ISOPERIMETRIC INEQUALITIES IN SIMPLICIAL COMPLEXES

ORI PARZANCHEVSKI, RON ROSENTHAL, RAN J. TESSLER

In graph theory there are intimate connections between the expansion properties of a graph and the spectrum of its Laplacian. In this paper we define a notion of combinatorial expansion for simplicial complexes of general dimension, and prove that similar connections exist between the combinatorial expansion of a complex, and the spectrum of the high dimensional Laplacian defined by Eckmann. In particular, we present a Cheeger-type inequality, and a high-dimensional Expander Mixing Lemma. As a corollary, using the work of Pach, we obtain a connection between spectral properties of complexes and Gromov's notion of geometric overlap. Using the work of Gundert and Wagner, we give an estimate for the combinatorial expansion and geometric overlap of random Linial–Meshulam complexes.

1. Introduction

It is a cornerstone of graph theory that the expansion properties of a graph are intimately linked to the spectrum of its Laplacian. In particular, the discrete Cheeger inequalities [47,15,4,3] relate the spectral gap of a graph to its Cheeger constant, and the Expander Mixing Lemma [20,2,7] relates the extremal values of the spectrum to pseudo-randomness in the graph, and to its mixing properties.

In this paper we define a notion of expansion for simplicial complexes, which generalizes the Cheeger constant and the notion of pseudo-randomness in graphs. We then study its relations to the spectrum of the high dimensional Laplacian defined by Eckmann [16], and present a high dimensional Cheeger inequality and a high dimensional Expander Mixing Lemma.

This study is closely related to the notion of *high dimensional expanders*. A family of graphs $\{G_i\}$ with uniformly bounded degrees is said to be a family of *expanders* if their Cheeger constants $h(G_i)$ are uniformly bounded away from zero. By the discrete Cheeger inequalities (1.3), this is equivalent to having their spectral gaps $\lambda(G_i)$ uniformly bounded away from zero. Thus, combinatorial expanders and spectral expanders are equivalent notions. We refer to [28,35] for the general background on expanders and their applications.

It is desirable to have a similar situation in higher dimensions, but at least as of now, it is not clear what is the "right" notion of "high dimensional expander". One generalization of the Cheeger constant to higher dimensions is the notion of *coboundary expansion*, originating in [31,25], and studied under various names in [38,39,12,27,46,41]. While in dimension one it coincides with the Cheeger constant, its combinatorial meaning is somewhat vague in higher dimensions. Furthermore, it is shown in [27] that there exist, in any dimension greater than one, complexes with spectral gaps bounded away from zero[1] and arbitrarily small coboundary expansion; In [46] the other direction is settled: there exist coboundary expanding complexes with arbitrarily small spectral gaps.

Another notion of expansion is Gromov's *geometric overlap property*, originating in [25] and studied in [39,19]. This notion was shown in [25,39] to be related to coboundary expansion. However, even in dimension one it is not equivalent to that of expander graphs.

Our definition of expansion suggests a natural notion of "combinatorial expanders", and we show that spectral expanders with complete skeletons are combinatorial expanders. A theorem of Pach [43] shows that this notion of combinatorial expansion is also connected to the geometric overlap property. As an application of our main theorems we analyze the Linial–Meshulam model of random complexes, and show that for suitable parameters they form combinatorial and geometric expanders.

1.1. Combinatorial expansion and the spectral gap

The *Cheeger constant* of a finite graph $G = (V, E)$ on n vertices is usually taken to be

$$\varphi(G) = \min_{\substack{A \subseteq V \\ 0 < |A| \leq \frac{n}{2}}} \frac{|E(A, V \setminus A)|}{|A|}$$

---

[1] The spectral gap of a complex is defined in Section 2.1.

where $E(A, B)$ is the set of edges with one vertex in A and the other in B. In this paper, however, we work with the following version:

$$(1.1) \qquad h(G) = \min_{0 < |A| < n} \frac{n|E(A, V \setminus A)|}{|A||V \setminus A|}.$$

Since $\varphi(G) \le h(G) \le 2\varphi(G)$, defining expanders by φ or by h is equivalent. The *spectral gap* of G, denoted $\lambda(G)$, is the second smallest eigenvalue of the *Laplacian* $\Delta^+ : \mathbb{R}^V \to \mathbb{R}^V$, which is defined by

$$(1.2) \qquad (\Delta^+ f)(v) = \deg(v)f(v) - \sum_{w \sim v} f(w).$$

The discrete Cheeger inequalities [47,15,4,3] relate the Cheeger constant and the spectral gap:

$$(1.3) \qquad \frac{h^2(G)}{8k} \le \lambda(G) \le h(G),$$

where k is the maximal degree of a vertex in G.[2] In particular, the bound $\lambda \le h$ shows that spectral expanders are combinatorial expanders. This has proved out to be of immense importance since the spectral gap is approachable by many mathematical tools (coming from linear algebra, functional analysis, representation theory and even number theory - see e.g. [34,35] and the references within). In contrast, the Cheeger constant is usually hard to analyze directly, and even to compute it for a given graph is NP-hard [5,36].

Moving on to higher dimension, let X be an (abstract) simplicial complex with vertex set V. This means that X is a collection of subsets of V, called *cells* (and also *simplexes*, *faces*, or *hyperedges*), which is closed under taking subsets, i.e., if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$. The *dimension* of a cell σ is $|\sigma| - 1$, and X^j denotes the set of *j-cells* (cells of dimension j). The dimension of X is the maximal dimension of a cell in it. The *degree* of a *j*-cell is the number of $(j+1)$-cells which contain it. Throughout this paper we denote by d the dimension of the complex at hand, and by n the number of vertices in it. We shall occasionally add the assumption that the complex has a *complete skeleton*, by which we mean that every possible *j*-cell with $j < d$ belongs to X.

We define the following generalization of the Cheeger constant:

Definition 1.1. For a finite *d*-complex X with n vertices V,

$$h(X) = \min_{V = \coprod_{i=0}^{d} A_i} \frac{n \cdot |F(A_0, A_1, \ldots, A_d)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_d|},$$

---

[2] For φ they are given by $\frac{\varphi^2(G)}{2k} \le \lambda(G) \le 2\varphi(G)$.

where the minimum is taken over all partitions of V into nonempty sets $A_0, \ldots, A_d$, and $F(A_0, \ldots, A_d)$ denotes the set of d-dimensional cells with one vertex in each A_i.

For $d=1$, this coincides with the Cheeger constant of a graph (1.1). To formulate an analogue of the Cheeger inequalities, we need a high-dimensional analogue of the spectral gap. Such an analogue is provided by the work of Eckmann on discrete Hodge theory [16]. In order to give the definition we shall need more terminology, and we defer this to Section 2.1[3]. The basic idea, however, is the same as for graphs, namely, the spectral gap $\lambda(X)$ is the smallest nontrivial eigenvalue of a suitable Laplace operator. The following theorem, whose proof appears in Section 4.1, generalizes the upper Cheeger inequality to higher dimensions:

Theorem 1.2 (Cheeger Inequality). *For a finite complex X with a complete skeleton, $\lambda(X) \leq h(X)$.*

Remarks. 1. If the skeleton of X is not complete, then $h(X) = 0$, since there exist some $\{v_0, \ldots, v_{d-1}\} \notin X^{d-1}$, and then $F(\{v_0\}, \{v_1\}, \ldots, \{v_{d-1}\}, V \setminus \{v_0, \ldots, v_{d-1}\}) = \emptyset$. This suggests that a different definition of h is called for, and we propose one in Section 5.
2. For a discussion of a possible lower Cheeger inequality, see Section 4.2.

In [31] Linial and Meshulam introduce and study the following model for random simplicial complexes: for a given $p = p(n) \in (0,1)$, $X(d,n,p)$ is a d-dimensional simplicial complex on n vertices, with a complete skeleton, and with every d-cell being included independently with probability p. Using the analysis of the spectrum of $X(d,n,p)$ in [27], we show the following:

Corollary 1.3. *The Linial–Meshulam complexes satisfy the following:*

1. *For large enough C, a.a.s. $h\left(X\left(d,n,\frac{C\log n}{n}\right)\right) \geq \left(C - O(\sqrt{C})\right)\log n$.*
2. *For small enough C, a.a.s. $h\left(X\left(d,n,\frac{C\log n}{n}\right)\right) = 0$.*

The proof appears in Section 4.5, as part of Corollary 4.6.

1.2. Mixing and pseudo-randomness

The Cheeger inequalities (1.1) bound the expansion along the partitions of a graph, in terms of its spectral gap. However, the spectral gap alone does

---

[3] The spectral gap appears in Definition 2.1, and is given alternative characterizations in Propositions 2.2 and 3.2.

not suffice to determine the expansion between arbitrary sets of vertices. For example, the bipartite Ramanujan graphs constructed in [32,37] are regular graphs with very large spectral gaps, which are bipartite. This means that they contain disjoint sets $A, B \subseteq V$ of size $\frac{n}{4}$, with $E(A, B) = \emptyset$. It turns out that control of the expansion between any two sets of vertices is possible by observing not only the smallest nontrivial eigenvalue of the Laplacian, but also the largest one[4]. In particular, the so-called Expander Mixing Lemma ([20,2,7], see also [28]) states that for a k-regular graph $G = (V, E)$, and $A, B \subseteq V$,

$$(1.4) \qquad \left| |E(A, B)| - \frac{k|A||B|}{n} \right| \le \rho \cdot \sqrt{|A||B|},$$

where ρ is the maximal absolute value of a nontrivial eigenvalue of $kI - \Delta^+$.

The deviation of $|E(A, B)|$ from its expected value $p|A||B|$, where $p = \frac{|E|}{\binom{n}{2}} \approx \frac{k}{n}$ is the edge density, is called the *discrepancy* of A and B. This is a measure of pseudo-randomness in a graph, a notion closely related to expansion (see e.g. [11]). In a similar fashion, if k is the average degree of a $(d-1)$-cell in X, we call the deviation

$$\left| |F(A_0, \ldots, A_d)| - \frac{|X^d|}{\binom{n}{d+1}} \cdot |A_0| \cdot \ldots \cdot |A_d| \right|$$

$$\approx \left| |F(A_0, \ldots, A_d)| - \frac{k|A_0| \cdot \ldots \cdot |A_d|}{n} \right|$$

the discrepancy of $A_0, \ldots, A_d$ (the question of using $\frac{|X^d|}{\binom{n}{d+1}}$ or $\frac{k}{n}$ is addressed in Remark 4.3). The following theorem generalizes the Expander Mixing Lemma to higher dimensions:

Theorem 1.4 (Mixing Lemma). *If X is a d-dimensional complex with a complete skeleton, then for any disjoint sets of vertices $A_0, \ldots, A_d$ one has*

$$\left| |F(A_0, \ldots, A_d)| - \frac{k \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \le \rho \cdot (|A_0| \cdot \ldots \cdot |A_d|)^{\frac{d}{d+1}},$$

where k is the average degree of a $(d-1)$-cell in X, and ρ is the maximal absolute value of a nontrivial eigenvalue of $kI - \Delta^+$.

Here Δ^+ is the Laplacian of X, which is defined in Section 2. The proof, and a formal definition of ρ, appear in Section 4.3.

---

[4] Graphs having both of them bounded are referred to as "two-sided expanders" in [48].

