# Mixing in high-dimensional expanders

Ori Parzanchevski

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**Abstract**

We establish a generalization of the Expander Mixing Lemma for arbitrary (finite) simplicial complexes. The original lemma states that concentration of the Laplace spectrum of a graph implies combinatorial expansion (which is also referred to as *mixing*, or *pseudo-randomness*). Recently, an analogue of this Lemma was proved for simplicial complexes of arbitrary dimension, provided that the skeleton of the complex is complete. More precisely, it was shown that a concentrated spectrum of the simplicial Hodge Laplacian implies a similar type of pseudo-randomness as in graphs. In this paper we remove the assumption of a complete skeleton, showing that simultaneous concentration of the Laplace spectra in all dimensions implies pseudo-randomness in any complex. We discuss various applications and present some open questions.

## 1 Introduction

The *spectral gap* of a finite graph $G = (V, E)$ is the smallest nontrivial eigenvalue of its Laplace operator. The so-called *discrete Cheeger inequalities* [Dod84, Tan84, AM85, Alo86] relate the spectral gap to expansion in the graph: If the spectral gap is large, then for any partition $V = A \cup B$ there is a large number of edges connecting a vertex in $A$ with a vertex in $B$. Nevertheless, a large spectral gap does not suffice to control the number of edges between any two sets of vertices. For example, there exist "bipartite expanders" (see e.g. [LPS88, MSS13]): regular graphs with a large spectral gap, which are bipartite, so that they contain $A, B \subseteq V$ of size $|V|/4$ with no edges between them. The *Expander Mixing Lemma* remedies this inconvenience, using not only the spectral gap but also the maximal eigenvalue of the Laplacian:

**Theorem** (Expander Mixing Lemma, [FP87, AC88, BMS93])**.** *Let $G = (V, E)$ be a graph on $n$ vertices. If the nontrivial spectrum of the Laplacian is contained within $[k(1-\varepsilon), k(1+\varepsilon)]$, then for any two sets of vertices $A, B$ one has*

$$\left| |E(A,B)| - \frac{k}{n}|A||B| \right| \leq \varepsilon k \sqrt{|A||B|},$$

*where $E(A,B)$ are the edges with one endpoint in $A$ and the other in $B$.*

If $k$ is the average degree of a vertex in $G$, then $\frac{k}{n}|A||B|$ is about the expected size of $|E(A,B)|$ (the exact value is $\frac{k}{n-1}|A||B|$). Thus, the Lemma states that a concentrated spectrum indicates a pseudo-random behavior. In light of the Expander Mixing Lemma, we call a graph whose nontrivial Laplace spectrum is contained in $[k(1-\varepsilon), k(1+\varepsilon)]$ a $(k, \varepsilon)$*-expander*[†].

[†] In [Tao11] this is referred to as a "*two-sided* $(k, \varepsilon)$*-expander*", as the spectrum is bounded on both sides.

In [PRT12] a generalization of the Expander Mixing Lemma is established for finite simplicial complexes of arbitrary dimension, assuming that they have a complete skeleton. Namely, that every possible cell of dimension smaller than the maximal one is in the complex. The Laplace operator which is studied in [PRT12] and in the current paper originates in Eckmann's work [Eck44]. It is a natural analogue of the Hodge Laplace operator in Riemmanian geometry, and it was studied in several prominent works [Gar73, Dod76, Żuk96, Fri98, KRS00, ABM05, DKM09, HJ13], sometimes under the name *combinatorial Laplacian*. For a complex of dimension $d$, Eckmann defines $d$ Laplace operators (see §2), with the $j$-th one acting on the cells of dimension $j$. We say that $X$ is a $(j,k,\varepsilon)$-*expander* if the nontrivial spectrum of the $j$-th Laplacian is contained within $[k(1-\varepsilon),k(1+\varepsilon)]$ (see §2.1 for the precise definition). One then has:

**Theorem** ([PRT12])**.** *Let $X$ be a $d$-dimensional complex on $n$ vertices with a complete skeleton, which is a $(d-1,k,\varepsilon)$-expander. For any disjoint $A_0,\ldots,A_d\subseteq V$,*

$$\left||F(A_0,\ldots,A_d)|-\frac{k}{n}|A_0|\cdot\ldots\cdot|A_d|\right|\le\varepsilon k\left(|A_0|\cdot\ldots\cdot|A_d|\right)^{\frac{d}{d+1}}$$

*where $F(A_0,\ldots,A_d)$ is the set of $d$-cells with one vertex in each $A_i$.*

In this paper we prove a mixing lemma for arbitrary (finite) complexes. Our main result is the following:

**Theorem 1.1.** *If a $d$-dimensional complex $X$ is a $(j,k_j,\varepsilon_j)$-expander for $0\le j\le d-1$, and $A_0,\ldots,A_d$ are disjoint sets of vertices in $X$ then*

$$\left||F(A_0,\ldots,A_d)|-\frac{k_0\ldots k_{d-1}}{n^d}|A_0|\cdot\ldots\cdot|A_d|\right|\le c_dk_0\ldots k_{d-1}\left(\varepsilon_0+\ldots+\varepsilon_{d-1}\right)\max|A_i|,$$

*where $c_d$ depends only on $d$.*

In order to understand $F(A_0,\ldots,A_d)$ in the case of general complexes, we study a more general counting problem:

**Definition 1.2.** Given $A_0,\ldots,A_\ell\subseteq V$, and $j\le\ell$, a *$j$-gallery in* $A_0,\ldots,A_\ell$ is a sequence of $j$-cells $\sigma_0,\ldots,\sigma_{\ell-j}\in X^j$, such that $\sigma_i$ is in $F(A_i,\ldots,A_{i+j})$, and $\sigma_i$ and $\sigma_{i+1}$ intersect in a $(j-1)$-cell (which must lie in $F(A_{i+1},\ldots,A_{i+j})$). We denote the set of $j$-galleries in $A_0,\ldots,A_\ell$ by $F^j(A_0,\ldots,A_\ell)$.

