be undefined. The definition of $\hat{v}$ is the following:

1. If $v$ is an input gate, labelled by $a$ (variable or a field element), then $\hat{v} \coloneqq a$.

2. If $v$ is a sum or a product gate and $v = v_1 + v_2$ or $v = v_1 \times v_2$, then $\hat{v} \coloneqq \hat{v_1} + \hat{v_2}$ or $\hat{v} \coloneqq \hat{v_1} \times \hat{v_2}$ respectively and assuming that $\hat{v_1}$ and $\hat{v_2}$ are both defined.

3. If $v$ is an inverse gate and $v = u^{-1}$, then $\hat{v} \coloneqq \hat{u}^{-1}$ if $\hat{u}$ is defined and $\hat{u} \neq 0$.

We say that $\Phi$ is a *correct circuit* if $\hat{u}$ is defined for every gate $u \in \Phi$.
Note that unlike the commutative case, a non-commutative rational expression might not be expressible as a ratio of two polynomials and in fact, there could be an unbounded number of nested inversions in such expressions [Reu96]. But still there are nontrivial rational identities in the non-commutative setting. One example is Hua's identity, $(x + xy^{-1}x)^{-1}$ is equivalent to the expression $x^{-1} - (x + y)^{-1}$. So it is highly nontrivial to even define the free skew field. The first construction of the free skew field was given by Amitsur [Ami66]. Subsequently, alternate constructions were given by Bergman [Ber70], Cohn [Coh71], Malcolmson [Mal78].

Let $\Phi$ be a circuit and $(B_1, \ldots, B_m)$ be a tuple of matrices. Then we will say that $\Phi$ is defined on $(B_1, \ldots, B_m)$ if all the inversions in $\Phi$ are possible when evaluated on $B_1, \ldots, B_m$. Amitsur proved that the set of all circuits (or equivalently formulas) modulo this equivalence relation form a skew field.

**Definition 3.1.** Let $\Phi$ be a non-commutative circuit with inversions.

$$\mathrm{Dom}(\Phi) \coloneqq \{(B_1, \ldots, B_m) : \Phi \text{ defined on } (B_1, \ldots, B_m)\}$$

Two circuits $\Phi_1$ and $\Phi_2$ are said to be equivalent if $\hat{\Phi}_1(B_1, \ldots, B_m) = \hat{\Phi}_2(B_1, \ldots, B_m)$ for all $(B_1, \ldots, B_m) \in \mathrm{Dom}(\Phi_1) \cap \mathrm{Dom}(\Phi_2)$ (and $\mathrm{Dom}(\Phi_1) \cap \mathrm{Dom}(\Phi_2) \neq \phi$).
We refer the reader to [KVV10, HW14] and references therein for further information about the free skew field.

### 3.1.2 Shrunk Subspace

The notion of a shrunk subspace will be very crucial for us.

**Definition 3.2.** Let $A_1, \ldots, A_m \in M_n(\mathbb{F})$. $A_1, \ldots, A_m$ are said to admit a shrunk subspace if there exist subspaces $V, W \subseteq \mathbb{F}^n$ s.t. $\dim(W) < \dim(V)$ and $A_i(V) \subseteq W$ for $1 \leq i \leq m$ ($V$ is the shrunk subspace here, shrunk to $W$ simultaneously by all $A_i$). $A_1, \ldots, A_m$ are said to admit a $c$-shrunk subspace if there exist subspaces $V, W \subseteq \mathbb{F}^n$ s.t. $\dim(W) \leq \dim(V) - c$ and $A_i(V) \subseteq W$ for $1 \leq i \leq m$.

### 3.1.3 Invariant Theory

In this section, we give the relevant background from Invariant Theory. We will give only a high level overview needed to understand the problems and the results in this paper. One can find much more in the books [KP96, DK02]. In this section, all the variables *commute*!
Fix a field $\mathbb{F}$. Let $G$ be a group and $V$ be its representation, namely a vector space on which $G$ acts. The group $G$ then also acts on the polynomial functions on $V$, $\mathbb{F}[V]$, which is also called the *coordinate ring* of $V$. In our case $V$ will be finite dimensional (say $m$), so $\mathbb{F}[V]$ will be simply be $\mathbb{F}[\mathbf{y}]$, where $\mathbf{y} = \{y_1, \ldots, y_m\}$. For a polynomial $p(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$, we will denote by $gp$ the action of $g \in G$ on $p$. A polynomial is said to be invariant if $gp = p$ for all $g \in G$. The following is a well studied example:

**Example 3.3.** *Let $G = S_m$ acts on the set of formal variables* $\mathbf{y}$ *(and hence on the vector spaces defined by them) by permuting them. Then the set of invariant polynomials are all the symmetric polynomials.*

The *null-cone* of an action is the set of all common zeroes of the non-constant invariant polynomials of the action.

We will be interested in the so called left-right action. Here a pair of matrices $(B, C) \in SL_n(\mathbb{F}) \times SL_n(\mathbb{F})$ acts on $(X_1, \ldots, X_m)$ to give $(BX_1C, \ldots, BX_mC)$. The variables are the entries of $X_1, \ldots, X_m$, so there are $mn^2$ variables in total.
Explicit generating sets for the invariant ring of polynomials of the left-right action have been well studied and the following is known about them:

**Theorem 3.4** ([DW00, DZ01, SdB01, ANS10])**.** *Over algebraically closed fields $\mathbb{F}$, the set*

$$\left\{ Det\left(\sum_{i=1}^{m} X_i \otimes B_i\right) : k \in \mathbb{N}, B_i \in M_k(\mathbb{F}) \right\}$$

*is a generating set for the ring of invariant polynomials of the left-right action.*

These are in fact special cases of more general results on invariant rings of arbitrary quivers. Explicit exponential bounds on the degree of polynomials required to generate the ring of invariant polynomials are known. It is in fact a very important open problem to obtain polynomial degree bounds for the generating set. See [HW14] for some applications which include eliminating divisions from non-commutative circuits.
The null-cone of the left-right action (over algebraically closed fields of arbitrary characteristic) was studied in [BD06, ANS10], who proved the following theorem:

**Theorem 3.5** ([BD06, ANS10])**.** *$(A_1, \ldots, A_m)$ is in the null-cone of the left-right action iff $A_1, \ldots, A_m$ admit a shrunk subspace.*

We state the degree bounds for the generating set of invariant polynomials in terms of the dimension of matrices required to test the null-cone property. But these are essentially equivalent to the degree bounds by [Der01] (over algebraically closed fields of characteristic zero).