A related measure of expansion in graphs is given by the convergence rate of the random walk on it. As for the discrepancy, it is not enough to bound the spectral gap but also the higher end of the Laplace spectrum in order to understand this expansion. For example, on bipartite graphs the random walk does not converge at all. In [44] we suggest a generalization of the notion of random walk to general simplicial complexes, and study its connection to the spectral properties of the complex.

1.3. Geometric overlap

If a graph $G = (V, E)$ has a large Cheeger constant, then given a mapping $\varphi \colon V \to \mathbb{R}$, there exists a point $x \in \mathbb{R}$ which is covered by many edges in the linear extension of φ to E (namely, $x = \mathrm{median}\left(\{\varphi(v) \mid v \in V\}\right)$). This observation led Gromov to define the *geometric overlap* of a complex [25]:

Definition 1.5. Let X be a d-dimensional simplicial complex. The overlap of X is defined by

$$\mathrm{overlap}(X) = \min_{\varphi \colon V \to \mathbb{R}^d} \max_{x \in \mathbb{R}^d} \frac{\#\{\sigma \in X^d \mid x \in \mathrm{conv}\{\varphi(v) \mid v \in \sigma\}\}}{|X^d|}.$$

In other words, X has overlap $\geq \varepsilon$ if for every simplicial mapping of X into $\mathbb{R}^d$ (a mapping induced linearly by the images of the vertices), some point in $\mathbb{R}^d$ is covered by at least an ε-fraction of the d-cells of X.

A theorem of Pach [43], together with Theorem 1.4 yield a connection between the spectrum of the Laplacian and the overlap property.

Corollary 1.6. *Let X be a d-complex with a complete skeleton, and denote the average degree of a $(d-1)$-cell in X by k. If the nontrivial spectrum of the Laplacian of X is contained in $[k-\varepsilon, k+\varepsilon]$, then*

$$\mathrm{overlap}(X) \geq \frac{c_d^d}{e^{d+1}} \left(c_d - \frac{\varepsilon(d+1)}{k} \right),$$

where c_d is Pach's constant from [43].

The proof appears in Section 4.4. As an application of this corollary, we show that Linial–Meshulam complexes have geometric overlap for suitable parameters:

Corollary 1.7. *There exist $\vartheta > 0$ such that for large enough C a.a.s. overlap $\left(X \left(d, n, \frac{C \cdot \log n}{n} \right) \right) > \vartheta$.*

Again, this is a part of Corollary 4.6, which is proved in Section 4.5.

Related work. Several authors have applied spectral techniques to the study of simplicial complexes. Eckmann's Laplacian appears in many of these, such as [24,14,21,30,1,13,27]. Another approach is taken in [23], which generalizes the bilinear form $\langle \Delta^+ f, f \rangle$ associated with a graph to a $(d+1)$-multilinear form defined by the d-cells of a complex; in [10] appears a Laplacian which acts on the various orderings of $(d-1)$-cells; and in [46] the authors study a Laplacian acting on the d-cells of a manifold, in the spirit of Poincaré duality.

The structure of the paper is as follows: in Section 2 we present the basic definitions relating to simplicial complexes and their spectral theory. Section 3 is devoted to proving basic properties of the high dimensional Laplacians. In Section 4 we prove the theorems and corollaries stated in the introduction, and discuss the possibility of a lower Cheeger inequality. Finally, Section 5 lists some open questions.

2. Notation and definitions

Recall that X denotes a finite d-dimensional simplicial complex with vertex set V of size n, and that X^j denotes the set of j-cells of X, where $-1 \le j \le d$. In particular, we have $X^{-1} = \{\emptyset\}$. For $j \ge 1$, every j-cell $\sigma = \{\sigma_0, \ldots, \sigma_j\}$ has two possible orientations, corresponding to the possible orderings of its vertices, up to an even permutation (1-cells and the empty cell have only one orientation). We denote an oriented cell by square brackets, and a flip of orientation by an overbar. For example, one orientation of $\sigma = \{x, y, z\}$ is $[x, y, z]$, which is the same as $[y, z, x]$ and $[z, x, y]$. The other orientation of σ is $\overline{[x, y, z]} = [y, x, z] = [x, z, y] = [z, y, x]$. We denote by $X_{\pm}^j$ the set of oriented j-cells (so that $|X_{\pm}^j| = 2|X^j|$ for $j \ge 1$ and $X_{\pm}^j = X^j$ for $j = -1, 0$).

We now describe the *discrete Hodge theory* due to Eckmann [16]. This is a discrete analogue of Hodge theory in Riemannian geometry, but in contrast, the proofs of the statements are all exercises in finite-dimensional linear algebra. Furthermore, it applies to any complex, and not only to manifolds.

The space of *j-forms on X*, denoted $\Omega^j(X)$, is the vector space of skew-symmetric functions on oriented j-cells:

$$\Omega^j = \Omega^j(X) = \left\{ f \colon X_{\pm}^j \to \mathbb{R} \mid f(\overline{\sigma}) = -f(\sigma) \; \forall \sigma \in X_{\pm}^j \right\}.$$

In particular, Ω^0 is the space of functions on V, and $\Omega^{-1} = \mathbb{R}^{\{\emptyset\}}$ can be identified in a natural way with $\mathbb{R}$. We endow each Ω^i with the inner product

$$(2.1) \qquad\qquad \langle f, g \rangle = \sum_{\sigma \in X^i} f(\sigma) g(\sigma)$$

(note that $f(\sigma)g(\sigma)$ is well defined even without choosing an orientation for σ).

For a cell σ (either oriented or non-oriented) and a vertex v, we write $v \sim \sigma$ if $v \notin \sigma$ and $\{v\} \cup \sigma$ is a cell in X (here we ignore the orientation of σ). If $\sigma = [\sigma_0, \ldots, \sigma_j]$ is oriented and $v \sim \sigma$, then $v\sigma$ denotes the oriented $(j+1)$-cell $[v, \sigma_0, \ldots, \sigma_j]$. An oriented $(j+1)$-cell $[\sigma_0, \ldots, \sigma_j]$ induces orientations on the j-cells which form its boundary, as follows: the face $\{\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_j\}$ is oriented as $(-1)^i [\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_k]$, where $(-1)\tau = \overline{\tau}$.

The jth *boundary operator* $\partial_j \colon \Omega^j \to \Omega^{j-1}$ is

$$(\partial_j f)(\sigma) = \sum_{v \sim \sigma} f(v\sigma).$$

The sequence (Ω^j, ∂_j) is a chain complex, i.e., $\partial_{j-1}\partial_j = 0$ for all j, and one denotes

$$\begin{aligned} Z_j &= \ker \partial_j & &j\text{-cycles} \\ B_j &= \operatorname{im} \partial_{j+1} & &j\text{-boundaries} \\ H_j &= Z_j / B_j & &\text{the } j^{\text{th}} (\text{reduced}) \text{ homology of } X \text{ (over } \mathbb{R}). \end{aligned}$$

The adjoint of ∂_j w.r.t. the inner product (2.1) is the *co-boundary operator* $\partial_j^* \colon \Omega^{j-1} \to \Omega^j$ given by

$$(\partial_j^* f)(\sigma) = \sum_{\substack{\tau \text{ is in the} \\ \text{boundary of } \sigma}} f(\tau) = \sum_{i=0}^{j} (-1)^i f(\sigma \setminus \sigma_i),$$

where $\sigma \setminus \sigma_i = [\sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots \sigma_j]$. Here the standard terms are

$$\begin{aligned} Z^j &= \ker \partial_{j+1}^* = B_j^\perp & &\text{closed } j\text{-forms (or cocycles)} \\ B^j &= \operatorname{im} \partial_j^* = Z_j^\perp & &\text{exact } j\text{-forms (or coboundaries)} \\ H^j &= Z^j / B^j & &\text{the } j^{\text{th}} (\text{reduced}) \text{ cohomology of } X \text{ (over } \mathbb{R}). \end{aligned}$$

The *upper*, *lower*, and *full Laplacians* $\Delta^+, \Delta^-, \Delta \colon \Omega^{d-1} \to \Omega^{d-1}$ are defined by

$$\Delta^+ = \partial_d \partial_d^*, \qquad \Delta^- = \partial_{d-1}^* \partial_{d-1}, \qquad \text{and} \qquad \Delta = \Delta^+ + \Delta^-,$$

respectively[5]. All the Laplacians decompose (as a direct sum of linear operators) with respect to the orthogonal decompositions $\Omega^{d-1} = B^{d-1} \oplus Z_{d-1} = B_{d-1} \oplus Z^{d-1}$. In addition, $\ker \Delta^+ = Z^{d-1}$ and $\ker \Delta^- = Z_{d-1}$.

The space of *harmonic* $(d-1)$-*forms* on X is $\mathcal{H}_{d-1} = \ker \Delta$. If $f \in \mathcal{H}_{d-1}$ then

$$0 = \langle \Delta f, f \rangle = \langle \partial_{d-1} f, \partial_{d-1} f \rangle + \langle \partial_d^* f, \partial_d^* f \rangle$$

which shows that $\mathcal{H}_{d-1} = Z^{d-1} \cap Z_{d-1}$. This gives the so-called *discrete Hodge decomposition*

$$\Omega^{d-1} = B^{d-1} \oplus \mathcal{H}_{d-1} \oplus B_{d-1} \qquad \text{(orthogonal sum)}.$$

In particular, it follows that the space of harmonic forms can be identified with the cohomology of X:

$$H^{d-1} = \frac{Z^{d-1}}{B^{d-1}} = \frac{B_{d-1}^{\perp}}{B^{d-1}} = \frac{B^{d-1} \oplus \mathcal{H}_{d-1}}{B^{d-1}} \cong \mathcal{H}_{d-1}.$$

The same holds for the homology of X, giving

$$(2.2) \qquad\qquad H^{d-1} \cong \mathcal{H}_{d-1} \cong H_{d-1}.$$

For comparison, the original Hodge decomposition states that for a Riemannian manifold M and $0 \le j \le \dim M$, there is an orthogonal decomposition

$$\Omega^j(M) = d(\Omega^{j-1}(M)) \oplus \mathcal{H}^j(M) \oplus \delta(\Omega^{j+1}(M))$$

where Ω^j are the smooth j-forms on M, d is the exterior derivative, δ its Hodge dual, and $\mathcal{H}^j$ the smooth harmonic j-forms on M. As in the discrete case, this gives an isomorphism between $\mathcal{H}^j(M)$ and $H^j_{dR}(M)$, the jth de-Rham cohomology of M.

Example. For $j = 0$, Z^0 consists of the locally constant functions (functions constant on connected components); B^0 consists of the constant functions; Z_0 of the functions whose sum vanishes, and B_0 of the functions whose sum on each connected component vanishes.

For $j = 1$, Z^1 are the forms whose sum along the boundary of every triangle in the complex vanishes; in B^1 lie the forms whose sum along every closed path vanishes; Z_1 are the *Kirchhoff forms*, also known as *flows*, those for which the sum over all edges incident to a vertex, oriented inward, is zero; and B_1 are the forms spanned (over $\mathbb{R}$) by oriented boundaries of triangles in the complex. The chain of simplicial forms in dimensions -1 to 2 is depicted in Figure 1.