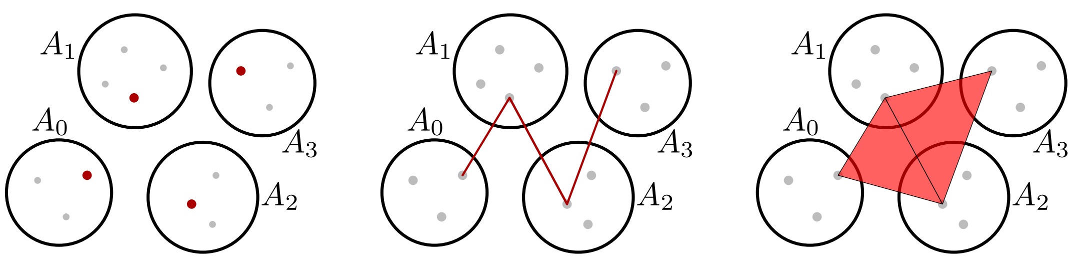


Figure 1.1: A 0-gallery, a 1-gallery and a 2-gallery in $A_0,\ldots,A_3$.

At the heart of our analysis is the following lemma, which estimates the size of $F^{j+1}(A_0,\ldots,A_\ell)$ in terms of that of $F^j(A_0,\ldots,A_\ell)$. Since $F(A_0,\ldots,A_\ell)$ is $F^\ell(A_0,\ldots,A_\ell)$ and $F^0(A_0,\ldots,A_\ell)=A_0\times\ldots\times A_\ell$, repeatedly applying the lemma allows us to estimate $|F(A_0,\ldots,A_d)|$ in terms of $\left|F^0(A_0,\ldots,A_d)\right|=|A_0|\cdot\ldots\cdot|A_d|$.

**Lemma 1.3** (Descent Lemma)**.** *Let $A_0,\ldots,A_\ell$ be sets of vertices in $X$, such that each $j+1$ tuple $A_i,A_{i+1},\ldots,A_{i+j+1}$ consists of disjoint sets. If $X$ is an $(i,k_i,\varepsilon_i)$-expander for $i=j-1,i=j$, then*

$$\left|\left|F^{j+1}\left(A_0,\ldots,A_\ell\right)\right|-\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left|F^j\left(A_0,\ldots,A_\ell\right)\right|\right| \\ \leq\left(\ell-j\right)k_j^{\ell-j}\left(\varepsilon_j+\varepsilon_{j-1}\right)\sqrt{\left|F\left(A_0,\ldots,A_j\right)\right|\left|F\left(A_{\ell-j},\ldots,A_\ell\right)\right|}.$$

Building upon this we obtain a more general pseudo-randomness result for galleries, of which Theorem 1.1 is a special case:

**Proposition 1.4.** *For any $j<\ell$, there exists $c_{j,\ell}$ such that if $X$ is an $(i,k_i,\varepsilon_i)$-expander for $0\leq i\leq j$ and $A_0,\ldots,A_\ell$ are disjoint sets of vertices in $X$ then*

$$\left|\left|F^{j+1}\left(A_0,\ldots,A_\ell\right)\right|-\frac{k_0k_1\ldots k_{j-1}k_j^{\ell-j}}{n^\ell}\prod_{i=0}^{\ell}\left|A_i\right|\right|\leq c_{j,\ell}k_0k_1\ldots k_{j-1}k_j^{\ell-j}\left(\varepsilon_0+\ldots+\varepsilon_j\right)\max\left|A_i\right|.$$

The proofs of the descent lemma and it's corollaries appear in §3, after giving the required definitions in §2. In §4 we discuss applications of the mixing lemma for geometric expansion (in the sense of [Gro10, FGL$^+$12]), chromatic bounds, isoperimetric constants and crossing numbers. We also present the idea of *ideal expanders* in this section, and list some open problems in §5.

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## 2 Simplicial Hodge theory

We describe here briefly the notions we shall need from simplicial Hodge theory. For a more detailed summary we refer the reader to [PRT12, §2].

Let $X$ be a $d$-dimensional simplicial complex, with vertex set $V$ of size $n$. For $-1\leq j\leq d$ we denote by $X^j$ the set of $j$-cells in $X$ (cells of size $j+1$), and by $X_\pm^j$ the set of oriented $j$-cells, i.e. ordered cells up to an even permutation. A *$j$-form* on $X$ is an antisymmetric function on oriented $j$-cells:

$$\Omega^j=\Omega^j\left(X\right)=\left\{f:X_\pm^j\to\mathbb{R}\,\middle|\,f\left(\overline{\sigma}\right)=-f\left(\sigma\right)\ \forall\sigma\in X_\pm^j\right\},$$

where $\overline{\sigma}$ is $\sigma$ endowed with the opposite orientation. In dimensions 0 and $-1$ there is only one orientation, and so $\Omega^0=\mathbb{R}^V$ and $\Omega^{-1}=\mathbb{R}^{\{\varnothing\}}\cong\mathbb{R}$. The $j^{\text{th}}$ *boundary operator* $\partial_j:\Omega^j\to\Omega^{j-1}$ is defined by $\left(\partial_jf\right)\left(\sigma\right)=\sum_{v\cup\sigma\in X^j}f\left(v\sigma\right)$. The sequence $\Omega^{-1}\overset{\partial_0}{\longleftarrow}\Omega^0\overset{\partial_1}{\longleftarrow}\ldots$ is a chain complex, i.e. $B_j\overset{def}{=}\operatorname{im}\partial_{j+1}\subseteq\ker\partial_j\overset{def}{=}Z_j$, and $H^j={}^{Z_j}/_{B_j}$ is the $j^{\text{th}}$ (real, reduced) homology group of $X$. We endow each $\Omega^j$ with the inner product $\langle f,g\rangle=\sum_{\sigma\in X^j}f\left(\sigma\right)g\left(\sigma\right)$, which gives rise to a dual *coboundary operator* $\delta_j=\partial_j^*:\Omega^{j-1}\to\Omega^j$. The real *cohomology* of $X$ is $H^j={}^{Z^j}/_{B^j}$, where $B^j\overset{def}{=}\operatorname{im}\delta_j\subseteq\ker\delta_{j+1}\overset{def}{=}Z^j$, and by the fundamental theorem of linear algebra one has $B_j^\perp=Z^j$ and $Z_j^\perp=B^j$.

The *upper, lower* and *full Laplacians* in dimension $j$ are $\Delta_j^+=\partial_{j+1}\delta_{j+1}$, $\Delta_j^-=\delta_j\partial_j$, and $\Delta_j=\Delta_j^++\Delta_j^-$, respectively. All of the Laplacians are self-adjoint and decompose with respect to the orthogonal decompositions $\Omega^j = B^j \oplus Z_j = B_j \oplus Z^j$, and the following properties are simple exercises (unlike their Riemannian counterparts):

$$Z^j = \ker \Delta_j^+ \quad B_j = \operatorname{im} \Delta_j^+ \quad Z_j = \ker \Delta_j^- \quad B^j = \operatorname{im} \Delta_j^-$$
$$Z^j \cap Z_j = \left(B_j \oplus B^j\right)^{\perp} = \ker \Delta_j \cong H_j \cong H^j \quad \text{(Discrete Hodge Theorem).}$$

The dimension of $\ker \Delta_j$ is the $j^{\text{th}}$ (reduced) *Betti number* of $X$, denoted by $\beta_j$.