**Theorem 3.6** ([Der01])**.** *Let $A_1, \ldots, A_m$ be $n \times n$ complex matrices such that they are not in the null-cone of the left-right action. Then there exist matrices $B_1, \ldots, B_m$ of dimension $n^2 4^{n^2}$ s.t.*

$$Det(A_1 \otimes B_1 + \cdots A_m \otimes B_m) \neq 0$$

**Theorem 3.7** ([IQS15])**.** *Let $A_1, \ldots, A_m$ be $n \times n$ complex matrices that are not in the null-cone of the left-right action. Then there exist matrices $B_1, \ldots, B_m$ of dimension $(n+1)!$ s.t.*

$$Det(A_1 \otimes B_1 + \cdots A_m \otimes B_m) \neq 0$$

By combining Theorem 3.22 and Theorem 3.7, we get the following theorem which will be very crucial for us:

**Theorem 3.8.** *Let $A_1, \ldots, A_m$ be $n \times n$ complex matrices such that they do not admit a shrunk subspace. Then there exist matrices $B_1, \ldots, B_m$ of dimension $(n+1)!$ s.t.*

$$Det(A_1 \otimes B_1 + \cdots A_m \otimes B_m) \neq 0$$

## 3.2 Problem Definitions

In this section, we formally state and give some background to all incarnations of the problems solved by Algorithm G. These problems arise from several disparate fields which are all magically interconnected. In most of these problems, the inputs will essentially be a tuple of $n \times n$ matrices with rational entries $A_1, \ldots, A_m$. We will denote the maximum bit size of the entries of these matrices by $b$. We give deterministic poly$(n, m, b)$ time algorithms for all of these problems.

### 3.2.1 SINGULAR

In the (non-commutative) SINGULAR problem, the input is a matrix $L \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ whose entries are linear polynomials and the question is to check whether $L$ is singular (not invertible) over $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$. The problem makes sense over any field but our results will hold when $\mathbb{F} = \mathbb{Q}$ (most of the theory works over $\mathbb{C}$ but to account for input size, we only work over the rationals).

**Problem 3.9.** *SINGULAR*
***Input**: $A_1, \ldots, A_m \in M_n(\mathbb{Q})$ interpreted as $L = \sum_{i=1}^{m} x_i A_i$*
***Output**: Is $L$ singular over the free skew field $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$?*

### 3.2.2 Computing the Non-Commutative Rank and Fullness Testing

We also study the problem of computing the rank of a matrix over the free skew field. It turns out that there are various interesting notions of rank which coincide. The two main ones for us will be the inner rank and the rank over the free skew field (which is unambiguously defined).

**Definition 3.10** (Inner Rank)**.** Suppose $L$ is a matrix in $\mathbb{F}\langle \mathbf{x} \rangle^{n \times p}$. The inner rank of $L$ is defined as the minimum $r$ such that $L = KM$, where $K \in \mathbb{F}\langle \mathbf{x} \rangle^{n \times r}$ and $M \in \mathbb{F}\langle \mathbf{x} \rangle^{r \times p}$.

It is clear that the inner rank of $L$ is at most $\min(n, p)$, since we can write $L = I_n \cdot L$ and $L = L \cdot I_p$. When the inner rank of $L$ is equal to $\min(n, p)$, we say that $L$ is a *full matrix*. The more formal definition is given below:

**Definition 3.11** (Fullness)**.** A matrix $L \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ is said to be *full* if it cannot be written as

$$L = KM$$

where $K \in \mathbb{F}\langle \mathbf{x} \rangle^{n \times r}$, $M \in \mathbb{F}\langle \mathbf{x} \rangle^{r \times n}$, for some $r < n$.

The other way to define the non-commutative rank of a matrix $L$ is by defining its rank over the free skew field, which we do now.

**Definition 3.12.** [Non-Commutative Rank] Given a matrix $L$ in $\mathbb{F}\langle \mathbf{x} \rangle^{n \times p}$, nc-rank$(L)$ is defined to be the rank of $L$ over $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$.

It is important to notice that even over skew fields, the rank of a matrix is defined unambiguously as the row rank or the column rank.

Now that we have defined the non-commutative rank of a matrix, we state the general problem of determining the non-commutative rank of matrices over $\mathbb{F}\langle \mathbf{x} \rangle$.

**Problem 3.13.** *Computing Noncommutative rank*
***Input**: Matrix $L$ in $\mathbb{Q}\langle \mathbf{x}\rangle^{n\times p}$. Each entry is given by a formula computing it which is of size $s$ and uses field elements (rational numbers) of bit size at most $b$.*
***Output**: nc-rank$(L)$*
***Input size**: $n+p+b+s$.*

Since we can compute the non-commutative rank of $L$, we can also determine whether $L$ is full. Thus, by solving Problem 3.13 we have solved the following problem:

**Problem 3.14.** *General Fullness testing*
***Input**: Formulas computing the entries of a matrix $M \in M_n(\mathbb{F}\langle \mathbf{x}\rangle)$.*
***Output**: Is $M$ full?*

Due to the following theorem of Cohn, which works over all fields, by testing fullness of a matrix we can also decide whether a matrix is invertible over the free skew field:

**Theorem 3.15** ([Coh95])**.** *Let $L \in \mathbb{F}\langle \mathbf{x}\rangle^{n\times n}$. Then $L$ is invertible over $\mathbb{F}\langle\langle \mathbf{x}\rangle\rangle$ if and only if $L$ is full.*

It is important to remark that the analogous statement for matrices of commutative polynomials is not true. The following skew symmetric matrix is a counterexample:

$$\begin{bmatrix} 0 & x & y \\ -x & 0 & 1 \\ -y & -1 & 0 \end{bmatrix}$$

It is full but not invertible (when $x$ and $y$ are commuting variables).