---

[5] More generally, one can define the jth *lower Laplacian* $\Delta_j^- : \Omega^j \to \Omega^j$ by $\Delta_j^- = \partial_j^* \partial_j$, and similarly for Δ_j^+ and Δ_j. For our purposes, Δ_{d-1}^-, Δ_{d-1}^+ and Δ_{d-1} are the relevant ones.
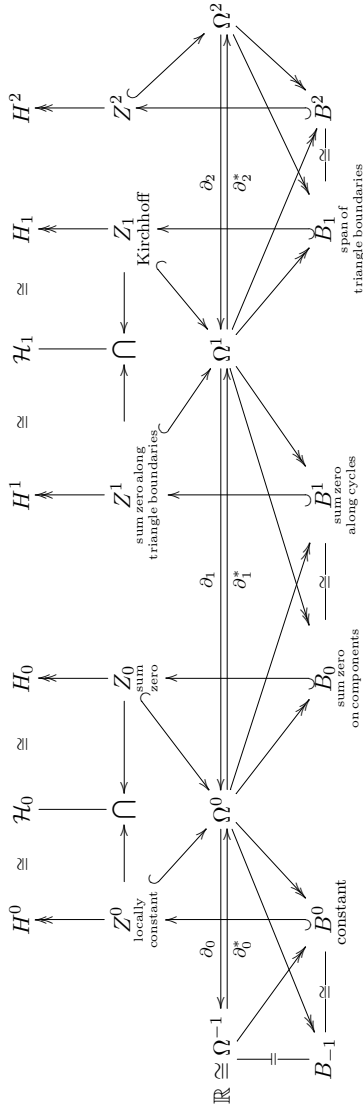
Figure 1. The lowermost part of the chain complex of simplicial forms.

2.1. Definition of the spectral gap

Every graph has a "trivial zero" in the spectrum of its upper Laplacian, corresponding to the constant functions. There can be more zeros in the spectrum, and these encode information about the graph (its connectedness),

while the first one does not. Similarly, for a d-dimensional complex, the space B^{d-1} is always in the kernel of the upper Laplacian, and considered to be its "trivial zeros". The existence of more zeros indicates a nontrivial $(d-1)$-cohomology, since it means that $B^{d-1} \subsetneq \ker \Delta^+ = Z^{d-1}$. As $(B^{d-1})^\perp = Z_{d-1}$, this leads to the following definition:

Definition 2.1. The spectral gap of a d-dimensional complex X, denoted $\lambda(X)$, is the minimal eigenvalue of the upper or the full Laplacian on $(d-1)$-cycles:

$$\lambda(X) = \min \operatorname{Spec}(\Delta\big|_{Z_{d-1}}) = \min \operatorname{Spec}(\Delta^+\big|_{Z_{d-1}})$$

(the equality follows from $\Delta\big|_{Z_{d-1}} \equiv \Delta^+\big|_{Z_{d-1}}.$)

The following proposition gives two more characterizations of the spectral gap. For complexes with a complete skeleton we shall obtain even more explicit characterizations in Proposition 3.2.

Proposition 2.2. Let $\operatorname{Spec} \Delta^+(X) = \{\lambda_0 \le \lambda_1 \le \ldots \le \lambda_{|X^{d-1}|-1}\}$.

1. If $\beta_j = \dim H_j$ is the jth (reduced) Betti number of X, then

$$\lambda(X) = \lambda_r \qquad \text{where} \qquad r = \left(|X^{d-1}| - \beta_{d-1}\right) - \left(|X^d| - \beta_d\right).$$

2. $\lambda(X)$ is the minimal nonzero eigenvalue of Δ^+, unless X has a nontrivial $(d-1)^{\text{th}}$-homology, in which case $\lambda(X) = 0$.

Remark. For a graph $G = (V, E)$, Definition 2.1 states that $\lambda(G)$ is the minimal eigenvalue of the Laplacian on a function which sums to zero. By Proposition 2.2 (1) we have $\lambda(G) = \lambda_r$, where $r = n - |E| - \beta_0 + \beta_1$. Since $\beta_0 + 1$ is the number of connected components in G, and β_1 is the number of cycles in G, by Euler's formula

$$r = n - |E| - \beta_0 + \beta_1 = \chi(G) - (\chi(G) - 1) = 1$$

and therefore $\lambda(G) = \lambda_1$. From (2) in Proposition 2.2 we obtain that $\lambda(G)$ is the minimal nonzero eigenvalue of G's Laplacian if G is connected, and zero otherwise.

Proof. Since Δ^+ decomposes w.r.t. $\Omega^{d-1} = B^{d-1} \oplus Z_{d-1}$, and $\Delta^+\big|_{B^{d-1}} \equiv 0$, the spectrum of Δ^+ consists of $r = \dim B^{d-1}$ zeros, followed by the spectral gap. By (2.2),

$$H_{d-1} \cong \mathcal{H}_{d-1} = Z^{d-1} \cap Z_{d-1} = \ker \Delta^+\big|_{Z_{d-1}}$$

so that $\lambda(X) = 0$ if and only if $H_{d-1} \neq 0$, i.e. X has a nontrivial $(d-1)^{\text{th}}$-homology. This also shows that if $H_{d-1} = 0$, then $\lambda(X)$ is the smallest nonzero eigenvalue of Δ^+. Finally, to compute $r = \dim B^{d-1}$, we observe that

$$\dim B^{j-1} = \dim Z^{j-1} - \dim H^{j-1} = \text{null } \partial_j^* - \beta_{j-1}$$
$$= \dim \Omega^{j-1} - \text{rank } \partial_j^* - \beta_{j-1} = |X^{j-1}| - \dim B^j - \beta_{j-1}$$

and therefore

$$r = \dim B^{d-1} = |X^{d-1}| - \dim B^d - \beta_{d-1}$$
$$= |X^{d-1}| - \left(|X^d| - \dim B^{d+1} - \beta_d\right) - \beta_{d-1}$$
$$= \left(|X^{d-1}| - \beta_{d-1}\right) - \left(|X^d| - \beta_d\right).$$
∎

3. Properties of the Laplacians

In this section we begin the study of the Laplacians and their spectra. We start by writing the Laplacians in a more explicit form.

For the upper Laplacian, if $f \in \Omega^{d-1}$ and $\sigma \in X^{d-1}$, then

$$(\Delta^+ f)(\sigma) = \sum_{v \sim \sigma} (\partial_{d-1}^* f)(v\sigma) = \sum_{v \sim \sigma} \sum_{i=0}^{d} (-1)^i f\left(v\sigma \setminus (v\sigma)_i\right)$$

$$= \sum_{v \sim \sigma} \left[f(\sigma) - \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus \sigma_i) \right]$$

(3.1)
$$= \deg(\sigma) f(\sigma) - \sum_{v \sim \sigma} \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus \sigma_i),$$

where we recall that $\deg(\sigma)$ is the number of d-cells containing σ. Let us introduce the following notation: for $\sigma, \sigma' \in X_{\pm}^{d-1}$ we denote $\sigma' \sim \sigma$ if there exists an oriented d-cell τ such that both σ and $\overline{\sigma'}$ are in the boundary of τ (as oriented cells). Using this notation we can express Δ^+ more elegantly as

(3.2)
$$(\Delta^+ f)(\sigma) = \deg(\sigma) f(\sigma) - \sum_{\sigma' \sim \sigma} f(\sigma').$$

For the lower Laplacian we have

(3.3) $$(\Delta^- f)(\sigma) = \sum_{i=0}^{d-1} (-1)^i (\partial_{d-1} f)(\sigma \setminus \sigma_i) = \sum_{i=0}^{d-1} (-1)^i \sum_{v \sim \sigma \setminus \sigma_i} f(v\sigma \setminus \sigma_i).$$

The following straightforward claim bounds the spectrum of the upper Laplacian:

Claim 3.1. *The spectrum of Δ^+ is contained in the interval $[0, (d+1)k]$, where k is the maximal degree in X.* ∎

3.1. Complexes with a complete skeleton

Complexes with a complete skeleton appear to be particularly well behaved, in comparison with the general case. The following proposition lists some observations regarding their Laplacians. These will be used in the proofs of the main theorems, and also to obtain simpler characterizations of the spectral gap in this case.

Proposition 3.1. *If X has a complete skeleton, then*

1. *If $\overline{X}$ is the complement complex of X, i.e., $\overline{X}^{d-1} = X^{d-1} = \binom{V}{d}$[6] and $\overline{X}^d = \binom{V}{d+1} \setminus X^d$, then*

   $$(3.4) \qquad \Delta^+_{\overline{X}} = n \cdot I - \Delta_X.$$

2. *The spectrum of Δ lies in the interval $[0, n]$.*
3. *The lower Laplacian of X satisfies*

   $$(3.5) \qquad \Delta^- = n \cdot \mathbb{P}_{B^{d-1}}$$

   where $\mathbb{P}_{B^{d-1}}$ is the orthogonal projection onto B^{d-1}.

Proof. By the completeness of the skeleton, the lower Laplacian (see (3.3)) can be written as

$$(\Delta^- f)(\sigma) = \sum_{i=0}^{d-1} (-1)^i \sum_{v \sim \sigma \setminus \sigma_i} f(v\sigma \setminus \sigma_i) = \sum_{i=0}^{d-1} (-1)^i \sum_{v \notin \sigma \setminus \sigma_i} f(v\sigma \setminus \sigma_i)$$

$$= d \cdot f(\sigma) + \sum_{v \notin \sigma} \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus \sigma_i).$$

To show (1) we observe that $v \sim \sigma$ in $\overline{X}$ iff $v \notin \sigma$ and $v \nsim \sigma$ (in X), so that

$$\left(\Delta_X f + \Delta^+_{\overline{X}} f\right)(\sigma) = (\Delta^-_X f)(\sigma) + (\Delta^+_X f)(\sigma) + \left(\Delta^+_{\overline{X}} f\right)(\sigma)$$

---

[6] $\binom{V}{j}$ denotes the set of subsets of V of size j.

$$= d \cdot f(\sigma) + \sum_{v \notin \sigma} \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus \sigma_i)$$

$$+ \deg(\sigma)f(\sigma) - \sum_{v \sim \sigma} \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus \sigma_i)$$

$$+ (n - d - \deg(\sigma))f(\sigma) - \sum_{\substack{v \notin \sigma \\ v \nsim \sigma}} \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus \sigma_i)$$

$$= nf(\sigma).$$

From (1) we conclude that $\operatorname{Spec} \Delta_X^+ = \{n - \gamma \mid \gamma \in \operatorname{Spec} \Delta_X\}$, and since Δ_X and Δ_X^+ are positive semidefinite, (2) follows. To establish (3), recall that $(B^{d-1})^\perp = Z_{d-1} = \ker \Delta^-$, and it is left to show that $\Delta^- f = nf$ for $f \in B^{d-1}$. Note that $B^{d-1} \subseteq Z^{d-1} = \ker \Delta_X^+$, and in addition, that since B^{d-1} only depends on X's $(d-1)$-skeleton,

$$B^{d-1}(X) = B^{d-1}(\overline{X}) \subseteq Z^{d-1}(\overline{X}) = \ker \Delta_{\overline{X}}^+.$$

Now from (1) it follows that for $f \in B^{d-1}$

$$\Delta_X^- f = \Delta_X^- f + \Delta_X^+ f = \Delta_X f = nf - \Delta_{\overline{X}}^+ f = nf$$

as desired. ∎

The next proposition offers alternative characterizations of the spectral gap:

Proposition 3.2. *If X has a complete skeleton, then*

1. The spectral gap of X is obtained by

(3.6) $$\lambda(X) = \min \operatorname{Spec} \Delta.$$

2. Furthermore, it is the $\binom{n-1}{d-1} + 1$ smallest eigenvalue of Δ^+.

Remarks. 1. For graphs (3.6) gives $\lambda(G) = \min \operatorname{Spec}(\Delta^+ + J)$, where

$$J = \Delta^- = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

2. In general (3.6) does not hold: for example, for the triangle complex ▶◀, $\lambda = \min \operatorname{Spec} \left(\Delta \big|_{Z_{d-1}} \right) = 3$ but $\min \operatorname{Spec} \Delta = 1$.