The combinatorial meaning of the Laplacians is better understood via the following adjacency relations on oriented cells:

(1) For two oriented $j$-cells $\sigma, \sigma'$, we denote $\sigma \pitchfork \sigma'$ if $\sigma$ and $\sigma'$ intersect in a common $(j-1)$-cell and induce the same orientation on it; for edges this means that they have a common origin or a common endpoint, and for vertices $v \pitchfork v'$ holds whenever $v \neq v'$.

(2) We denote $\sigma \sim \sigma'$ if $\sigma \pitchfork \sigma'$, and in addition the $(j+1)$-cell $\sigma \cup \sigma'$ is in $X$. For vertices this is the standard relation of neighbors in a graph.

Using these relations, the Laplacians can be expressed as follows (here the degree of a $j$-cell is the number of $(j+1)$-cells which contain it):

$$\left(\Delta_j^+ \varphi\right)(\sigma) = \deg(\sigma)\,\varphi(\sigma) - \sum_{\sigma' \sim \sigma} \varphi(\sigma')$$
$$\left(\Delta_j^- \varphi\right)(\sigma) = (j+1)\,\varphi(\sigma) + \sum_{\sigma' \pitchfork \sigma} \varphi(\sigma')$$
$$\left(\Delta_j \varphi\right)(\sigma) = (\deg \sigma + j + 1)\,\varphi(\sigma) + \sum_{\substack{\sigma' \pitchfork \sigma \\ \sigma' \nsim \sigma}} \varphi(\sigma')$$

We also define adjacency operators on $\Omega^j$ which correspond to the $\sim$ and $\pitchfork$ relations:

$$\left(\mathcal{A}_j^{\sim} \varphi\right)(\sigma) = \sum_{\sigma' \sim \sigma} \varphi(\sigma'), \qquad \left(\mathcal{A}_j^{\pitchfork} \varphi\right)(\sigma) = \sum_{\sigma' \pitchfork \sigma} \varphi(\sigma'),$$

so that $\Delta_j^- = (j+1) \cdot I + \mathcal{A}_j^{\pitchfork}$ and $\Delta_j^+ = D_j - \mathcal{A}_j^{\sim}$, where $D_j$ is the degree operator $(D_j f)(\sigma) = \deg(\sigma) f(\sigma)$. Let us remark that these operators can be used to define stochastic processes on $j$-cells whose properties relate to the homology of the complex - see [PR12, MS13].

## 2.1 Spectrum

The spectra we are primarily interested in are those of $\Delta_j^+$, for $0 \leq j \leq d-1$. Since $\left(\Omega^j, \delta_j\right)$ is a co-chain complex, $B^j = \operatorname{im} \delta_j$ must be contained in the kernel of $\Delta_j^+ = \partial_{j+1} \delta_{j+1}$, and the zero eigenvalues which correspond to forms in $B^j$ are considered to be the *trivial spectrum* of $\Delta_j^+$. As $\left(B^j\right)^{\perp} = Z_j$, we call $\operatorname{Spec} \Delta_j^+\big|_{Z_j}$ the *nontrivial spectrum* of $\Delta_j^+$. Note that zero is a nontrivial eigenvalue of $\Delta_j^+$ precisely when $Z_j \cap Z^j \neq 0$, i.e. $\beta_j \neq 0$. For example, the constant functions on $V$ form the trivial eigenfunctions of $\Delta_0^+$. The nontrivial spectrum of $\Delta_j^+$ corresponds to $Z_0$, which are the functions whose sum on all vertices vanish, and zero is a nontrivial eigenvalue of $\Delta_0^+$ if and only if the complex is disconnected.

**Definition.** The complex $X$ is a $(j, k, \varepsilon)$-*expander* if $\varepsilon < 1$ and $\operatorname{Spec} \Delta_j^+\big|_{Z_j} \subseteq [k(1-\varepsilon), k(1+\varepsilon)]$. Given $\overline{k} = (k_0, \ldots, k_{d-1})$ and $\overline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{d-1})$, we say that $X$ is a $\left(\overline{k}, \overline{\varepsilon}\right)$-*expander* if it is a $(j, k_j, \varepsilon_j)$-expander for all $j$.

The restriction $\varepsilon_j < 1$ in the definition ensures that $X$ has trivial $j$-th homology, i.e. $\beta_j = 0$. While some of our results hold for general $\varepsilon$ (e.g. Lemma 1.3), or for any global bound on it (e.g. Theorem 1.1), we shall need the stronger assumption for later applications.

We remark that it is sometimes convenient to consider the Laplacian $\Delta_{-1}^{+}$ as well. This operator acts on $\Omega^{-1} \cong \mathbb{R}$ as multiplication by $n = |V|$, so that every complex is automatically a $(-1, n, 0)$-expander.

# 3 The main theorems

In this section we assume that $X$ is a $d$-complex on $n$ vertices, and prove the Descent Lemma (Lemma 1.3) and the mixing lemmas it implies.

*Proof of the Descent Lemma.* To any disjoint sets of vertices $A_0, \ldots, A_j$, we associate the characteristic $j$-form $\mathbb{1}_{A_0 \ldots A_j} \in \Omega^j$, which takes $\pm 1$ on $j$-cells in $F(A_0, \ldots, A_j)$ (according to their orientation), and vanishes elsewhere:

$$\mathbb{1}_{A_0 \ldots A_j}(\sigma) = \begin{cases} \operatorname{sgn}(\pi) & \exists \pi \in \operatorname{Sym}_{\{0 \ldots j\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \le i \le j \\ 0 & \text{otherwise.} \end{cases}$$

Restriction of $j$-forms to $F(A_0, \ldots, A_j)$ gives an orthogonal projection operator in $\Omega^j$, which we denote by $\mathbb{P}_{A_0 \ldots A_j}$:

$$\left(\mathbb{P}_{A_0 \ldots A_j} \varphi\right)(\sigma) = \begin{cases} \varphi(\sigma) & \sigma \in F(A_0, \ldots, A_j) \\ 0 & \text{otherwise.} \end{cases}$$