Going back to Definition 3.12, let us assume for the sake of simplicity that $L$ is square, i.e. $n = p$, although the results hold for rectangular matrices as well.

As stated in Theorem 1.17, Cohn proved that nc-rank$(L)$ is equal to the inner rank of $L$ (there it is stated for $L$ which have linear polynomials as entries but it is in fact true for general $L$ as seen by the Higman's trick (see section 4) of reducing general $L$ to the linear case). Theorem 1.17 also states that a linear matrix $L$ has nc-rank$(L) \leq r$ iff there exist invertible matrices $B, C$ over $\mathbb{F}$ s.t. $BLC$ has a block of zeroes of size $i \times j$ where $i + j \geq 2n - r$. This can be easily translated into the following theorem:

**Theorem 3.16.** *A linear matrix $L = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{F}\langle x_1, \ldots, x_m\rangle)$ has nc-rank$(L) \leq r$ iff $A_1, \ldots, A_m$ admit a $n - r$-shrunk subspace.*

By proposition 3.25, we also get that a linear matrix $L = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{F}\langle \mathbf{x}\rangle)$ has nc-rank$(L) \leq r$ iff the operator defined by $A_1, \ldots, A_m$ is $(n-r)$-rank-decreasing. In section 4, we will give two different deterministic polynomial time algorithms for computing noncommutative rank, one by an algebraic reduction to fullness testing and the other via a "quantum" reduction from checking if an operator is $c$-rank-decreasing to checking if an operator is rank-decreasing.

### 3.2.3 Testing Shrunk Subspaces

**Problem 3.17.** *Testing shrunk subspaces*
***Input:*** *$A_1, \ldots, A_m \in M_n(\mathbb{Q})$.*
***Output:*** *Do $A_1, \ldots, A_m$ admit a shrunk subspace?*

Another theorem due to Cohn is the following (works over all fields):

**Theorem 3.18** ([Coh95])**.** *A matrix $L = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ is not full if and only if there are invertible matrices $U, V \in M_n(\mathbb{F})$ s.t. $ULV$ has a $r \times s$ block of zeroes for some $r, s$ with $r + s > n$.*

The above theorem is an analogue of Hall's bipartite matching theorem where the $r \times s$ block of zeroes corresponds to a set of size $r$ on one side of the bipartition and set of size $s$ on the other side of the bipartition with no edges between them. If $r + s > n$, this prevents the graph from having a perfect matching. The following theorem is an easy corollary of Theorem 3.18.

**Theorem 3.19.** *A matrix $L = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ is not full if and only if $A_1, \ldots, A_m$ admit a shrunk subspace.*

So an extremely hard looking problem about inverting matrices in the free skew field is equivalent to a simply stated problem in (commutative) linear algebra!

**Remark 3.20.** *$A_1, \ldots, A_m$ shrink a subpace of $\mathbb{C}^n$ iff $A_1, \ldots, A_m$ shrink a subspace of $\mathbb{Q}^n$ since $\sum_{i=1}^{m} x_i A_i$ is invertible over $\mathbb{C}\langle\langle \mathbf{x} \rangle\rangle$ iff it is invertible over $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$.*

### 3.2.4 Null-Cone of the Left-Right Action

**Problem 3.21.** *Testing membership in the null-cone of the left-right action*
***Input:*** *$A_1, \ldots, A_m \in M_n(\mathbb{Q})$.*
***Output:*** *Does $(A_1, \ldots, A_m)$ lie in the null-cone of the left-right action?*

As we saw in section 3.1.3, testing if $(A_1, \ldots, A_m)$ is in the null-cone of the left-right action is the same as testing if $A_1, \ldots, A_m$ admit a shrunk subspace (over algebraically closed fields).

**Theorem 3.22** ([BD06, ANS10])**.** *$(A_1, \ldots, A_m)$ is in the null-cone of the left-right action iff $A_1, \ldots, A_m$ admit a shrunk subspace.*

The above theorem can also be obtained in a round-about manner through infinite generating sets of the left-right action. The tuple $(A_1, \ldots, A_m)$ is in the null-cone of the left-right action iff

$$\mathrm{Det}\left(\sum_{i=1}^{m} A_i \otimes B_i\right) = 0$$

for all $k \in \mathbb{N}, B_i \in M_k(\mathbb{F})$. This is equivalent to the matrix $\sum_{i=1}^{m} x_i A_i$ being singular over the free skew field $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$ (this essentially follows from Amitsur's construction of the free skew field [Ami66], see [HW14]) which is equivalent to $A_1, \ldots, A_m$ admitting a shrunk subspace, because of Theorem 3.15 and Theorem 3.19.

In [ANS10], several fundamental problems about the left-right action were connected to Geometric Complexity Theory. Most of these problems seem to require polynomial degree bounds for the generating set of the invariant ring of polynomials. However, we are able to bypass proving these polynomial degree bounds for determining membership in the null-cone over $\mathbb{Q}$, and it is plausible that our methods can solve other problems for the left-right action, e.g. the orbit closure intersection problem.

### 3.2.5 Testing rank decreasing property of quantum operators

This problem will only make sense over $\mathbb{Q}$ (or more generally over $\mathbb{C}$).