Proof. (1) First, since Δ decomposes w.r.t. $\Omega^{d-1} = B^{d-1} \oplus Z_{d-1}$ we have

$$\operatorname{Spec}\Delta = \operatorname{Spec}\Delta\big|_{B^{d-1}} \cup \operatorname{Spec}\Delta\big|_{Z_{d-1}} = \operatorname{Spec}\Delta^-\big|_{B^{d-1}} \cup \operatorname{Spec}\Delta^+\big|_{Z_{d-1}}.$$

By Proposition 3.1, $\operatorname{Spec}\Delta^-\big|_{B^{d-1}} = \{n\}$ and $\operatorname{Spec}\Delta \subseteq [0, n]$, which implies that

$$\lambda = \min \operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}}\right) = \min \operatorname{Spec}\Delta.$$

(2) The Euler characteristic satisfies

$$\sum_{i=-1}^{d} (-1)^i |X^i| = \chi(X) = \sum_{i=-1}^{d} (-1)^i \beta_i.$$

Therefore, by Proposition 2.2 we have $\lambda = \lambda_r$, with

$$r = (|X^{d-1}| - \beta_{d-1}) - (|X^d| - \beta_d) + (-1)^d \sum_{i=-1}^{d} (-1)^i (|X^i| - \beta_i)$$

$$= \sum_{i=-1}^{d-2} (-1)^{d+i} (|X^i| - \beta_i).$$

Since the $(d-1)$-skeleton is complete, $|X^i| = \binom{n}{i+1}$ and $\beta_i = 0$ for $-1 \leq i \leq d-2$, and so

$$r = \sum_{i=-1}^{d-2} (-1)^{d+i} \binom{n}{i+1} = \binom{n-1}{d-1}. \qquad \blacksquare$$

We finish with a note on the density of d-cells in X:

Proposition 3.3. *Let $\mathfrak{D}$ denote the d-cell density of X, $\mathfrak{D} = \frac{|X^d|}{\binom{n}{d+1}}$, let k denote the average degree of a $(d-1)$-cell, and let λ_{avg} denote the average over the spectrum of $\Delta^+\big|_{Z_{d-1}}$. Then*

$$\mathfrak{D} = \frac{\lambda_{avg}}{n} = \frac{k}{n-d}.$$

Proof. On the one hand

$$\mathfrak{D} = \frac{|X^d|}{\binom{n}{d+1}} = \frac{|X^{d-1}|\frac{k}{d+1}}{\binom{n}{d+1}} = \frac{\binom{n}{d}\frac{k}{d+1}}{\binom{n}{d+1}} = \frac{k}{n-d}.$$

On the other,

$$\binom{n}{d} k = |X^{d-1}|k = \sum_{\sigma \in X^{d-1}} \deg \sigma = \text{trace } \Delta^+ = \sum_{\lambda \in \text{Spec } \Delta^+} \lambda = \sum_{\lambda \in \text{Spec } \Delta^+|_{Z_{d-1}}} \lambda$$

and by Proposition 3.2

$$\lambda_{avg} = \frac{1}{\binom{n}{d} - \binom{n-1}{d-1}} \sum_{\lambda \in \text{Spec } \Delta^+|_{Z_{d-1}}} \lambda = \frac{1}{\binom{n-1}{d}} \sum_{\lambda \in \text{Spec } \Delta^+|_{Z_{d-1}}} \lambda = \frac{n}{n-d} \cdot k. \ \blacksquare$$

4. Proofs of the main theorems

4.1. A Cheeger-type inequality

This section is devoted to the proof of Theorem 1.2: For a complex with a complete skeleton, the Cheeger constant is bounded from below by the spectral gap.

Proof of Theorem 1.2. Recall that we seek to show

$$\min \text{Spec}\left(\Delta^+\big|_{Z_{d-1}}\right) = \lambda(X) \le h(X) = \min_{V = \coprod_{i=0}^d A_i} \frac{n \cdot |F(A_0, A_1, \ldots, A_d)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_d|}.$$

Let $A_0, \ldots, A_d$ be a partition of V which realizes the minimum in h. We define $f \in \Omega^{d-1}$ by

(4.1)
$$f\left([\sigma_0 \sigma_1 \ldots \sigma_{d-1}]\right)$$
$$= \begin{cases} \text{sgn}(\pi) \left| A_{\pi(d)} \right| & \exists \pi \in \text{Sym}_{\{0 \ldots d\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \le i \le d-1 \\ 0 & \text{else, i.e. } \exists k, i \ne j \text{ with } \sigma_i, \sigma_j \in A_k. \end{cases}$$

Note that $f(\pi'\sigma) = \text{sgn}(\pi')f(\sigma)$ for any $\pi' \in \text{Sym}_{\{0 \ldots d-1\}}$ and $\sigma \in X^{d-1}$. Therefore, f is a well-defined skew-symmetric function on oriented $(d-1)$-cells, i.e., $f \in \Omega^{d-1}$. Figure 2 illustrates f for $d = 1, 2$.

We proceed to show that $f \in Z_{d-1}$. Let $\sigma = [\sigma_0, \sigma_1, \ldots, \sigma_{d-2}] \in X_{\pm}^{d-2}$. As we assumed that X^{d-1} is complete,

$$(\partial_{d-1}f)(\sigma) = \sum_{v \sim \sigma} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]) = \sum_{v \notin \sigma} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]).$$
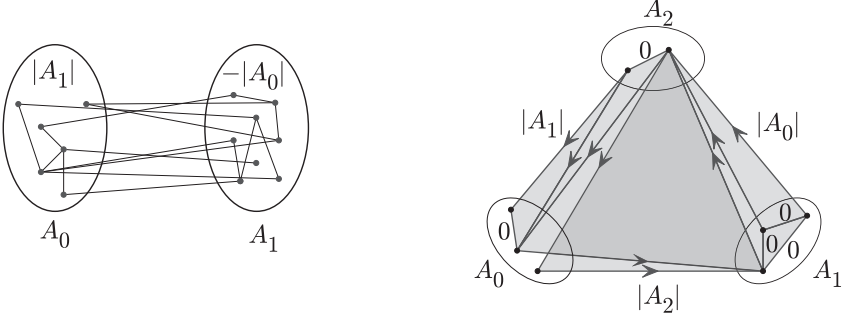
Figure 2. The form $f \in \Omega^{d-1}$ defined in (4.1), for complexes of dimensions one and two.

If for some k and $i \neq j$ we have $\sigma_i, \sigma_j \in A_k$, this sum vanishes. On the other hand, if there exists $\pi \in \mathrm{Sym}_{\{0...d\}}$ such that $\sigma_i \in A_{\pi(i)}$ for $0 \leq i \leq d-2$ then

$$(\partial_{d-1} f)(\sigma) = \sum_{v \in A_{\pi(d-1)}} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]) + \sum_{v \in A_{\pi(d)}} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}])$$

$$= \sum_{v \in A_{\pi(d-1)}} (-1)^{d-1} \operatorname{sgn} \pi |A_{\pi(d)}| + \sum_{v \in A_d} (-1)^d \operatorname{sgn} \pi |A_{\pi(d-1)}|$$

$$= (-1)^{d-1} \operatorname{sgn} \pi \left(|A_{\pi(d-1)}||A_{\pi(d)}| - |A_{\pi(d)}||A_{\pi(d-1)}| \right) = 0$$

and in both cases $f \in Z_{d-1}$. Thus, by Rayleigh's principle

$$(4.2) \qquad \lambda(X) = \min \operatorname{Spec} \left(\Delta^+ \big|_{Z_{d-1}} \right) \leq \frac{\langle \Delta^+ f, f \rangle}{\langle f, f \rangle} = \frac{\langle \partial_d^* f, \partial_d^* f \rangle}{\langle f, f \rangle}.$$

The denominator is

$$\langle f, f \rangle = \sum_{\sigma \in X^{d-1}} f(\sigma)^2,$$

and a $(d-1)$-cell σ contributes to this sum only if its vertices are in different blocks of the partition, i.e., there are no k and $i \neq j$ with $\sigma_i, \sigma_j \in A_k$. In this case, there exists a unique block, A_i, which does not contain a vertex of σ, and σ contributes $|A_i|^2$ to the sum. Since X^{d-1} is complete, there are $|A_0| \cdot \ldots \cdot |A_{i-1}| \cdot |A_{i+1}| \cdot \ldots \cdot |A_d|$ non-oriented $(d-1)$-cells whose vertices are in distinct blocks and which do not intersect A_i, hence

$$\langle f, f \rangle = \sum_{i=0}^{d} \left(\prod_{j \neq i} |A_j| \right) |A_i|^2 = n \prod_{i=0}^{d} |A_i|.$$

To evaluate the numerator in (4.2), we first show that for $\sigma \in X^d$

$$(4.3) \qquad |(\partial_d^* f)(\sigma)| = \begin{cases} n & \sigma \in F(A_0, \ldots, A_d) \\ 0 & \sigma \notin F(A_0, \ldots, A_d) \end{cases}.$$

First, let $\sigma \notin F(A_0, \ldots, A_d)$. If σ has three vertices from the same A_i, or two pairs of vertices from the same blocks (i.e. $\sigma_i, \sigma_j \in A_k$ and $\sigma_{i'}, \sigma_{j'} \in A_{k'}$), then for every summand in

$$(4.4) \qquad (\partial_d^* f)(\sigma) = \sum_{i=0}^{d} (-1)^i f(\sigma \setminus \sigma_i),$$

the cell $\sigma \setminus \sigma_i$ has two vertices from the same block, and therefore $(\partial_d^* f)(\sigma) = 0$. Next, assume that σ_j and σ_k (with $j < k$) is the only pair of vertices in σ which belong to the same block. The only non-vanishing terms in (4.4) are $i = j$ and $i = k$, i.e.,

$$(\partial_d^* f)(\sigma) = (-1)^j f(\sigma \setminus \sigma_j) + (-1)^k f(\sigma \setminus \sigma_k).$$