We start our analysis by observing that for disjoint sets $A_0, \ldots, A_{j+1}$ the form $(-1)^j \mathbb{P}_{A_0 \ldots A_j} \mathcal{A}_j^{\sim} \mathbb{1}_{A_1 \ldots A_{j+1}}$ vanishes outside $F(A_0, \ldots, A_j)$, and assigns to each $j$-cell therein its number of $\sim$-neighbors in $F(A_1, \ldots, A_{j+1})$. As these neighbors are in correspondence with $(j+1)$-cells in $F(A_0, \ldots, A_{j+1})$, we obtain

$$|F(A_0, \ldots, A_{j+1})| = \left|\left\langle \mathbb{1}_{A_0 \ldots A_j}, \mathbb{P}_{A_0 \ldots A_j} \mathcal{A}_j^{\sim} \mathbb{1}_{A_1 \ldots A_{j+1}} \right\rangle\right|. \tag{3.1}$$

Next, let $\varphi$ be a $j$-form which is supported on $F(A_1, \ldots, A_{j+1})$, and which assigns to each $j$-cell $\sigma$ the number of $(j+1)$-galleries in $A_1, \ldots, A_\ell$ whose first cell contains $\sigma$. By the same considerations as above, $(-1)^j \mathbb{P}_{A_0 \ldots A_j} \mathcal{A}_j^{\sim} \varphi$ assigns to every $j$-cell $\tau$ in $F(A_0, \ldots, A_j)$ the number of $(j+1)$-galleries in $A_0, \ldots, A_\ell$ whose first $(j+1)$ cell contains $\tau$. Therefore, $\left|\left\langle \mathbb{1}_{A_0 \ldots A_j}, \mathbb{P}_{A_0 \ldots A_j} \mathcal{A}_j^{\sim} \varphi \right\rangle\right| = \left|F^{j+1}(A_0, \ldots, A_\ell)\right|$, and we conclude by induction that

$$\left|F^{j+1}(A_0, \ldots, A_\ell)\right| = \left|\left\langle \mathbb{1}_{A_0 \ldots A_j}, \left(\prod_{i=0}^{\ell-j-1} \mathbb{P}_{A_i \ldots A_{i+j}} \mathcal{A}_j^{\sim}\right) \mathbb{1}_{A_{\ell-j} \ldots A_\ell} \right\rangle\right|. \tag{3.2}$$

Since $A_i, \ldots, A_{i+j+1}$ are disjoint, $\mathbb{1}_{A_i \ldots A_{i+j}}$ and $\mathbb{1}_{A_{i+1} \ldots A_{i+j+1}}$ are supported on different cells, so that $\mathbb{P}_{A_i \ldots A_{i+j}} T \mathbb{P}_{A_{i+1} \ldots A_{i+j+1}} = 0$ for any diagonal operator $T$. Thus, all the $\mathcal{A}_j^{\sim}$ in (3.2) can be replaced by $\mathcal{A}_j^{\sim} + T$, and taking $T = k_j I - D_j$ we obtain

$$\left|F^{j+1}(A_0, \ldots, A_\ell)\right| = \left|\left\langle \mathbb{1}_{A_0 \ldots A_j}, \left(\prod_{i=0}^{\ell-j-1} \mathbb{P}_{A_i \ldots A_{i+j}} \left(k_j I - \Delta_j^{+}\right)\right) \mathbb{1}_{A_{\ell-j} \ldots A_\ell} \right\rangle\right|. \tag{3.3}$$

Our next step is to approximate this quantity using the lower $j$-th Laplacian. Denoting $E = k_jI - \Delta_j^+ - \frac{k_j}{k_{j-1}}\Delta_j^-$, the orthogonal decomposition $\Omega^j = Z_j \oplus B^j$ gives

$$E = k_j\left(\mathbb{P}_{Z_j} + \mathbb{P}_{B^j}\right) - \Delta_j^+ - \frac{k_j}{k_{j-1}}\Delta_j^- = k_j\mathbb{P}_{Z_j} - \Delta_j^+ + \frac{k_j}{k_{j-1}}\left(k_{j-1}\mathbb{P}_{B^j} - \Delta_j^-\right).$$

We first observe that $\left\|k_j\mathbb{P}_{Z_j} - \Delta_j^+\right\| \leq k_j\varepsilon_j$ follows from $\operatorname{Spec}\Delta_j^+\big|_{Z_j} \subseteq \left[k_j\left(1-\varepsilon_j\right), k_j\left(1+\varepsilon_j\right)\right]$ and $\Delta_j^+\big|_{B^j} \equiv 0$. For the lower Laplacian, we have

$$\begin{aligned}\operatorname{Spec}\Delta_j^-\big|_{B^j} &= \operatorname{Spec}\Delta_j^-\big|_{Z_j^\perp} = \operatorname{Spec}\Delta_j^-\backslash\{0\} \overset{(*)}{=} \operatorname{Spec}\Delta_{j-1}^+\backslash\{0\} = \operatorname{Spec}\Delta_{j-1}^+\big|_{(Z^{j-1})^\perp} \\ &= \operatorname{Spec}\Delta_{j-1}^+\big|_{B_{j-1}} \subseteq \operatorname{Spec}\Delta_{j-1}^+\big|_{Z_{j-1}} \subseteq \left[k_{j-1}\left(1-\varepsilon_{j-1}\right), k_{j-1}\left(1+\varepsilon_{j-1}\right)\right],\end{aligned} \tag{3.4}$$

where $(*)$ follows from the fact that $\Delta_j^- = \partial_j^*\partial_j$ and $\Delta_{j-1}^+ = \partial_j\partial_j^*$. As $\Delta_j^-$ vanishes on $Z_j$, we have in total $\left\|k_{j-1}\mathbb{P}_{B^j} - \Delta_j^-\right\| \leq k_{j-1}\varepsilon_{j-1}$, so that

$$\|E\| \leq \left\|k_j\mathbb{P}_{Z_j} - \Delta_j^+\right\| + \frac{k_j}{k_{j-1}}\left\|k_{j-1}\mathbb{P}_{B^j} - \Delta_j^-\right\| \leq k_j\left(\varepsilon_{j-1} + \varepsilon_j\right). \tag{3.5}$$