**Definition 3.23.** A completely positive operator is called rank decreasing if there exists a $X \succeq 0$ s.t.

$$\text{Rank}(T(X)) < \text{Rank(X)}$$

We will restrict our attention quantum operators with Kraus operators having rational entries.

**Problem 3.24.** *Testing if quantum operators are rank decreasing*
***Input**: $A_1, \ldots, A_m$ interpreted as a quantum operator $T$ with Kraus operators $A_1, \ldots, A_m$.*
***Output**: Is $T$ rank decreasing?*

**Proposition 3.25** ([Gur04])**.** *Suppose $T$ is a cptp map whose Kraus operators are $A_1, \ldots, A_m$. Then $T$ is c-rank-decreasing iff $A_1, \ldots, A_m$ admit a c-shrunk subspace.*

The connection essentially is that if $A_1, \ldots, A_m$ shrink a subspace $V$, then $T$ will decrease the rank of the projector onto $V$, $\Pi_V$.

*Proof.* First suppose that $A_1, \ldots, A_m$ have a shrunk subspace $V$ such that each $A_i$ maps $V$ into $W$, where $\dim(W) \leq \dim(V) - c$. Suppose $\dim(V) = p$ and $v_1, \ldots, v_p$ be an orthonormal basis for $V$. Let $\Pi_V$ be the projection onto $V$. Then $\text{Rank}(\Pi_V) = p$. We will show that $\text{Rank}(T(\Pi_V)) \leq p - c$ which will show that $T$ is $c$-rank-decreasing.

$$T(\Pi_V) = \sum_{i=1}^{m} \sum_{j=1}^{p} A_i v_j v_j^\dagger A_i^\dagger$$

Since for all $i, j$, $A_i v_j \in W$, it is clear that $\text{Im}(T(\Pi_V)) \subseteq W$ and hence $\text{Rank}(T(\Pi_V)) \leq p - c$.

Now suppose that $T$ is $c$-rank-decreasing i.e. there exists a hermitian $X$, $X \succeq 0$ s.t. $\text{Rank}(T(X)) \leq \text{Rank}(X) - c$. We will show that $\text{Im}(X)$ is a $c$-shrunk subspace for $A_1, \ldots, A_m$. For this it suffices to show that for all $i$, $A_i(\text{Im}(X)) \subseteq \text{Im}(T(X))$. This is true because:

$$\begin{aligned} A_i(\text{Im}(X)) = A_i(\text{Im}(X^{1/2})) &= \text{Im}(A_i X^{1/2}) \\ &= \text{Im}(A_i X A_i^\dagger) \\ &\subseteq \text{Im}\left(\sum_{j=1}^{m} A_j X A_j^\dagger\right) \\ &= \text{Im}(T(X)) \end{aligned}$$

The third equality follows from the fact that $\mathrm{Im(A)} = \mathrm{Im}(AA^\dagger)$ for any matrix $A$. The containment follows from the following claim:

**Claim 3.26.** *Let $A$ and $B$ be two psd matrices. Then $Im(A) \subseteq Im(A+B)$.*

*Proof.* Suppose $\mathrm{Im}(A) \subseteq \mathrm{Im}(A+B)$ is not true. Then there exists a vector $u$ s.t. $u^\dagger Au > 0$ but $u^\dagger(A+B)u = 0$, which is clearly not possible since $u^\dagger Bu \geq 0$. □

□

**Remark 3.27.** *Because of Remark 3.20, $T$ decreases the rank of a psd matrix with rational entries iff it decreases the rank of a psd matrix with complex entries.*

### 3.2.6 Matrix Inverse Identity Testing

INVERSE is the following computational model: Given a matrix $L \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ whose entries are linear polynomials ($L = A_1 + \sum_{i=2}^m x_i A_i$), the output is the top right entry of $L^{-1}$ ($A$ should be invertible over the free skew field $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$). Note that the output is a rational function in the non-commuting variables $\mathbf{x} = x_1, \ldots, x_m$. This model is exponentially more powerful than non-commutative formulas with divisions [HW14], can (efficiently) simulate non-commutative algebraic branching programs (without division) [HW14] and can be (efficiently) simulated by non-commutative circuits with division [HW14]. However it is not known whether INVERSE can simulate non-commutative circuits (with or without division) or not.
We are interested in the identity testing of INVERSE, namely given an invertible (over $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$) linear matrix $L$ as input, check whether the top right entry of $L^{-1}$ is zero. It is worth noting that the word problem for this computational model, whether, given two linear matrices $L$ and $M$, the top right entries of $L^{-1}$ and $M^{-1}$ are the same or not, can be reduced to the identity testing problem. The difference between the top right entries of $L^{-1}$ and $M^{-1}$ can be written as the top right entry of $N^{-1}$ for a slightly larger linear matrix $N$ [HW14].

The problem makes sense for any underlying field $\mathbb{F}$. However our results will hold for $\mathbb{F} = \mathbb{Q}$.

**Problem 3.28.** *INVERSE identity testing*
***Input**: $A_1, \ldots, A_m$ interpreted as $L = A_1 + \sum_{i=2}^m x_i A_i$.*
***Ouput**: Is the top right entry of $L^{-1}$ zero?*

In [HW14], it was shown that INVERSE identity testing can be reduced to SINGULAR. Let $u$ be the $1 \times n$ vector $(1, 0, \ldots, 0)$ and $v$ be $(0, 0, \ldots, 1)$.

**Proposition 3.29** ([HW14])**.** *Suppose $L \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ is invertible (over $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$). Then the top right entry of $L^{-1}$ is zero iff*

$$M = \begin{bmatrix} v^T & L \\ 0 & -u \end{bmatrix}$$

*is singular.*

Previously we only considered singularity testing for matrices of the form $\sum_{i=1}^m x_i B_i$ but now we are looking at matrices of the form $B_1 + \sum_{i=2}^m x_i B_i$. However $B_1 + \sum_{i=2}^m x_i B_i$ is singular iff $\sum_{i=1}^m x_i B_i$ is singular. This is best seen from the shrunk subspace perspective, these matrices are singular iff $B_1, \ldots, B_m$ admit a shrunk subspace.