Since the value of f on a simplex depends only on the blocks to which its vertices belong,

$$\begin{aligned}
f(\sigma \setminus \sigma_j) &= f([\sigma_0\sigma_1 \ldots \sigma_{j-1}\sigma_{j+1} \ldots \sigma_{k-1}\sigma_k\sigma_{k+1} \ldots \sigma_d]) \\
&= f([\sigma_0\sigma_1 \ldots \sigma_{j-1}\sigma_{j+1} \ldots \sigma_{k-1}\sigma_j\sigma_{k+1} \ldots \sigma_d]) \\
&= f\left((-1)^{k-j+1}[\sigma_0\sigma_1 \ldots \sigma_{j-1}\sigma_j\sigma_{j+1} \ldots \sigma_{k-1}\sigma_{k+1} \ldots \sigma_d]\right) \\
&= (-1)^{k-j+1} f(\sigma \setminus \sigma_k),
\end{aligned}$$

so that $(\partial_d^* f)(\sigma) = (-1)^j (-1)^{k-j+1} f(\sigma \setminus \sigma_k) + (-1)^k f(\sigma \setminus \sigma_k) = 0$. The remaining case is $\sigma \in F(A_0, \ldots, A_d)$. Here, there exists $\pi \in \text{Sym}_{\{0 \ldots d\}}$ with $\sigma_i \in A_{\pi(i)}$ for $0 \le i \le d$. Observe that

$$f(\sigma \setminus \sigma_i) = \text{sgn}\left(\pi \cdot (d \; d-1 \; d-2 \ldots i)\right)|A_{\pi(i)}| = (-1)^{d-i} \text{sgn} \, \pi |A_{\pi(i)}|$$

and therefore

$$(\partial_d^* f)(\sigma) = \sum_{i=0}^{d}(-1)^i f(\sigma \setminus \sigma_i) = (-1)^d \text{sgn} \, \pi \sum_{i=0}^{d} |A_{\pi(i)}| = (-1)^d \text{sgn} \, \pi \cdot n.$$

This establishes (4.3), which implies that

$$\langle \partial_d^* f, \partial_d^* f \rangle = \sum_{\sigma \in X^d} |(\partial_d^* f)(\sigma)|^2 = n^2 |F(A_0, \ldots, A_d)|$$

and in total

$$\lambda(X) \leq \frac{\langle \partial_d^* f, \partial_d^* f \rangle}{\langle f, f \rangle} = \frac{n|F(A_0, \ldots, A_d)|}{\prod_{i=0}^d |A_i|} = h(X). \qquad \blacksquare$$

4.2. Towards a lower Cheeger inequality

The first observation to be made regarding a lower Cheeger inequality, is that no bound of the form $C \cdot h(X)^m \leq \lambda(X)$ can be found. Had such a bound existed, one would have that $\lambda(X)=0$ implies $h(X)=0$, but a counterexample to this is provided by the minimal triangulation of the Möbius strip (Figure 3).
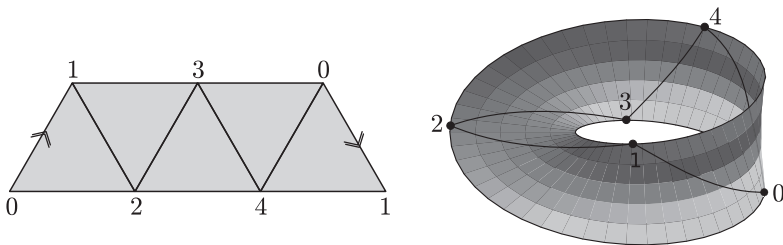

Figure 3. A triangulation of the Möbius strip for which $h(X)=1\frac{1}{4}$ but $\lambda(X)=0$.

Nevertheless, numerical experiments hint that a bound of the form $C \cdot h(X)^2 - c \leq \lambda(X)$ should hold, where C and c depend on the dimension and the maximal degree of a $(d-1)$-cell in X.

An attempt towards an upper bound for the Cheeger constant can be made by connecting it to "local Cheeger constants", as follows. For every $\tau \in X^{d-2}$ we consider the *link* of τ (see Figure 4),

$$\mathrm{lk}\,\tau = \{\sigma \in X \mid \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in X\}.$$

Since $\dim \tau = d-2$, the link of τ is a graph, and there is a $1-1$ correspondence between vertices (edges) of $\mathrm{lk}\,\tau$ and $(d-1)$-cells (d-cells) of X which contain τ. We have the following bound for the Cheeger constant of X:

Proposition 4.1. *The bound* $h(X) \leq \frac{h(\mathrm{lk}\,\tau)}{1 - \frac{d-1}{n}}$ *holds for any d-complex* X *and* $\tau \in X^{d-2}$.
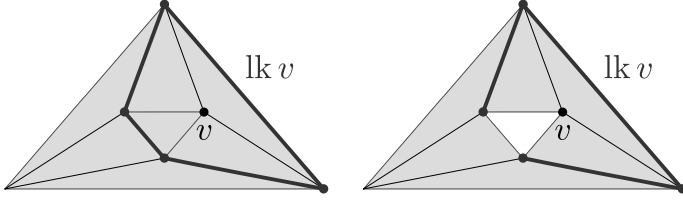
Figure 4. Two examples for the link of a vertex in a triangle complex.

Proof. Write $\tau = [\tau_0, \tau_1, \ldots, \tau_{d-2}]$ and denote $A_i = \{\tau_i\}$ for $0 \le i \le d-2$. Due to the correspondence between $(\mathrm{lk}\,\tau)^j$ and cells in X^{d-1+j} containing τ,

$$h(\mathrm{lk}\,\tau) = \min_{B \coprod C = (\mathrm{lk}\,\tau)^0} \frac{|E_{\mathrm{lk}\,\tau}(B,C)| \cdot |(\mathrm{lk}\,\tau)^0|}{|B| \cdot |C|}$$

$$= \min_{B \coprod C = (\mathrm{lk}\,\tau)^0} \frac{|F(A_0, \ldots, A_{d-2}, B, C)| \cdot |(\mathrm{lk}\,\tau)^0|}{|B| \cdot |C|}.$$

Assume that the minimum is attained by $B = B_0$ and $C = C_0$. We define

$$A_{d-1} = B_0, \qquad A_d = V \setminus \left(\bigcup_{i=0}^{d-1} A_i \right).$$

Now $A_0, \ldots, A_d$ is a partition of V, and

$$F(A_0, \ldots, A_{d-2}, B_0, C_0) = F(A_0, \ldots, A_{d-2}, A_{d-1}, A_d)$$

since no d-cell containing τ has a vertex in $A_d \setminus C_0$. In addition,

$$\frac{|(\mathrm{lk}\,\tau)^0||A_d|}{n|C_0|} \ge \frac{|(\mathrm{lk}\,\tau)^0||A_d| - |A_{d-1}|(|A_d| - |C_0|)}{n|C_0|}$$

$$= \frac{[n - (d-1) - (|A_d| - |C_0|)]|A_d| - |A_{d-1}|(|A_d| - |C_0|)}{n|C_0|}$$

$$= \frac{(n - (d-1))|A_d| - (|A_{d-1}| + |A_d|)(|A_d| - |C_0|)}{n|C_0|}$$

$$= \frac{(n - (d-1))[|A_d| - (|A_d| - |C_0|)]}{n|C_0|} = 1 - \frac{d-1}{n},$$

which implies

$$h(\mathrm{lk}\,\tau) = \frac{F(A_0, \ldots, A_{d-2}, A_{d-1}, A_d)|(\mathrm{lk}\,\tau)^0|}{|B_0| \cdot |C_0|}$$

$$= \frac{F(A_0, \ldots, A_{d-2}, A_{d-1}, A_d)n}{|A_0| \cdot \ldots \cdot |A_d|} \cdot \frac{|(\mathrm{lk}\,\tau)^0||A_d|}{n|C_0|}$$

$$\geq h(X) \cdot \frac{|(\mathrm{lk}\,\tau)^0||A_d|}{n|C_0|} \geq \left(1 - \frac{d-1}{n}\right) h(X). \qquad \blacksquare$$

Since $\mathrm{lk}\,\tau$ is a graph, its Cheeger constant can be bounded using the lower inequality in (1.1). The degree of a vertex in $\mathrm{lk}\,\tau$ corresponds to the degree of a $(d-1)$-cell in X, and therefore

$$(4.5) \qquad \frac{\left(1 - \frac{d-1}{n}\right)^2}{8k} h^2(X) \leq \frac{h(\mathrm{lk}\,\tau)^2}{8k} \leq \frac{h(\mathrm{lk}\,\tau)^2}{8k_\tau} \leq \lambda(\mathrm{lk}\,\tau)$$

where k is the maximal degree of a $(d-1)$-cell in X, and k_τ of a vertex in $\mathrm{lk}\,\tau$.

We now see that a bound of the spectral gap of links by that of the complex would yield a lower Cheeger inequality. Such a bound was indeed discovered by Garland in [24], and was studied further by several authors [49,1,27]. The following lemma appears in [27], for a normalized version of the Laplacian. We give here, without proof, its form for the Laplacian we use.

Lemma 4.2 ([24,27]). *Let X be a d-dimensional simplicial complex. Given $f \in \Omega^{d-1}, \sigma \in X^{d-1}, \tau \in X^{d-2}$ define a function $f_\tau : (\mathrm{lk}\,\tau)^0 \to \mathbb{R}$ by $f_\tau(v) = f(v\tau)$, and an operator $\Delta_\tau^+ : \Omega^{d-1}(X) \to \Omega^{d-1}(X)$ by*

$$(\Delta_\tau^+ f)(\sigma) = \begin{cases} \deg_\tau(\sigma)f(\sigma) - \sum\limits_{\substack{\sigma' \sim \sigma \\ \tau \subseteq \sigma'}} f(\sigma') & \tau \subset \sigma \\ 0 & \tau \not\subseteq \sigma \end{cases}$$

where $\deg_\tau(\sigma) = \#\{\sigma' \sim \sigma \mid \tau \subseteq \sigma'\} = \deg_{\mathrm{lk}\,\tau}(\sigma \setminus \tau)$. The following then hold:

1. *$\Delta^+ = \left(\sum_{\tau \in X^{d-2}} \Delta_\tau^+\right) - (d-1)D$, where $(Df)(\sigma) = \deg(\sigma)f(\sigma)$.*
2. *$\langle \Delta_\tau^+ f, f \rangle = \langle \Delta_{\mathrm{lk}\,\tau}^+ f_\tau, f_\tau \rangle$.*
3. *If $f \in Z_{d-1}$ then $f_\tau \in Z_0(\mathrm{lk}\,\tau)$.*
4. *$\sum_{\tau \in X^{d-2}} \langle f_\tau, f_\tau \rangle = d \langle f, f \rangle$.*

Assume now that $f \in Z_{d-1}$ is a normalized eigenfunction for $\lambda(X)$, i.e. $\langle f, f \rangle = 1$ and $\Delta^+ f = \lambda(X)f$. Using the lemma we find that

$$\lambda(X) = \langle \Delta^+ f, f \rangle \overset{(1)}{=} \sum_{\tau \in X^{d-2}} \langle \Delta_\tau^+ f, f \rangle - (d-1)\langle Df, f \rangle$$

$$\overset{(2)}{=} \sum_{\tau \in X^{d-2}} \langle \Delta_{\mathrm{lk}\,\tau}^+ f_\tau, f_\tau \rangle - (d-1)\langle Df, f \rangle \geq \sum_{\tau \in X^{d-2}} \langle \Delta_{\mathrm{lk}\,\tau}^+ f_\tau, f_\tau \rangle - (d-1)k$$

$$\overset{(3)}{\geq} \sum_{\tau \in X^{d-2}} \lambda(\mathrm{lk}\,\tau)\langle f_\tau, f_\tau\rangle - (d-1)k \overset{(4)}{=} d \min_{\tau \in X^{d-2}} \lambda(\mathrm{lk}\,\tau) - (d-1)k.$$

By (4.5) we obtain the bound

$$\frac{d\left(1 - \frac{d-1}{n}\right)^2}{8k} h^2(X) - (d-1)k \leq \lambda(X).$$

Sadly, this bound is trivial, as it is not hard to show that the l.h.s. is non-positive for every complex X. A possible line of research would be to find a stronger relation between the spectral gap of the complex and that of its links, for the case of complexes with a complete skeleton (Garland's work applies to general ones).