Let us denote for brevity $\mathbb{P}_i = \mathbb{P}_{A_i,\ldots,A_{i+j}}$. Using $k_jI - \Delta_j^+ = \frac{k_j}{k_{j-1}}\Delta_j^- + E$, allowing to translate $\Delta_j^-$ by diagonal (in fact, scalar) operators, we find that

$$\begin{aligned}&\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_i\left(k_jI - \Delta_j^+\right)\right)\mathbb{P}_{\ell-j} = \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_i\left(\frac{k_j}{k_{j-1}}\Delta_j^- + E\right)\right)\mathbb{P}_{\ell-j} \\ &= \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_i\Delta_j^-\right)\mathbb{P}_{\ell-j} \\ &\quad + \sum_{m=1}^{\ell-j}\left(\tfrac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_i\Delta_j^-\right)\mathbb{P}_{\ell-j-m}E\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_i\left(\frac{k_j}{k_{j-1}}\Delta_j^- + E\right)\right)\mathbb{P}_{\ell-j} \\ &= \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_i\mathcal{A}_j^{\pitchfork}\right)\mathbb{P}_{\ell-j} \\ &\quad + \sum_{m=1}^{\ell-j}\left(\tfrac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_i\left(\Delta_j^- - k_{j-1}I\right)\right)\mathbb{P}_{\ell-j-m}E\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_i\left(k_jI - \Delta_j^+\right)\right)\mathbb{P}_{\ell-j}\end{aligned}$$

and plugging this into (3.3) gives

$$\begin{aligned}\left|F^{j+1}\left(A_0,\ldots,A_\ell\right)\right| = \Bigg|&\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left\langle \mathbb{1}_{A_0\ldots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\mathcal{A}_j^{\pitchfork}\right)\mathbb{1}_{A_{\ell-j}\ldots A_\ell}\right\rangle \\ &+ \sum_{m=1}^{\ell-j}\left(\tfrac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left\langle \mathbb{1}_{A_0\ldots A_j}, \begin{matrix}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(\Delta_j^- - k_{j-1}I\right)\right)\mathbb{P}_{A_{\ell-j-m}\ldots A_{\ell-m}}E\cdot \\ \cdot\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(k_jI - \Delta_j^+\right)\right)\mathbb{1}_{A_{\ell-j}\ldots A_\ell}\end{matrix}\right\rangle\Bigg|.\end{aligned}$$

Let us call the term on the first line the main term, and the one on the second line the error term. Note that the form $(-1)^j \mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\pitchfork}\mathbb{1}_{A_1\ldots A_{j+1}}$ assigns to every $j$-cell in $F(A_0,\ldots,A_j)$ the number of $j$-cells in $F(A_1,\ldots,A_{j+1})$ with which it intersects, so that by definition

$$\left|\left\langle \mathbb{1}_{A_0\ldots A_j}, \mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\pitchfork}\mathbb{1}_{A_1\ldots A_{j+1}}\right\rangle\right| = \left|F^j(A_0,\ldots,A_{j+1})\right|$$

(compare this with (3.1)). By the same arguments as for $\mathcal{A}^{\sim}$ one sees that

$$\left|F^j(A_0,\ldots,A_\ell)\right| = \left|\left\langle \mathbb{1}_{A_0\ldots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\mathcal{A}_j^{\pitchfork}\right)\mathbb{1}_{A_{\ell-j}\ldots A_\ell}\right\rangle\right|,$$

so that the main term is precisely $\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left|F^j(A_0,\ldots,A_\ell)\right|$, our estimate for $\left|F^{j+1}(A_0,\ldots,A_\ell)\right|$. Turning to the error term, since $\operatorname{Spec}\Delta_j^+ \subseteq [0,k_j(1+\varepsilon_j)]$ we have $\left\|k_jI-\Delta_j^+\right\| \le k_j$, and similarly from (3.4) we obtain $\left\|\Delta_j^- - k_{j-1}I\right\| \le k_{j-1}$. Together with (3.5) this implies that the error term is bounded by

$$\begin{aligned}&\sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left\|\mathbb{1}_{A_0\ldots A_j}\right\| k_{j-1}^{\ell-j-m}k_j\left(\varepsilon_{j-1}+\varepsilon_j\right)k_j^{m-1}\left\|\mathbb{1}_{A_{\ell-j}\ldots A_\ell}\right\| \\ &= (\ell-j)\,k_j^{\ell-j}\left(\varepsilon_{j-1}+\varepsilon_j\right)\sqrt{\left|F(A_0,\ldots,A_j)\right|\left|F(A_{\ell-j},\ldots,A_\ell)\right|},\end{aligned}$$

which concludes the proof.

We remark that a slightly better bound is possible here: one can replace $k_jI-\Delta_j^+$ in the error term by $\frac{k_j(1+\varepsilon_j)}{2}I-\Delta_j^+$, which is bounded by $\frac{k_j(1+\varepsilon_j)}{2}$, and likewise for $\Delta_j^-$ and $k_{j-1}$. For example, putting $\varepsilon=\max(\varepsilon_{j-1},\varepsilon_j)$ this gives the bound

$$\frac{(\ell-j)}{2^{\ell-j-2}}k_j^{\ell-j}\varepsilon(1+\varepsilon)^{\ell-j-1}\sqrt{\left|F(A_0,\ldots,A_j)\right|\left|F(A_{\ell-j},\ldots,A_\ell)\right|}$$

which might be useful when $\varepsilon$ is small and $\ell \gg j$.

□

Using the Descent Lemma repeatedly gives Proposition 1.4, which for $j=d-1$, $\ell=d$ implies Theorem 1.1:

*Proof of Proposition 1.4.* We denote $m=\max|A_i|$ and assume by induction that the proposition holds for $j-1$ (and any $j\le\ell$), i.e. that

$$\left|F^j(A_0,\ldots,A_\ell) - \frac{k_0\ldots k_{j-2}k_{j-1}^{\ell-j+1}}{n^\ell}\prod_{i=0}^{\ell}|A_i|\right| \le c_{j-1,\ell}k_0\ldots k_{j-2}k_{j-1}^{\ell-j+1}\left(\varepsilon_0+\ldots+\varepsilon_{j-1}\right)m. \tag{3.6}$$

For $j=0$ this indeed holds, in the sense that

$$\left|F^0(A_0,\ldots,A_\ell) - \frac{k_{-1}^\ell}{n^\ell}\prod_{i=0}^{\ell}|A_i|\right| = 0 \tag{3.7}$$

(recall that every complex is a $(-1,n,0)$-expander). Let us denote by $\mathcal{E}$ the discrepancy $\left|\left|F^{j+1}(A_0,\ldots,A_\ell)\right| - \frac{k_0k_1\ldots k_{j-1}k_j^{\ell-j}}{n^\ell}\prod_{i=0}^{\ell}|A_i|\right|$. Combining the Descent Lemma with (3.6) (or