### 3.2.7 Non-commutative Rational Identity Testing

In this section, we look at the problem of non-commutative rational identity testing (RIT) of formulas. Given an element of the free skew field $\mathbb{F}\langle\langle \mathbf{x} \rangle\rangle$, represented as a formula over non-commutative variables with addition, multiplication and inversion gates, we want to check if the element is zero or not. Note that the word problem over the free skew field, where we are given two rational expressions (represented as formulas) and want to check if they represent the same element in the free skew field, can be easily reduced to non-commutative RIT.

**Problem 3.30.** *Non-commutative RIT*
***Input**: Non-commutative formula $\Phi$ (with inversion gates) of size $s$.*
***Output**: Does $\Phi$ represents zero or not?*
***Input size**: $s + b$, where $b$ is the maximum bit size of field elements (rational numbers) used in the formula.*

Non-commutative RIT can be reduced to INVERSE identity testing because of the following theorem.

**Theorem 3.31** ([HW14])**.** *For each non-commutative formula (with inversions) $\Phi$ whose size is $s$ and uses rational numbers of bit size at most $b$, there exists $s' \leq 2s$ and a $s' \times s'$ matrix $L_\Phi$, whose entries are either variables or rational numbers of bit size at most $b$ such that the rational functions computed by $\Phi$ and the top right entry of $L_\Phi^{-1}$ are the same.*

## 3.3 Results

**Theorem 3.32.** *There are deterministic polynomial time algorithms for Problems 3.9, 3.17, 3.21, 3.24, 3.13, 3.28, 3.30. For problems 3.9, 3.14, 3.17, 3.21, 3.24, 3.28, the complexity is poly$(n, b)$. For 3.13, the complexity is poly$(n, p, s, b)$, whereas for 3.30, it is poly$(s, b)$.*

# 4 Computing the Noncommutative Rank

In this section, we show how to compute the non-commutative rank of any (not necessarily square) matrix with linear entries over the free skew field $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$. This will be achieved in two ways: the first, in Subsection 4.2, by reducing this problem to testing singularity of a certain square matrix with linear entries, and the second, in Subsection 4.3, by a purely quantum approach which in a sense mimics the reduction from maximum matching to perfect matching.

In fact, we solve a more general problem. Subsection 4.1 starts with a reduction of computing nc-rank of a matrix with *polynomial* entries (given by formulae), to the problem of computing the nc-rank of a matrix with linear entries, via the so-called "Higman's trick" (Proposition 4.2). We give the simple quantitative analysis of this reduction, which as far as we know does not appear in the literature and may be useful elsewhere. This reduction, with the two above, allow computing the non-commutative rank of any matrix in time polynomial in the description of its entries.

## 4.1 Higman's Trick

Before stating the full version of the effective Higman trick, we need to define the bit complexity of a formula computing a non-commutative polynomial.

**Definition 4.1** (Bit Complexity of a Formula)**.** Let $\Phi$ be a non-commutative formula without divisions such that each of its gates computes a polynomial in $\mathbb{Q}\langle \mathbf{x} \rangle$ (i.e., the inputs to the formula are either rational numbers or non-commutative variables). The *bit complexity* of $\Phi$ is the maximum bit complexity of any rational input appearing in the formula $\Phi$.

With this definition in hand, we can state and prove Higman's trick, which first appeared in [Hig40]. In the proposition below, it will be useful to have the following notation to denote the direct sum of two matrices $A$ and $B$:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where the zeros in the top right and bottom left corners are of appropriate dimensions. Before stating and proving Higman's trick, let us work through a small example which showcases the essence of the trick.

Suppose we want to know the nc-rank of matrix $\begin{pmatrix} 1 & x \\ y & z+xy \end{pmatrix}$. The problem here is that this matrix is not linear, and we need to have a linear matrix. How can we convert this matrix into a linear matrix while preserving the rank, or the complement of the rank? To do this, we need to remove the multiplication happening in $z + xy$.

Notice that the complement of its rank does not change after the following transformation:

$$\begin{pmatrix} 1 & x \\ y & z+xy \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & 0 \\ y & z+xy & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the complement of the rank does not change after we perform elementary row or column operations, we can first add $x \cdot$ (third row) to the second row, and then subtract $y \cdot$ (third column) to the second column, to obtain:

$$\begin{pmatrix} 1 & x & 0 \\ y & z+xy & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & 0 \\ y & z+xy & x \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & 0 \\ y & z & x \\ 0 & -y & 1 \end{pmatrix}$$

The complement of the rank of this last matrix is the same as the complement of the rank of our original matrix! In particular, if this last matrix is full rank, it implies that our original matrix is also full rank. This is the essence of Higman's trick. We now proceed to its full version.

**Proposition 4.2** (Effective Higman's Trick)**.** *Let $A \in \mathbb{Q}\langle \mathbf{x} \rangle^{m \times n}$ be a matrix where each entry $a_{ij}$ is computed by a non-commutative formula of size $\leq s$ and bit complexity $\leq b$ without divisions. Let $k$ be the total number of multiplication gates used in the computation of the entries of $A$. There exist matrices $P \in \mathbb{GL}_{m+k}(\mathbb{Q}\langle \mathbf{x} \rangle)$, $Q \in \mathbb{GL}_{n+k}(\mathbb{Q}\langle \mathbf{x} \rangle)$ such that $P(A \oplus I_k)Q$ is a matrix with linear entries and coefficients with bit complexity bounded by $b$. Moreover, given access to the formulas computing the entries, one can construct $P$ and $Q$ efficiently in time poly$(m, n, s, b)$. Since $P$ and $Q$ are non-singular matrices, the* co-rank *and the* co-nc-rank *of $P(A \oplus I_k)Q$ are the same as the* co-rank *and the* co-nc-rank *of $A$.*

*Proof.* Let $\mathsf{Mult}(a_{ij})$ be the number of multiplication gates in the formula computing entry $a_{ij}$ and

$$T = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \mathsf{Mult}(a_{ij}).$$

That is, $T$ is the total number of multiplication gates used to compute all entries of the matrix $A$.