4.3. The Mixing Lemma

Here we prove Theorem 1.4. We begin by formulating it precisely.

Theorem (1.4). *Let X be a d-dimensional complex with a complete skeleton. Fix $\alpha \in \mathbb{R}$, and write $\mathrm{Spec}(\alpha I - \Delta^+) = \{\mu_0 \geq \mu_1 \geq \ldots \geq \mu_m\}$ (where $m = \binom{n}{d} - 1$). For any disjoint sets of vertices $A_0, \ldots, A_d$ (not necessarily a partition), one has*

$$\left| |F(A_0, \ldots, A_d)| - \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \leq \rho_\alpha \cdot (|A_0| \cdot \ldots \cdot |A_d|)^{\frac{d}{d+1}}$$

where

$$\rho_\alpha = \max\left\{ \left|\mu_{\binom{n-1}{d-1}}\right|, |\mu_m| \right\} = \left\| (\alpha I - \Delta^+)\big|_{Z_{d-1}} \right\|.$$

Remark 4.3. Which α should one take in practice? In the introduction we state the theorem for $\alpha = k$, the average degree of a $(d-1)$-cell, so that it generalize the familiar form of the Expander Mixing Lemma for k-regular graphs. However, the expectation of $|F(A_0, \ldots, A_d)|$ in the random settings is actually $\mathfrak{D}|A_0| \cdot \ldots \cdot |A_d|$, where $\mathfrak{D}$ is the d-cell density $\frac{|X^d|}{\binom{n}{d}}$. Therefore, $\alpha = n\mathfrak{D} = \frac{nk}{n-d}$ is actually a more accurate choice. This becomes even clearer upon observing that we seek to minimize $\rho_\alpha = \left\| (\alpha I - \Delta^+)\big|_{Z_{d-1}} \right\|$, since Proposition 3.3 shows that the spectrum of $\Delta^+\big|_{Z_{d-1}}$ is centered around $\lambda_{avg} = n\mathfrak{D} = \frac{nk}{n-d}$. While for a fixed d the choice between k and $\frac{nk}{n-d}$ is negligible, this should be taken into account when d depends on n.

Proof. For any disjoint sets of vertices $A_0, \ldots, A_{d-1}$, define $\delta_{A_0,\ldots,A_{d-1}} \in \Omega^{d-1}$ by

$$\delta_{A_0,\ldots,A_{d-1}}(\sigma)$$
$$= \begin{cases} \operatorname{sgn}(\pi) & \exists \pi \in \operatorname{Sym}_{\{0\ldots d-1\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \leq i \leq d-1 \\ 0 & \text{else} \end{cases}.$$

Since the skeleton of X is complete,

$$(4.6) \qquad \|\delta_{A_0,\ldots,A_{d-1}}\| = \sqrt{\sum_{\sigma \in X^{d-1}} \delta^2_{A_0,\ldots,A_{d-1}}(\sigma)} = \sqrt{|A_0| \cdot \ldots \cdot |A_{d-1}|}.$$

Now, let $A_0, \ldots, A_d$ be disjoint subsets of V (not necessarily a partition), and denote

$$\varphi = \delta_{A_0,A_1,A_2,\ldots,A_{d-1}}$$
$$\psi = \delta_{A_d,A_1,A_2,\ldots,A_{d-1}}.$$

Let σ be an oriented $(d-1)$-cell with one vertex in each of $A_0, A_1, \ldots, A_{d-1}$. We shall denote this by $\sigma \in F(A_0, \ldots, A_{d-1})$, ignoring the orientation of σ. There is a correspondence between d-cells in $F(A_0, \ldots, A_d)$ containing σ, and neighbors of σ which lie in $F(A_d, A_1, \ldots, A_{d-1})$. Furthermore, for such a neighbor σ' we have $\varphi(\sigma) = \psi(\sigma')$, since σ and σ' must share the vertices which belong to $A_1, \ldots, A_{d-1}$. Therefore, denoting again $(Df)(\sigma) = \deg(\sigma) f(\sigma)$ we have by (3.2)

$$(4.7)$$
$$\begin{aligned} \langle \varphi, (D - \Delta^+)\psi \rangle &= \sum_{\sigma \in X^{d-1}} \varphi(\sigma)\big((D - \Delta^+)\psi\big)(\sigma) \\ &= \sum_{\sigma \in X^{d-1}} \sum_{\sigma' \sim \sigma} \varphi(\sigma)\psi(\sigma') \\ &= \sum_{\sigma \in F(A_0\ldots A_{d-1})} \sum_{\sigma' \sim \sigma} \varphi(\sigma)\psi(\sigma') \\ &= \sum_{\sigma \in F(A_0\ldots A_{d-1})} \#\{\sigma' \in F(A_d, A_1, \ldots, A_{d-1}) \mid \sigma' \sim \sigma\} \\ &= \sum_{\sigma \in F(A_0\ldots A_{d-1})} \#\{\tau \in F(A_0, A_1, \ldots, A_d) \mid \sigma \subseteq \tau\} \\ &= |F(A_0, A_1, \ldots, A_d)|. \end{aligned}$$

Notice that since the A_i are disjoint, φ and ψ are supported on different $(d-1)$-cells, so that for any $\alpha \in \mathbb{R}$

$$(4.8) \qquad \langle \varphi, (D - \Delta^+)\psi \rangle = \langle \varphi, -\Delta^+ \psi \rangle = \langle \varphi, (\alpha I - \Delta^+)\psi \rangle.$$

As Δ^+ decomposes w.r.t. the orthogonal decomposition $\Omega^{d-1} = B^{d-1} \oplus Z_{d-1}$, and since $B^{d-1} \subseteq Z^{d-1} = \ker \Delta^+$,

$$
\begin{aligned}
|F(A_0, A_1, \ldots, A_d)| &= \langle \varphi, (\alpha I - \Delta^+)\psi \rangle \\
&= \langle \varphi, (\alpha I - \Delta^+)(\mathbb{P}_{B^{d-1}}\psi + \mathbb{P}_{Z_{d-1}}\psi) \rangle \\
&= \langle \varphi, \alpha \mathbb{P}_{B^{d-1}}\psi + (\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi \rangle \\
(4.9) \qquad &= \alpha \langle \varphi, \mathbb{P}_{B^{d-1}}\psi \rangle + \langle \varphi, (\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi \rangle.
\end{aligned}
$$

We proceed to evaluate each of these terms separately. Using (3.5) and (3.4) we find that

$$\alpha \langle \varphi, \mathbb{P}_{B^{d-1}}\psi \rangle = \frac{\alpha}{n}\langle \varphi, \Delta^-\psi \rangle = \frac{\alpha}{n}\left\langle \varphi, \left(nI - \Delta_X^+ - \Delta_{\overline{X}}^+\right)\psi \right\rangle$$

and by (4.7) and (4.8) this implies

$$
\begin{aligned}
\alpha \langle \varphi, \mathbb{P}_{B^{d-1}}\psi \rangle &= \frac{\alpha}{n}\left\langle \varphi, \left(nI - \Delta_X^+\right)\psi \right\rangle + \frac{\alpha}{n}\left\langle \varphi, -\Delta_{\overline{X}}^+\psi \right\rangle \\
&= \frac{\alpha}{n}|F_X(A_0, A_1, \ldots, A_d)| + \frac{\alpha}{n}|F_{\overline{X}}(A_0, A_1, \ldots, A_d)| \\
(4.10) \qquad &= \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n}.
\end{aligned}
$$

We turn to the second term in (4.9). First, we recall from Proposition 3.2 that $\dim B^{d-1} = \binom{n-1}{d-1}$. Since $B^{d-1} \subseteq \ker \Delta^+$, we can assume that in $\mathrm{Spec}(\alpha I - \Delta^+) = \{\mu_0 \geq \mu_1 \geq \ldots \geq \mu_m\}$ the first $\binom{n-1}{d-1}$ values correspond to B^{d-1}, and the rest to $(B^{d-1})^\perp = Z_{d-1}$. Thus,

$$(4.11) \qquad \rho_\alpha = \left\| (\alpha I - \Delta^+)\big|_{Z_{d-1}} \right\| = \max\left\{ \left| \mu_{\binom{n-1}{d-1}} \right|, |\mu_m| \right\}$$

and therefore

$$
\begin{aligned}
|\langle \varphi, (\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi \rangle| &\leq \|\varphi\| \cdot \|(\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi\| \\
&\leq \|\varphi\| \cdot \left\|(\alpha I - \Delta^+)\big|_{Z_{d-1}}\right\| \cdot \|\mathbb{P}_{Z_{d-1}}\psi\| \\
(4.12) \qquad &\leq \rho_\alpha \cdot \|\varphi\| \cdot \|\psi\| = \rho_\alpha \sqrt{|A_0||A_d|}|A_1||A_2|\ldots|A_{d-1}|,
\end{aligned}
$$

where the last step is by (4.6). Together, (4.9), (4.10) and (4.12) give

$$\left| |F(A_0, A_1, \ldots, A_d)| - \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \leq \rho_\alpha \sqrt{|A_0||A_d|}|A_1||A_2|\ldots|A_{d-1}|.$$

Since $A_0, \ldots, A_d$ play the same role, one can also obtain the bound

$$\rho_\alpha \sqrt{|A_{\pi(0)}||A_{\pi(d)}||A_{\pi(1)}||A_{\pi(2)}|\ldots|A_{\pi(d-1)}|},$$

for any $\pi \in \mathrm{Sym}_{\{0..d\}}$. Taking the geometric mean over all such π gives

$$\left| |F(A_0, A_1, \ldots, A_d)| - \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \leq \rho_\alpha \cdot (|A_0||A_1|\ldots|A_d|)^{\frac{d}{d+1}}. \quad \blacksquare$$

Remark. The estimate (4.12) is somewhat wasteful. As is done in graphs, a slightly better one is

$$|\langle \varphi, (\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi\rangle| = |\langle \mathbb{P}_{Z_{d-1}}\varphi, (\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi\rangle|$$
$$\leq \rho_\alpha \cdot \|\mathbb{P}_{Z_{d-1}}\varphi\| \cdot \|\mathbb{P}_{Z_{d-1}}\psi\|,$$

and we leave it to the curious reader to verify that this gives

$$|\langle \varphi, (\alpha I - \Delta^+)\mathbb{P}_{Z_{d-1}}\psi\rangle|$$
$$\leq \frac{\rho_\alpha}{n} \sqrt{|A_0|\left(n - \sum_{i=1}^{d}|A_i|\right)|A_d|\left(n - \sum_{i=0}^{d-1}|A_i|\right)|A_1|\ldots|A_{d-1}|}.$$

4.4. Gromov's geometric overlap

Here we prove Corollary 1.6, which gives a bound on the geometric overlap of a complex in terms of the width of its spectrum.