(3.7), for $j=0$) multiplied by $\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}$ gives

$$\begin{aligned}\mathcal{E} \leq & (\ell-j)\, k_j^{\ell-j} \left(\varepsilon_j + \varepsilon_{j-1}\right) \sqrt{\left|F\left(A_0,\ldots,A_j\right)\right| \left|F\left(A_{\ell-j},\ldots,A_\ell\right)\right|} \\ & + c_{j-1,\ell} k_0 k_1 \ldots k_{j-1} k_j^{\ell-j} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) m.\end{aligned}$$

To bound $|F(A_0,\ldots,A_j)|$ we use (3.6) with $\ell=j$, which gives

$$\begin{aligned}\left|F^j\left(A_0,\ldots,A_j\right)\right| & \leq \frac{k_0\ldots k_{j-1}}{n^j} \prod_{i=0}^{j} |A_i| + c_{j-1,j} k_0 \ldots k_{j-1} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) m \\ & \leq \left[1 + c_{j-1,j}\left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right)\right] k_0 \ldots k_{j-1} m \\ & \leq \left(1 + j c_{j-1,j}\right) k_0 \ldots k_{j-1} m.\end{aligned}$$

(here we have used $\varepsilon_i < 1$, but any global bound for $\varepsilon_i$ would do). The same holds for $|F(A_{\ell-j},\ldots,A_\ell)|$, hence

$$\begin{aligned}\mathcal{E} & \leq (\ell-j)\, k_j^{\ell-j} \left(\varepsilon_j + \varepsilon_{j-1}\right) \left(1 + j c_{j-1,j}\right) k_0 \ldots k_{j-1} m \\ & \quad + c_{j-1,\ell} k_0 k_1 \ldots k_{j-1} k_j^{\ell-j} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) m \\ & = k_0 k_1 \ldots k_{j-1} k_j^{\ell-j} \left[c_{j-1,\ell}\left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) + (\ell-j)\left(1 + j c_{j-1,j}\right)\left(\varepsilon_j + \varepsilon_{j-1}\right)\right] m \\ & \leq \underbrace{\left[c_{j-1,\ell} + (\ell-j)\left(1 + j c_{j-1,j}\right)\right]}_{c_{j,\ell}} k_0 k_1 \ldots k_{j-1} k_j^{\ell-j} \left(\varepsilon_0 + \ldots + \varepsilon_j\right) m.\end{aligned}$$

as desired. □

# 4 Applications

**Geometric overlap.** The following notion of geometric expansion for graphs and complexes originates in Gromov's work [Gro10] (see also [FGL$^+$12, MW14]):

**Definition 4.1.** Let $X$ be a $d$-dimensional simplicial complex. The *geometric overlap* of $X$ is

$$\operatorname{overlap} X = \min_{\varphi: V \to \mathbb{R}^d} \max_{x \in \mathbb{R}^d} \frac{\#\left\{\sigma \in X^d \,\middle|\, x \in \operatorname{conv}\left\{\varphi(v) \,|\, v \in \sigma\right\}\right\}}{|X^d|}.$$

In words, $X$ has overlap $\geq \varepsilon$ if for every simplicial mapping of $X$ into $\mathbb{R}^d$ (a mapping induced linearly by the images of the vertices), some point in $\mathbb{R}^d$ is covered by at least an $\varepsilon$-fraction of the $d$-cells of $X$.

A theorem of Pach [Pac98] relates pseudo-randomness and geometric overlap, and allows us to show the following:

**Proposition 4.2.** *If $X$ is a $d$-dimensional $(\overline{k},\overline{\varepsilon})$-expander with $\mathcal{E} = \varepsilon_0 + \ldots + \varepsilon_{d-1} < 1$, then*

$$\operatorname{overlap} X > \mathcal{P}_d d! \left(1 - \mathcal{E}\right) \left(\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d \mathcal{E}\right),$$

*where $\mathcal{P}_d$ is Pach's constant [Pac98], and $c_d$ is the constant in Theorem 1.1.*

Thus, a family of $d$-complexes with $\varepsilon_0+\ldots+\varepsilon_{d-1}$ small enough is a family of geometric expanders. For the proof we shall need the following lemma, which relates the Laplace spectrum to cell density:

**Lemma 4.3.** *Let $X$ be a $d$-complex with $\beta_j = 0$ for $0 \le j \le d-1$, and let $\lambda_j$ be the average nontrivial eigenvalue of $\Delta_j^+$, for $-1 \le j \le d-1$ (in particular $\lambda_{-1} = n$). Then, for $0 \le m \le d$ the number of $m$-cells is*

$$\left|X^m\right| = \frac{\lambda_{m-1}}{m+1} \cdot \prod_{j=-1}^{m-2} \left(\frac{\lambda_j}{j+2} - 1\right) = \frac{\lambda_{m-1}(n-1)}{m+1} \cdot \prod_{j=0}^{m-2} \left(\frac{\lambda_j}{j+2} - 1\right), \tag{4.1}$$

*and the average degree of an $m$-cell is*

$$\operatorname{avg}\{\deg\sigma \,|\, \sigma \in X^m\} = \lambda_m \left(1 - \frac{m+1}{\lambda_{m-1}}\right). \tag{4.2}$$

*Proof.* Since the trivial spectrum of $\Delta_j^+$ consists of zeros,

$$\left|X^m\right| = \frac{1}{m+1} \sum_{\sigma \in X^{m-1}} \deg\sigma = \frac{1}{m+1} \operatorname{tr} D_{m-1} = \frac{1}{m+1} \operatorname{tr} \Delta_{m-1}^+ = \frac{\lambda_{m-1}}{m+1} \dim Z_{m-1}.$$

Thus, (4.1) is equivalent to the assertion that

$$\dim Z_{m-1} = \prod_{j=-1}^{m-2} \left(\frac{\lambda_j}{j+2} - 1\right).$$

This is true for $m = 0$, and by induction, together with the triviality of the $(m-2)$-th homology we find that

$$\begin{aligned} \dim Z_{m-1} &= \dim \Omega^{m-1} - \dim B_{m-2} = \left|X^{m-1}\right| - \dim Z_{m-2} \\ &= \frac{\lambda_{m-2}}{m} \prod_{j=-1}^{m-3} \left(\frac{\lambda_j}{j+2} - 1\right) - \prod_{j=-1}^{m-3} \left(\frac{\lambda_j}{j+2} - 1\right) = \prod_{j=-1}^{m-2} \left(\frac{\lambda_j}{j+2} - 1\right) \end{aligned}$$

as desired. Formula (4.2) follows from (4.1), as $\operatorname{avg}\{\deg\sigma \,|\, \sigma \in X^m\} = \frac{(m+2)\left|X^{m+1}\right|}{\left|X^m\right|}$. ☐