We prove this proposition by induction on $T$, for matrices of all dimensions. The base case, when $T = 0$, is trivial, as in this case $A$ itself has linear entries. Suppose now that the proposition is true for all matrices (regardless of their dimensions) which can be computed by formulas using $< T$ multiplication gates.

Let $A$ be our matrix, which can be computed using $T$ multiplications. W.l.o.g., we can assume that $\mathsf{Mult}(a_{mn}) \geq 1$. Then, by finding a multiplication gate in the formula for $a_{mn}$ that has no other multiplication gate as an ancestor, we can write $a_{mn}$ in the form $a_{mn} = a + b \cdot c$, where

$$\mathsf{Mult}(a_{mn}) = \mathsf{Mult}(a) + \mathsf{Mult}(b) + \mathsf{Mult}(c) + 1.$$

Hence, the matrix

$$B = \left(I_{m-1} \oplus \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)(A \oplus 1)\left(I_{n-1} \oplus \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\right)$$

is such that

$$b_{ij} = \begin{cases} a_{ij}, \text{ if } i \leq m, j \leq n \text{ and } (i,j) \neq (m,n) \\ a, \text{ if } (i,j) = (m,n) \\ b, \text{ if } (i,j) = (m,n+1) \\ -c, \text{ if } (i,j) = (m+1,n) \\ 1, \text{ if } (i,j) = (m+1,n+1) \\ 0 \text{ otherwise} \end{cases}$$

Therefore, the number of multiplications needed to compute $B$ is given by

$$\begin{aligned} \sum_{\substack{1 \leq i \leq m+1 \\ 1 \leq j \leq n+1}} \mathsf{Mult}(b_{ij}) &= \left(\sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \mathsf{Mult}(a_{ij})\right) - \mathsf{Mult}(a_{mn}) + \mathsf{Mult}(a) + \mathsf{Mult}(b) + \mathsf{Mult}(c) \\ &= T - \mathsf{Mult}(a_{mn}) + \mathsf{Mult}(a) + \mathsf{Mult}(b) + \mathsf{Mult}(c) \\ &= T - 1 \end{aligned}$$

Since $B$ is an $(m+1) \times (n+1)$ matrix which can be computed by using a total of $T-1$ multiplication gates, by the induction hypothesis, there exist $P' \in \mathbb{GL}_{m+1+(T-1)}(\mathbb{Q}\langle \mathbf{x} \rangle) = \mathbb{GL}_{m+T}(\mathbb{Q}\langle \mathbf{x} \rangle)$ and $Q' \in \mathbb{GL}_{n+1+(T-1)}(\mathbb{Q}\langle \mathbf{x} \rangle) = \mathbb{GL}_{n+T}(\mathbb{Q}\langle \mathbf{x} \rangle)$ such that $P'(B \oplus I_{T-1})Q'$ is a linear matrix. Since

$$\begin{aligned} B \oplus I_{T-1} &= \left(I_{m-1} \oplus \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \oplus I_{T-1}\right)(A \oplus I_T)\left(I_{n-1} \oplus \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \oplus I_{T-1}\right) \\ &= R(A \oplus I_T)S, \end{aligned}$$

where $R = \left(I_{m-1} \oplus \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \oplus I_{T-1}\right) \in \mathbb{GL}_{m+T}(\mathbb{Q}\langle \mathbf{x} \rangle)$ and $S = \left(I_{n-1} \oplus \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \oplus I_{T-1}\right) \in \mathbb{GL}_{n+T}(\mathbb{Q}\langle \mathbf{x} \rangle)$, we have that

$$P'(B \oplus I_{T-1})Q' = (P'R)(A \oplus I_T)(SQ').$$

Setting $P = P'R$ and $Q = SQ'$ proves the inductive step and completes the proof. Since we only use subformulas of the formulas computing the entries of $A$, the bound on the bit complexity does not change. $\square$

## 4.2 Classical Reduction

Having shown the effective version of Higman's trick, we can now compute the nc-rank of a matrix over $\mathbb{Q}\langle\mathbf{x}\rangle$. We begin with a lemma which will tell us that we can reduce the problem of computing the nc-rank of a matrix by testing fullness of a smaller matrix with polynomial entries.

**Lemma 4.3** (Reduction to Fullness Testing)**.** *Let $M \in \mathbb{F}\langle\mathbf{x}\rangle^{m\times n}$ be any matrix. In addition, let $U = (u_{ij})$ and $V = (v_{ij})$ be generic matrices in new, non-commuting variables $u_{ij}$ and $v_{ij}$, of dimensions $r \times m$ and $n \times r$, respectively. Then, nc-rank$(M) \geq r$ iff the matrix $UMV$ is full.*

*Proof.* Since nc-rank$(M) \geq r$, there exists an $r \times r$ minor of $M$ of full rank. Let $Q$ be such a minor of $M$. W.l.o.g.,[20] we can assume that $Q$ is the $[r] \times [r]$ principal minor of $M$. Hence, we have that

$$UMV = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} Q & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

where $U_1$ and $V_1$ are $r \times r$ matrices and the others are matrices with the proper dimensions.

Letting $U' = \begin{pmatrix} I_r & 0 \end{pmatrix}$ and $V' = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$, the equality above becomes:

$$U'MV' = Q.$$

As

$$r \geq \text{nc-rank}(UMV) \geq \text{nc-rank}(U'MV') = \text{nc-rank}(Q) = r,$$

we obtain that $UMV$ is full, as we wanted. Notice that the second inequality comes from the fact that rank does not increase after restrictions of the new variables. □

Notice that we do not know the rank nc-rank$(M)$ a priori. Therefore, our algorithm will try all possible values of $r \in [n]$ and output the maximum value of $r$ for which we find a full matrix.