Proof of Corollary 1.6. Given $\varphi\colon V \to \mathbb{R}^d$, choose arbitrarily some partition of V into equally sized parts $P_0, \ldots, P_d$. By Pach's theorem [43], there exist $c_d > 0$ and $Q_i \subseteq P_i$ of sizes $|Q_i| = c_d|P_i|$ such that some $x \in \mathbb{R}^{d+1}$ satisfies $x \in \mathrm{conv}\{\varphi(v) \mid v \in \sigma\}$ for all $\sigma \in F(Q_0, \ldots, Q_d)$. By the Mixing Lemma (Theorem 1.4),

$$|F(Q_0, \ldots, Q_d)| \geq \frac{k \cdot |Q_0| \cdot \ldots \cdot |Q_d|}{n} - \varepsilon \cdot (|Q_0| \cdot \ldots \cdot |Q_d|)^{\frac{d}{d+1}}$$
$$= \left(\frac{c_d n}{d+1}\right)^d \left(\frac{k c_d}{d+1} - \varepsilon\right).$$

On the other hand,

$$|X^d| = |X^{d-1}|\frac{k}{d+1} = \binom{n}{d}\frac{k}{d+1} \leq \left(\frac{en}{d}\right)^d \frac{k}{d+1}.$$

As this holds for every φ,

$$\text{overlap}(X) \geq \left(\frac{c_d d}{e(d+1)} \right)^d \left(c_d - \frac{\varepsilon(d+1)}{k} \right) \geq \frac{c_d^d}{e^{d+1}} \left(c_d - \frac{\varepsilon(d+1)}{k} \right). \quad \blacksquare$$

Remark 4.4. Following Remark 4.3, if $\operatorname{Spec} \Delta^+\big|_{Z_{d-1}} \subseteq [\lambda_{avg} - \varepsilon', \lambda_{avg} + \varepsilon']$ then using the Mixing Lemma with $\alpha = \lambda_{avg} = \frac{nk}{n-d}$ one has

$$|F(Q_0, \ldots, Q_d)| \geq \frac{k \cdot |Q_0| \cdot \ldots \cdot |Q_d|}{n-d} - \varepsilon' \cdot (|Q_0| \cdot \ldots \cdot |Q_d|)^{\frac{d}{d+1}}$$

$$\geq \left(\frac{c_d n}{d+1} \right)^d \left(\frac{nk c_d}{(n-d)(d+1)} - \varepsilon' \right)$$

so that

$$\text{overlap}(X) \geq \frac{c_d^d n}{e^{d+1}(n-d)} \left(c_d - \frac{\varepsilon'(d+1)}{\lambda_{avg}} \right).$$

4.5. Expansion in random complexes

In this section we prove Corollaries 1.3 and 1.7, regarding the expansion of random Linial–Meshulam complexes. The main idea is the following lemma, which is a variation on the analysis in [27] of the spectrum of $D - \Delta^+$ for $X = X(d, n, p)$.

Lemma 4.5. Let $c > 0$. There exists $\gamma = O\left(\sqrt{C}\right)$ (as $C \to \infty$) such that $X = X\left(d, n, \frac{C \cdot \log n}{n}\right)$ satisfies

$$\operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}} \right) \subseteq [(C - \gamma) \log n, (C + \gamma) \log n]$$

with probability at least $1 - n^{-c}$.

Proof. We denote $p = \frac{C \cdot \log n}{n}$. For C large enough we shall find $\gamma = O\left(\sqrt{C}\right)$ such that

$$(4.13) \qquad \left\| (\Delta^+ - pn \cdot I)\big|_{Z_{d-1}} \right\| \leq \gamma \log n$$

holds with probability at least $1 - n^{-c}$. This implies the Lemma, by

$$\operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}} \right) \subseteq [pn - \gamma \log n, pn + \gamma \log n] = [(C - \gamma) \log n, (C + \gamma) \log n].$$

To show (4.13) we use

$$\left\| (\Delta^+ - pn \cdot I) \big|_{Z_{d-1}} \right\| = \left\| (\Delta^+ - p(n-d)I - pdI + D - D) \big|_{Z_{d-1}} \right\|$$

$$(4.14) \qquad \leq \left\| (D - p(n-d)I) \big|_{Z_{d-1}} \right\| + \left\| (D - \Delta^+ + pdI) \big|_{Z_{d-1}} \right\|$$

and bound each term separately. For the first, we have

$$\left\| (D - (n-d)pI) \big|_{Z_{d-1}} \right\| \leq \| D - (n-d)pI \| = \max_{\sigma \in X^{d-1}} \deg \sigma - (n-d)p|.$$

Since $\deg \sigma \sim B(n-d, p)$, a Chernoff type bound (e.g. [29, Theorem 1]) gives that for every $t > 0$

$$\mathrm{Prob}(\deg \sigma - (n-d)p| > t) \leq 2e^{-\frac{t^2}{2(n-d)p + \frac{2t}{3}}}.$$

By a union bound on the degrees of the $(d-1)$-cells we get

$$(4.15) \qquad \mathrm{Prob}\left(\max_{\sigma \in X^{d-1}} \deg \sigma - (n-d)p| > t \right) \leq 2\binom{n}{d} e^{-\frac{t^2}{2(n-d)p + \frac{2t}{3}}},$$

and a straightforward computation shows that there exists $\alpha = \alpha(c, d) > 0$ such that for $t = \alpha\sqrt{np \log n}$, the r.h.s. in (4.15) is bounded by $\frac{1}{2n^c}$ for large enough C and n. In total this implies

$$(4.16) \qquad \mathrm{Prob}\left(\left\| (D - (n-d)pI) \big|_{Z_{d-1}} \right\| \leq \alpha\sqrt{C} \log n \right) \geq 1 - \frac{1}{2n^c}.$$

In order to understand the last term in (4.14) we follow [27], which shows that $(D_X - \Delta_X^+)\big|_{Z_{d-1}}$ is close to p times $\left(D_{K_n^d} - \Delta_{K_n^d}^+ \right)\big|_{Z_{d-1}}$, where K_n^d is the complete d-complex on n vertices. Note that $D_{K_n^d} = (n-d) \cdot I$ and $\Delta_{K_n^d}^+\big|_{Z_{d-1}} = n \cdot I$, and that $Z_{d-1}(X) = Z_{d-1}(K_n^d)$ as both have the same $(d-1)$-skeleton. In the proof of Theorem 7 in [27] (which uses an idea from [42]), it is shown that

$$\mathrm{Prob}\left(\left\| (D_X - \Delta_X^+ + pdI) \big|_{Z_{d-1}} \right\| \geq t \right)$$

$$= \mathrm{Prob}\left(\left\| (D_X - \Delta_X^+) \big|_{Z_{d-1}} - p \left(D_{K_n^d} - \Delta_{K_n^d}^+ \right) \big|_{Z_{d-1}} \right\| \geq t \right)$$

$$\leq 2\binom{n}{d} e^{-\frac{t^2}{8pnd + 4t}}.$$

Again, there exists $\beta = \beta(c,d) > 0$ such that for $t = \beta\sqrt{np\log n}$, the r.h.s. is bounded by $\frac{1}{2n^c}$ for large enough C and n. Consequently,

$$\text{Prob}\left(\left\|(D - \Delta^+ + pdI)\big|_{Z_{d-1}}\right\| \leq \beta\sqrt{C}\log n\right) \geq 1 - \frac{1}{2n^c},$$

so that

$$\text{Prob}\left(\left\|(\Delta^+ - pnI)\big|_{Z_{d-1}}\right\| \leq (\alpha + \beta)\sqrt{C}\log n\right) \geq 1 - n^{-c},$$

and $\gamma = (\alpha + \beta)\sqrt{C}$ gives the required result. ∎

We obtain the following corollary, which implies in particular Corollaries 1.3 and 1.7.

Corollary 4.6. *Let* $X = X\left(d, n, \frac{C \cdot \log n}{n}\right)$.

1. *Given* $c > 0$, *there exist a constant* $H = C - O\left(\sqrt{C}\right)$ *such that for large enough* n

   (4.17) $$\text{Prob}(h(X) \geq H \cdot \log n) \geq 1 - n^{-c},$$

   and for any $\vartheta < \left(\frac{c_d}{e}\right)^{d+1}$ *(where* c_d *is Pach's constant [43]), for* C *and* n *large enough*

   $$\text{Prob}(\text{overlap}(X) > \vartheta) \geq 1 - n^{-c}.$$

2. *If* $C < 1$ *then* $\text{Prob}(h(X) = 0) \overset{n\to\infty}{\longrightarrow} 1$.

Proof. (1) Since $\lambda(X) \leq h(X)$ (Theorem 1.2), (4.17) follows from Lemma 4.5 with $H = C - \gamma$. We turn to the geometric overlap. From Lemma 4.5 it follows that for C large enough a.a.s. $\text{Spec}\,\Delta^+\big|_{Z_{d-1}} \subseteq [(C - \gamma)\log n, (C + \gamma)\log n]$. Therefore, $\text{Spec}\left(\Delta^+\big|_{Z_{d-1}}\right) \subseteq [\lambda_{avg} - \varepsilon', \lambda_{avg} + \varepsilon']$ with $\varepsilon' = 2\gamma\log n$. By Remark 4.4,

$$\text{overlap}(X) \geq \frac{c_d^d n}{e^{d+1}(n-d)}\left(c_d - \frac{2\gamma\log n(d+1)}{\lambda_{avg}}\right)$$

$$\geq \frac{c_d^d}{e^{d+1}}\left(c_d - \frac{2\gamma(d+1)}{C - \gamma}\right) \overset{C\to\infty}{\longrightarrow} \left(\frac{c_d}{e}\right)^{d+1}.$$

(2) Choose some $\tau \in X^{d-2}$. It was observed in [27] that $\text{lk}\,\tau \sim G\left(n - d + 1, \frac{C \cdot \log n}{n}\right)$ (where $G(n,p) = X(1,n,p)$ is the Erdős–Rényi model), and $G\left(n, \frac{C \cdot \log n}{n}\right)$ has isolated vertices a.a.s. for $C < 1$ [17,18]. These correspond to isolated $(d-1)$-cells in X (cells of degree zero), whose existence implies $h(X) = 0$ (and thus also $\lambda(X) = 0$). ∎

5. Open questions

Non-complete skeleton.   The proof of the generalized mixing lemma assumes that the skeleton is complete. This raises the following question:

Question: Can the discrepancy in X be bounded for general simplicial complexes?

As remarked after the statement of Theorem 1.2, one always has $h(X)=0$ for X with a non-complete skeleton. This calls for a refined definition, and a natural candidate is the following:

$$\widetilde{h}(X) = \min_{V=\coprod_{i=0}^{d} A_i} \frac{n \cdot |F(A_0, A_1, \ldots, A_d)|}{|F^{\partial}(A_0, A_1, \ldots, A_d)|},$$

where $F^{\partial}(A_0, A_1, \ldots, A_d)$ denotes the set of $(d-1)$-spheres (i.e. copies of the $(d-1)$-skeleton of the d-simplex) having one vertex in each A_i. For a complex X with a complete skeleton, $\widetilde{h}(X)=h(X)$ as $F^{\partial}(A_0, \ldots, A_d)=A_0 \times \ldots \times A_d$. It is not hard to see that a lower Cheeger inequality does not hold here: consider any non-minimal triangulation of the $(d-1)$-sphere, and attach a single d-simplex to one of the $(d-1)$-cells on it. The obtained complex has $\lambda=0$, and $\widetilde{h}=n$. However, we conjecture that the upper bound still holds:

Question: Does the inequality $\lambda(X) \leq \widetilde{h}(X)$ holds for every d-complex?

Note added in proof.   By the time this paper is in press, this question was solved in the affirmative in [26].