We can now proceed:

*Proof of Proposition 4.2.* Let $\varphi$ be a simplicial map $X \to \mathbb{R}^d$, and divide $V = X^0$ arbitrarily into parts $P_0, \ldots, P_{d+1}$ of size $\frac{n}{d+1}$. Pach's theorem [Pac98] then states that there exist $Q_i \subseteq P_i$ of size $|Q_i| = \mathcal{P}_d |P_i|$ and a point $x \in \mathbb{R}^{d+1}$, such that $x \in \operatorname{conv}\{\varphi(v) \,|\, v \in \sigma\}$ for all $\sigma \in F(Q_0, \ldots, Q_d)$. Denoting $\mathcal{K} = k_0 \cdot \ldots \cdot k_{d-1}$ and $\mathcal{E} = \varepsilon_0 + \ldots + \varepsilon_{d-1}$, we have by Theorem 1.1

$$|F(Q_0, \ldots, Q_d)| \ge \frac{\mathcal{K}}{n^d} \left(\frac{\mathcal{P}_d n}{d+1}\right)^{d+1} - \frac{c_d \mathcal{K} \mathcal{E} \mathcal{P}_d n}{d+1} = \frac{\mathcal{K} \mathcal{P}_d n}{d+1} \left[\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d \mathcal{E}\right],$$

and by the lemma above

$$\left|X^d\right| = \frac{\lambda_{d-1}}{d+1} \cdot \prod_{j=-1}^{d-2} \left(\frac{\lambda_j}{j+2} - 1\right) \le \prod_{j=-1}^{d-1} \frac{\lambda_j}{j+2} \le n \prod_{j=0}^{d-1} \frac{k_j(1+\varepsilon_j)}{j+2} \le \frac{n\mathcal{K}}{(d+1)!\,(1-\mathcal{E})}.$$

This means $x$ is covered by the $\varphi$-images of at least a $\mathcal{P}_d d!\,(1-\mathcal{E})\left(\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d\mathcal{E}\right)$-fraction of the $d$-cells, and as this is true for all $\varphi$ the proposition follows. ☐

**Crossing numbers.** This is another invariant, of somewhat similar flavor: The *$m$-dimensional crossing number* of a $d$-complex $X$ is the minimal $c$ such that for any simplicial map $X \to \mathbb{R}^m$ there are at least $c$ pairs of disjoint $d$-cells in $X$ whose images in $\mathbb{R}^m$ intersect. In [GW13, §8.1] Gundert and Wagner use the mixing lemma from [PRT12] to give a lower bound for the $2d$-dimensional crossing number of a $d$-complex with a complete skeleton, and their arguments hold for the general case using Theorem 1.1 and Lemma 4.3.

**Chromatic numbers.** A $d$-complex $X$ is *weakly $c$-colorable* if there is a coloring of its vertices by $c$ colors so that no $d$-cell is monochromatic. The *weak chromatic number* of $X$, denoted $\chi(X)$, is the smallest $c$ for which $X$ is $c$-colorable. We will use the mixing property to show that spectral expansion implies a chromatic bound, as is done for graphs in [LPS88]. These results are weaker than Hoffman's chromatic bound for graphs [Hof70], as they require a two-sided spectral bound, and the chromatic bound obtained is not optimal. A chromatic bound for complexes which generalizes Hoffman's result was recently obtained in [Gol13].

**Proposition 4.4.** *If $X$ is a $d$-dimensional $(\overline{k},\overline{\varepsilon})$-expander, then*

$$\chi(X) \geq \frac{1}{(d+1)\sqrt[d]{c_d(\varepsilon_0+\ldots+\varepsilon_{d-1})}},$$

*where $c_d$ is the constant from Theorem 1.1.*

*Proof.* Coloring $X$ by $\chi = \chi(X)$ colors, there is necessarily a monochromatic set of vertices of size at least $\frac{n}{\chi}$. Take $\frac{n}{\chi}$ of these vertices and partition them arbitrarily into $d+1$ sets $A_0,\ldots,A_d$ of equal size. By assumption we have $F(A_0,\ldots,A_d)=\varnothing$, so that Theorem 1.1 reads

$$\frac{k_0\ldots k_{d-1}}{n^d}\prod_{i=0}^{d}|A_i| \leq c_d k_0\ldots k_{d-1}(\varepsilon_0+\ldots+\varepsilon_{d-1})\max|A_i|,$$

and since $|A_i| = \frac{n}{\chi\cdot(d+1)}$, the conclusion follows. □

**Isoperimetric bounds.** The Cheeger inequalities for graphs relate $\min \operatorname{Spec} \Delta_0^+\big|_{Z_0}$ to the Cheeger constant, which is the minimum of $\frac{|E(A,B)|}{|A||B|}$ over all partitions of the vertices into (nonempty) sets $A$ and $B$. In [PRT12] one side of these inequalities is generalized to complexes with a complete skeleton. Namely, for any partition $A_0,\ldots,A_d$ of the vertices in a $d$-complex, the quantity $\frac{|F(A_0,\ldots,A_d)|}{|A_0|\cdot\ldots\cdot|A_d|}$ is bounded from below in terms of $\min \operatorname{Spec} \Delta_{d-1}^+\big|_{Z_{d-1}}$. In [GP14] we generalize this result to complexes with non-complete skeleton, obtaining:

**Theorem** ([GP14])**.** *Let $X$ be a $d$-complex on $n$ vertices, which is a $(j,k_j,\varepsilon_j)$-expander for $0\leq j\leq d-2$, and let $\lambda_{d-1} = \min \operatorname{Spec} \Delta_{d-1}^+\big|_{Z_{d-1}}$. Then for any partition $A_0,\ldots,A_d$ of the vertices of $X$,*

$$\frac{|F(A_0,\ldots,A_d)|\,n^d}{|A_0|\cdot\ldots\cdot|A_d|} \geq k_0\cdot\ldots\cdot k_{d-2}\cdot\lambda_{d-1}\left(1-\varepsilon_{d-2}-C_d(\varepsilon_0+\ldots+\varepsilon_{d-2})\frac{n^{d+1}}{|A_0|\cdot\ldots\cdot|A_d|}\right),$$

*where $C_d$ depends only on $d$.*

The proof relies on both Theorem 1.1 and Proposition 1.4 of the present paper, which serve to replace the assumption of a complete $(d-1)$-skeleton with a pseudo-random one.