For each $r \times r$ matrix $UMV$, we can use the effective Higman's trick to convert $UMV$ into a $s \times s$ matrix with linear entries. With this matrix, we can use the truncated Gurvits' algorithm to check whether the matrix we just obtained is full. Since we have this test, we will be able to output the correct rank. Algorithm 3 is the precise formulation of the procedure just described.

**Theorem 4.4.** *Let $M \in \mathbb{Q}\langle\mathbf{x}\rangle^{m\times n}$ be s.t. each entry of $M$ is a polynomial computed by a formula of size bounded by $s$ and bit complexity bounded by $b$. There exists a deterministic algorithm that finds the non-commutative rank of $M$ in time poly$(m, n, s, b)$.*

*Proof.* To prove this theorem, it is enough to show that Algorithm 3 is correct and it runs with the desired runtime.

Without loss of generality, we can assume that $n \leq m$. Therefore we have that nc-rank$(M) \leq n$. By Lemma 4.3, if $r \leq$ nc-rank$(M)$, then matrix $M_r$ will be of full rank (and therefore will not have a shrunk subspace, by Theorem 1.4). Since $M_r = U_r M V_r$, from the formulas computing the entries of $M$ we obtain formulas of size at most $2smn$ computing the entries of $M_r$. Moreover, the bit complexities of these formulas will still be bounded by $b$, as multiplication by generic matrices do not mix any of the polynomials of $M$.

[20] Notice that we can make the following assumption just to simplify notation. In actuality, we do not know where the full rank minor is located in $M$.

Input: $M \in \mathbb{Q}\langle \mathbf{x} \rangle^{m \times n}$ s.t. each entry of $M$ is a polynomial computed by a formula of size bounded by $s$ and bit complexity bounded by $b$.
Output: nc-rank($M$)

1. For $1 \leq r \leq \min(m, n)$, let $U_r$ and $V_r$ be $r \times m$ and $n \times r$ generic matrices in new, non-commuting variables $u_{ij}^{(r)}$, $v_{ij}^{(r)}$.
2. Let $M_r = U_r M V_r$.
3. Apply the effective Higman's trick to $M_r$ to obtain a matrix $N_r$ with linear entries on the variables $x_1, \ldots, x_m$ and $u_{ij}^{(r)}$, $v_{ij}^{(r)}$.
4. Use Algorithm $G'$ to test whether $N_r$ has a shrunk subspace.
5. Output the maximum value of $r$ for which $N_r$ has **no** shrunk subspace.

**Algorithm 3:** Noncommutative Rank Algorithm

By Proposition 4.2 and the fact that the size of the formulas computing the entries of $M_r$ are bounded by $2smn$, we have that $N_r$ is a linear matrix of dimensions $(k + r) \times (k + r)$, where $k \leq 2s(mn)^2$ and the bit complexity of the coefficients bounded by $b$. Moreover, $N_r = P(M_r \oplus I_k)Q$ implies that $N_r$ is full if, and only if, $M_r$ is full, which is true if, and only if, nc-rank($M$) $\geq r$.

Now, by Theorem 3.32, we have a deterministic polynomial time algorithm to determine whether $N_r$ has a shrunk subspace. By Theorem 1.4, this algorithm tests if the matrix $N_r$ is full rank. Moreover, it outputs that $N_r$ has no shrunk subspace iff $N_r$ is full. Hence, if $r \leq$ nc-rank($M$), $N_r$ will be full, and the maximum such $r$ will be exactly when $r =$ nc-rank($M$). Therefore, by outputting the maximum $r$ for which $N_r$ is full, i.e. has no shrunk subspace, we compute nc-rank($M$). This proves that our algorithm is correct. Notice that the runtime is polynomial in the input size, as we perform at most $n$ applications of the Higman trick and of Algorithm $G'$. This completes the proof. □

### 4.3 The quantum reduction

Here we present a different reduction from computing non-commutative rank to fullness testing from a quantum viewpoint. We will only work with square matrices though. As we saw, by Higman's trick, we can assume the matrices to be linear. So we are given a matrix $L = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$. As we saw in section 3.2.2, nc-rank($L$) $\leq r$ iff the operator defined by $A_1, \ldots, A_m$ is $n - r$-rank-decreasing. So we just want to check whether a completely positive operator is $c$-rank-decreasing and we will do this by using an algorithm for checking if an operator is rank-decreasing as a black box using the following lemma:

**Lemma 4.5.** *Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely positive operator. Define an operator $\overline{T} : M_{n+c-1}(\mathbb{C}) \to M_{n+c-1}(\mathbb{C})$ as follows:*

$$\overline{T}\left(\begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix}\right) = \begin{bmatrix} T(X_{1,1}) + tr(X_{2,2}) I_n & 0 \\ 0 & tr(X_{1,1}) I_{c-1} \end{bmatrix}$$

*Here $X_{1,1}$, $X_{1,2}$, $X_{2,1}$, $X_{2,2}$ are $n \times n$, $n \times c-1$, $c-1 \times n$, $c-1 \times c-1$ matrices respectively.*

*Then $\overline{T}$ is completely positive and $T$ is $c$-rank-decreasing iff $\overline{T}$ is rank-decreasing. Note that we are considering $c \leq n$.*