Inverse Mixing Lemma   In [6] Bilu and Linial prove an Inverse Mixing Lemma for graphs:

Theorem ([6]). *Let G be a k-regular graph on n vertices. Suppose that for any disjoint $A, B \subseteq V$*

$$\left| E(A, B) - \frac{k|A||B|}{n} \right| \leq \rho \sqrt{|A||B|}.$$

Then the nontrivial eigenvalues of $kI - \Delta_G^+$ are bounded, in absolute value, by $O\left(\rho\left(1+\log\left(\frac{k}{\rho}\right)\right)\right)$.

Question: Can one prove a generalized Inverse Mixing Lemma for simplicial complexes?

Random simplicial complexes   In the random graph model $G = G(n,p) = X(1,n,p)$, taking $p = \frac{k}{n}$ with a fixed k gives disconnected G a.a.s. However, random k-regular graphs are a.a.s. connected, and in fact are excellent expanders (see e.g. [22,45]). In higher dimension, $X = X\left(d,n,\frac{k}{n}\right)$ has a.a.s. a nontrivial $(d-1)$-homology, and also $h(X) = 0$ (by Corollary 4.6 (2)). It is thus natural to ask about the expansion quality of k-regular d-complexes, but since it is not clear whether such complexes even exist, we say that a *k-semiregular* complex is a complex with $k - \sqrt{k} \le \deg\sigma \le k + \sqrt{k}$ for all $\sigma \in X^{\dim X - 1}$, and ask:

Question: Are $\lambda(X)$, $h(X)$ and overlap(X) bounded away from zero with high probability, for X a random k-semiregular d-complex with a complete skeleton?

A Riemannian analogue   In Riemannian geometry, the Cheeger constant of a Riemannian manifold M is concerned with its partitions into two submanifolds along a common boundary of codimension one. The original Cheeger inequalities, due to Cheeger [9] and Buser [8], relate the Cheeger constant to the smallest eigenvalue of the Laplace–Beltrami operator on $C^\infty(M) = \Omega^0(M)$.

Question: Can one define an isoperimetric quantity which concerns partitioning of M into $d+1$ parts, and relate it to the spectrum of the Laplace–Beltrami operator on $\Omega^{d-1}(M)$, the space of smooth $(d-1)$-forms?

Ramanujan complexes   *Ramanujan Graphs* are expanders which are spectrally optimal in the sense of the Alon–Boppana theorem [40], and therefore excellent combinatorial expanders. Such graphs were constructed in [32] as quotients of the Bruhat–Tits tree associated with $\mathrm{PSL}_2(\mathbb{Q}_p)$ by certain arithmetic lattices. Analogue quotients of the Bruhat–Tits buildings associated with $\mathrm{PSL}_d\left(\mathbb{F}_q\big((t)\big)\right)$ are constructed in [33], and termed *Ramanujan Complexes*. It is natural to ask whether these complexes are also optimal expanders in the spectral and combinatorial senses.

References

[1] R. AHARONI, E. BERGER and R. MESHULAM: Eigenvalues and homology of flag complexes and vector representations of graphs, *Geometric and functional analysis* **15** (2005), 555–566.

[2] N. ALON and F. R. K. CHUNG: Explicit construction of linear sized tolerant networks, *Discrete Mathematics* **72** (1988), 15–19.

[3] N. ALON: Eigenvalues and expanders, *Combinatorica* **6** (1986), 83–96.

[4] N. ALON and V. D. MILMAN: λ_1, isoperimetric inequalities for graphs, and super-concentrators, *Journal of Combinatorial Theory, Ser. B* **38** (1985), 73–88.

[5] M. BLUM, R. M. KARP, O. VORNEBERGER, C. H. PAPADIMITRIOU and M. YAN-NAKAKIS: Complexity of testing whether a graph is a superconcentrator, *Information Processing Letters* **13** (1981), 164–167.

[6] Y. BILU and N. LINIAL: Lifts, discrepancy and nearly optimal spectral gap, *Combinatorica* **26** (2006), 495–519.

[7] R. BEIGEL, G. MARGULIS and D. A. SPIELMAN: Fault diagnosis in a small constant number of parallel testing rounds, in: *Proceedings of the fifth annual ACM symposium on Parallel algorithms and architectures,* ACM, 1993, 21–29.

[8] P. BUSER: A note on the isoperimetric constant, *Ann. Sci. École Norm. Sup.* **15** (1982), 213–230.

[9] J. CHEEGER: A lower bound for the smallest eigenvalue of the Laplacian, *Problems in analysis* **195** (1970), 199.

[10] F. CHUNG: *The Laplacian of a hypergraph, Expanding Graphs* (Joel Friedman, ed.), DIMACS, **10**, AMS, 1993, 21–36.

[11] F. R. K. CHUNG: *Spectral graph theory*, CBMS, Amer Mathematical Society, 1997.

[12] D. DOTTERRER and M. KAHLE: Coboundary expanders, *Journal of Topology and Analysis* **4** (2012), 499–514.

[13] A. DUVAL, C. KLIVANS and J. MARTIN: Simplicial matrix-tree theorems, *Transactions of the American Mathematical Society* **361** (2009), 6073–6114.

[14] J. DODZIUK: Finite-difference approach to the Hodge theory of harmonic forms, *American Journal of Mathematics* **98** (1976), 79–104.

[15] J. DODZIUK: Difference equations, isoperimetric inequality and transience of certain random walks, *Trans. Amer. Math. Soc.* **284** (1984).

[16] B. ECKMANN: Harmonische funktionen und randwertaufgaben in einem komplex, *Commentarii Mathematici Helvetici* **17** (1944), 240–255.

[17] P. ERDŐS and A. RÉNYI: On random graphs, *Publicationes Mathematicae Debrecen* **6** (1959), 290–297.

[18] P. ERDŐS and A. RÉNYI: On the evolution of random graphs, *Bull. Inst. Internat. Statist* **38** (1961), 343–347.

[19] J. FOX, M. GROMOV, V. LAFFORGUE, A. NAOR and J. PACH: Overlap properties of geometric expanders, *J. Reine Angew. Math.* **671** (2012), 49–83.

[20] J. FRIEDMAN and N. PIPPENGER: Expanding graphs contain all small trees, *Combinatorica* **7** (1987), 71–76.

[21] J. FRIEDMAN: Computing Betti numbers via combinatorial Laplacians, *Algorithmica* **21** (1998), 331–346.

[22] J. FRIEDMAN: A proof of Alon's second eigenvalue conjecture and related problems, *Mem. Amer. Math. Soc.* **195** (2008).

[23] J. FRIEDMAN and A. WIGDERSON: On the second eigenvalue of hypergraphs, *Combinatorica* **15** (1995), 43–65.

[24] H. GARLAND: p-adic curvature and the cohomology of discrete subgroups of p-adic groups, *The Annals of Mathematics* **97** (1973), 375–423.

[25] M. GROMOV: Singularities, expanders and topology of maps. Part 2: from combinatorics to topology via algebraic isoperimetry, *Geometric And Functional Analysis* **20** (2010), 416–526.

[26] A. GUNDERT and M. SZEDLÁK: Higher dimensional discrete Cheeger inequalities, *Annual Symposium on Computational Geometry* (New York, NY, USA), SOCG14, ACM, 2014.

[27] A. GUNDERT and U. WAGNER: On Laplacians of random complexes, in: *Proceedings of the 2012 symposuim on Computational Geometry,* ACM, 2012, 151–160.

[28] S. HOORY, N. LINIAL and A. WIGDERSON: Expander graphs and their applications, *Bulletin of the American Mathematical Society* **43** (2006), 439–562.

[29] S. JANSON: On concentration of probability, *Contemporary combinatorics* **10** (2002), 1–9.

[30] W. KOOK, V. REINER and D. STANTON: Combinatorial Laplacians of matroid complexes, *Journal of the American Mathematical Society* **13** (2000), 129–148.

[31] N. LINIAL and R. MESHULAM: Homological connectivity of random 2-complexes, *Combinatorica* **26** (2006), 475–487.

[32] A. LUBOTZKY, R. PHILLIPS and P. SARNAK: Ramanujan graphs, *Combinatorica* **8** (1988), 261–277.

[33] A. LUBOTZKY, B. SAMUELS and U. VISHNE: Ramanujan complexes of type $\tilde{A}_d$, *Israel Journal of Mathematics* **149** (2005), 267–299.

[34] A. LUBOTZKY: *Discrete groups, expanding graphs and invariant measures*, vol. 125, Birkhauser, 2010.

[35] A. LUBOTZKY: Expander graphs in pure and applied mathematics, *Bull. Amer. Math. Soc.* **49** (2012), 113–162.

[36] D. W. MATULA and F. SHAHROKHI: Sparsest cuts and bottlenecks in graphs, *Discrete Applied Mathematics* **27** (1990), 113–123.

[37] A. MARCUS, D. A. SPIELMAN and N. SRIVASTAVA: Interlacing families I: Bipartite Ramanujan graphs of all degrees, arXiv preprint arXiv:1304.4132, (2013).

[38] R. MESHULAM and N. WALLACH: Homological connectivity of random k-dimensional complexes, *Random Structures & Algorithms* **34** (2009), 408–417.

[39] J. MATOUŠEK and U. WAGNER: On Gromov's method of selecting heavily covered points, *Discrete & Computational Geometry* **52** (2014), 1–33.

[40] A. NILLI: On the second eigenvalue of a graph, *Discrete Mathematics* **91** (1991), 207–210.

[41] I. NEWMAN and Y. RABINOVICH: On multiplicative λ-approximations and some geometric applications, in: *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms,* SODA '12, SIAM, 2012, 51–67.

[42] R. I. OLIVEIRA: Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges, Arxiv preprint ArXiv:0911.0600v2 (2010).

[43] J. PACH: A Tverberg-type result on multicolored simplices, *Computational Geometry* **10** (1998), 71–76.

[44] O. PARZANCHEVSKI and R. ROSENTHAL: Simplicial complexes: spectrum, homology and random walks, arXiv preprint arXiv:1211.6775 (2012).

[45] D. PUDER: Expansion of random graphs: New proofs, new results, *Inventiones mathematicae* (2014), 1–64.

[46] J. STEENBERGEN, C. KLIVANS and S. MUKHERJEE: A Cheeger-type inequality on simplicial complexes, *Advances in Applied Mathematics* **56** (2014), 56–77.

[47] R. M. TANNER: Explicit concentrators from generalized n-gons, *SIAM Journal on Algebraic and Discrete Methods* **5** (1984), 287.

[48] T. TAO: Basic theory of expander graphs, *http://terrytao.wordpress.com/2011/12/02/245b-notes-1-basic-theory-of-expander-graphs/*, 2011.

[49] A. Żuk: La propriété (T) de Kazhdan pour les groupes agissant sur les polyedres, in: *Comptes rendus de l'Académie des sciences, Série 1, Mathématique* **323** (1996), 453–458.

Ori Parzanchevski

Einstein Institute of Mathematics
Hebrew University
Jerusalem 91904, Israel

Current address:

School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540
`parzan@ias.edu`

Ron Rosenthal

Einstein Institute of Mathematics
Hebrew University
Jerusalem 91904, Israel

Current address:

Departement Mathematik
ETH Zürich
8092 Zürich, Switzerland
`ron.rosenthal@math.ethz.ch`

Ran J. Tessler

Einstein Institute of Mathematics
Hebrew University
Jerusalem 91904, Israel
`ran.tessler@mail.huji.ac.il`