**Ramanujan complexes.** Ramanujan graphs, constructed in [LPS88, Mar88, MSS13] are spectrally optimal expanders. Ramanujan complexes, a natural high dimensional counterpart, were defined and studied in [CSŻ03, Li04, LSV05, Sar07], but as yet not from the point of view of the Hodge Laplacian. It is natural to suspect that they are high-dimensional expanders, in the sense of the current paper, but it turns out that the picture if more complicated. For the two-dimensional case, the nontrivial 1-dimensional spectrum is concentrated in two narrow strips:

**Theorem** ([GP14]). *Let $X$ be a two-dimensional Ramanujan complex with $n$ vertices, constant vertex degree $k_0 = 2\left(q^2+q+1\right)$ and constant edge degree $k_1 = q+1$ ($q$ is any prime power). If $X$ is not three-colorable then*

$$\operatorname{Spec}\Delta_0^+\big|_{Z_0} \subseteq \left[k_0\left(1-\frac{3}{q}\right), k_0\left(1+\frac{3}{q}\right)\right]$$

$$\operatorname{Spec}\Delta_1^+\big|_{Z_1} \subseteq \left[k_1\left(1-\frac{2}{\sqrt{q}}\right), k_1\left(1+\frac{2}{\sqrt{q}}\right)\right] \cup \left[2k_1\left(1-\frac{4}{q}\right), 2k_1\left(1+\frac{4}{q}\right)\right] \cup \{3k_1\},$$

*and if $X$ is three-colorable then $\Delta_0^+$ has in addition the eigenvalue $\frac{3k_0}{2}$.*

The $\Delta_1^+$-eigenform corresponding to the eigenvalue $3k_1$ is a *disorientation* (see [PR12, HJ13, GP14]), a two-dimensional analogue of graph bipartition. It is well understood, and would not prevent us from conducting an analysis as the one carried out in this paper. What prevents such an analysis is the strip around $2k_1$, which requires different arguments. In [GP14] we circumvent this problem by observing a quadratic polynomial in $\Delta_1^+$, and conclude a pseudo-random behavior for two-galleries of length four, namely $F^2(A,B,C,D)$.

**Ideal expanders.** This is not quite an application, but rather a useful intuition. Let us say that $X$ is an *ideal $\overline{k}$-expander* if it is a $(j,k_j,0)$-expander for $0 \le j \le d-1$. In this case, the Descent Lemma tell us that

$$F^{j+1}(A_0,\ldots,A_\ell) = \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j} \left|F^j(A_0,\ldots,A_\ell)\right|,$$

and the number of $j$-galleries between disjoint sets of vertices is completely determined by their sizes:

$$\left|F^j(A_0,\ldots,A_\ell)\right| = \frac{k_0k_1\ldots k_{j-2}k_{j-1}^{\ell-j+1}}{n^\ell}\prod_{i=0}^{\ell}|A_i| \tag{4.3}$$

(in particular, $|F(A_0,\ldots,A_d)| = \frac{k_0\ldots k_{d-1}}{n^d}|A_0|\ldots|A_d|$). For

$$k_j = \begin{cases} n & 0 \le j < m \\ 0 & m \le j < d, \end{cases}$$

an example of an ideal $\overline{k}$-expander is given by $K_n^{(m)}$, the $m$-th skeleton of the complete complex on $n$ vertices. For this complex (4.3) holds trivially, and perhaps disappointingly, these are the only examples of ideal expanders: if $X$ is an ideal $\overline{k}$-expander on $n$ vertices, and $X^{(j)} = K_n^{(j)}$ (which holds for $j=0$), one has $k_0 = \ldots = k_{j-1} = n$, and also $k_j \le n$ by [PRT12, prop. 3.2(2)]. For any vertices $v_0,\ldots,v_{j+1}$, $\frac{k_0\ldots k_j}{n^{j+1}} = |F(\{v_0\},\ldots,\{v_{j+1}\})| \in \{0,1\}$ then forces either $k_j = n$, which implies that $X^{(j+1)} = K_n^{(j+1)}$ as well, or $k_j = 0$, which means that $X$ has no $(j+1)$-cells at all.

While ideal $\overline{k}$-expanders do not actually exist, save for the trivial examples $\overline{k} = (n, \ldots, n, 0, \ldots)$, they provide a conceptual way to think of expanders in general: $(\overline{k}, \overline{\varepsilon})$-expanders spectrally approximate the ideal (nonexistent) $\overline{k}$-expander, and the mixing lemma asserts that they approximate it combinatorially as well. This point of view seems close in spirit to that of *spectral sparsification* [ST11], which proved to be fruitful in both graphs and complexity theory.

# 5 Questions

Several natural questions arise from this study:

- In [GW12] it is shown that random complexes in the Linial-Meshulam model [LM06] have spectral concentration for appropriate parameters (see also [PRT12, §4.5]). These are complexes with a complete skeleton, which are high-dimensional analogues of Erdős–Rényi graphs. Is there a similar model for general complexes, for which the skeletons are not complete (preferably, where the expected degrees of cells are only logarithmic in the number of vertices), with concentrated spectrum?
- A well known source of excellent expanders are random regular graphs (see, e.g. [Fri08, Pud14]). Can one construct a model for random regular complexes, and are these complexes high-dimensional expanders? This is interesting even for a weak notion of regularity, such as having a bounded fluctuation of degrees, or having all links of vertices isomorphic.
- Bilu and Linial [BL06] have established a converse to the Expander Mixing Lemma, which shows that pseudo-randomness and two-sided spectral concentration are (almost) equivalent. For complexes with a complete skeleton, a converse to the mixing lemma from [PRT12] was recently established in [CMRT14]. Do these converse theorems admit a generalization to the general case?
- Another generalization of the high-dimensional Cheeger inequality for complexes without a complex skeleton appears in [GS14]. Rather than comparing $|F(A_0, \ldots, A_d)|$ with $|A_0| \cdot \ldots \cdot |A_d|$, it is compared with the number of $(d-1)$-spheres with one vertex at each $A_i$ (which for a complete skeleton is $|A_0| \cdot \ldots \cdot |A_d|$), and the smallest nontrivial eigenvalue of $\Delta_{d-1}^{+}$ is used to relate them. Can one prove a mixing lemma along this line of thought, using $\operatorname{Spec} \Delta_0^{+}$ alone?

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School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540
E-mail: `parzan@ias.edu`