*Proof.* A well known characterization due to Choi [Cho75] states that $\overline{T}$ is completely positive iff $\sum_{i,j=1}^{n+c-1} E_{i,j} \otimes \overline{T}(E_{i,j})$ is psd. Here $E_{i,j}$ is the matrix with 1 at $i,j$ position and 0 everywhere else. Now

$$\overline{T}(E_{i,j}) = \begin{cases} \begin{bmatrix} T(E_{i,j}) & 0 \\ 0 & I_{c-1} \end{bmatrix} & 1 \leq i = j \leq n \\ \begin{bmatrix} T(E_{i,j}) & 0 \\ 0 & 0 \end{bmatrix} & 1 \leq i, j \leq n, i \neq j \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & 1 \leq i \leq n, n+1 \leq j \leq n+c-1 \text{ or } n+1 \leq i \leq n+c-1, 1 \leq j \leq n \\ \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} & n+1 \leq i = j \leq n+c-1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & n+1 \leq i, j \leq n+c-1, i \neq j \end{cases}$$

From here it is easy to verify that $\sum_{i,j=1}^{n+c-1} E_{i,j} \otimes \overline{T}(E_{i,j})$ is psd given that $\sum_{i,j=1}^{n} E_{i,j} \otimes T(E_{i,j})$ is psd. Now suppose that $\overline{T}$ is rank-decreasing. This can only happen if $X_{1,1} = 0$ or $X_{2,2} = 0$, otherwise

$$\overline{T}\left(\begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix}\right) = \begin{bmatrix} T(X_{1,1}) + \operatorname{tr}(X_{2,2}) I_n & 0 \\ 0 & \operatorname{tr}(X_{1,1}) I_{c-1} \end{bmatrix}$$

is full rank. If $X_{1,1} = 0$, then

$$\begin{bmatrix} 0 & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix}$$

can be psd (and hermitian) only if $X_{1,2} = X_{2,1} = 0$. In this case a $c-1$ ranked matrix is mapped to rank $n$ matrix. So $X_{2,2}$ has to be zero. Then again by the psd condition $X_{1,2} = X_{2,1} = 0$. So

$$\overline{T}\left(\begin{bmatrix} X_{1,1} & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} T(X_{1,1}) & 0 \\ 0 & \operatorname{tr}(X_{1,1}) I_{c-1} \end{bmatrix}$$

and $X_{1,1} \neq 0$ and

$$\operatorname{Rank}\left(\begin{bmatrix} X_{1,1} & 0 \\ 0 & 0 \end{bmatrix}\right) > \operatorname{Rank}\left(\begin{bmatrix} T(X_{1,1}) & 0 \\ 0 & \operatorname{tr}(X_{1,1}) I_{c-1} \end{bmatrix}\right)$$

Hence $\operatorname{Rank}(T(X_{1,1})) \leq \operatorname{Rank}(X_{1,1}) - c$. This proves one direction. Now suppose that $T$ is $c$-rank-decreasing and $\operatorname{Rank}(T(X)) \leq \operatorname{Rank}(X) - c$, then

$$\operatorname{Rank}\left(\overline{T}\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right)\right) = \operatorname{Rank}\left(\begin{bmatrix} T(X) & 0 \\ 0 & \operatorname{tr}(X) I_{c-1} \end{bmatrix}\right) < \operatorname{Rank}\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right)$$

This proves the lemma. □

**Remark 4.6.** *This seems to be the "quantum" analogue of obtaining a maximum matching oracle based on a perfect matching oracle: add c-1 dummy vertices to both sides of the bipartite graph and connect them to everything. Then the new graph has a perfect matching iff the original graph had a matching of size* $\geq n - c + 1$.

**Remark 4.7.** *Here we didn't specify a set of Kraus operators for the operator* $\overline{T}$ *which seem to be needed to run Algorithms 1 and 2 but Kraus operators can be obtained by looking at the eigenvectors of* $\sum_{i,j=1}^{n+c-1} E_{i,j} \otimes \overline{T}(E_{i,j})$. *Alternatively Algorithms 1 and 2 can also be interpreted as acting directly on the Choi-Jamiolkowski state of* $\overline{T}$ *i.e.* $\sum_{i,j=1}^{n+c-1} E_{i,j} \otimes \overline{T}(E_{i,j})$.

# 5 Conclusion and Open Problems

In this paper we gave a polynomial time algorithm for computing the non-commutative rank of a symbolic matrix over any subfield of the complex numbers. We stated its different incarnations and implications to the many different areas in which this problem arises (indeed we feel that expositing these many connections, some essential to the present result, may yield better future interaction between them with possible more benefits) . We note that our algorithm bypasses the fact that the best degree bound on the left-right action is exponential, and yet how the fact that it is not more than exponential is essential to guarantee a polynomial run-time of our algorithm. We further note again that despite the purely algebraic nature of the problem our algorithm is purely analytic, generating a sequence of complex matrices and testing its convergence.

We collect now the most obvious directions for future research, some of them already mentioned in the paper.

- Obtain the same result over finite fields. Our algorithm does not lend itself to such generalization.
- Improve the exponential degree bound on the left-right action (over any field). This is perhaps the most basic open problem in the area. Needless to say, a polynomial upper bound will yield a randomized polynomial time algorithm for the same problem over the corresponding field, and solve the division elimination problem of [HW14].
- Solve the natural search problems which decision version is solved by Algorithm $G$. For example, *find* the implied shrunk subspaces, Hall-blocker and factorization for singular symbolic matrices.
- Improve the approximation guarantee of *commutative rank* of symbolic matrices (we provided a factor of 2). More generally, explore further connection the commutative and non-commutative PIT problems. We feel that beyond the many connections between commutative and non-commutative settings that arise here, this different angle of looking at the commutative PIT problem, relating it to its non-commutative cousin, may help in the major quest of finding an efficient deterministic algorithm for it.
- Find an analog of the Kabanets-Impagliazzo [KI04] result, which will enable to convert our deterministic identity testing result to *some* natural lower bound for non-commutative circuits. A natural computational model, which by [HW14] is exponentially more powerful than

formulae with division but not stronger than circuits with division, is simply (an entry of) the inverse of a symbolic matrix with linear entries. Is there a natural polynomial which requires exponential size in this model?

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