

SPECTRUM AND COMBINATORICS OF RAMANUJAN TRIANGLE COMPLEXES

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ABSTRACT. Ramanujan graphs have extremal spectral properties, which imply a remarkable combinatorial behavior. In this paper we compute the high-dimensional Laplace spectrum of Ramanujan triangle complexes, and show that it implies a combinatorial expansion property, and a pseudo-randomness result. For this purpose we prove a Cheeger-type inequality and a mixing lemma of independent interest.

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1. INTRODUCTION

A k -regular graph is said to be *Ramanujan* if its nontrivial Laplace spectrum (see (2.1)) is contained in the interval $\mathcal{I}_k = [k - 2\sqrt{k-1}, k + 2\sqrt{k-1}]$. The reason for these particular values is the following theorem, which says that asymptotically, they are the best that one can hope for:

Theorem (Alon-Boppana, [LPS88, Nil91, GZ99, Li01]). *Let $\{G_i\}$ be a collection of k -regular graphs, $\text{Spec}(G_i)$ the nontrivial Laplace spectrum of G_i , and $\lambda(G_i)$, $\Lambda(G_i)$ the minimum and maximum of $\text{Spec}(G_i)$, respectively.*

- (1) *If there are infinitely many different G_i then $\liminf_i \lambda(G_i) \leq k - 2\sqrt{k-1}$.*
- (2) *If $\{\text{girth}(G_i)\}$ is unbounded then the closure of $\bigcup_i \text{Spec}(G_i)$ contains \mathcal{I}_k , and in particular $\limsup_i \Lambda(G_i) \geq k + 2\sqrt{k-1}$.*

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Bounds on the Laplace spectrum of a graph are extremely useful. The so-called “discrete Cheeger inequalities” relate $\lambda(G)$ to combinatorial expansion, and the “expander mixing lemma” connects spectral concentration (namely $\Lambda(G) - \lambda(G)$) to pseudo-randomness. Thus, Ramanujan graphs are excellent *expanders*: we refer to [HLW06, Tao11, Lub12] for detailed surveys on expander graphs, their properties and applications.

Ramanujan graphs were first constructed in the works [LPS88, Mar88], using the Ramanujan–Petersson conjecture for GL_2 , which was proved by Eichler and Deligne in characteristic zero, and by Drinfeld in positive characteristic (the latter being used to construct Ramanujan graphs in [Mor94]). The constructed graphs are certain quotients of the Bruhat-Tits building associated with PGL_2 over a nonarchimedean local field. This building is a k -regular tree, and this gives the real reason for the values which appear in the Alon-Boppana theorem: the interval \mathcal{I}_k is precisely the L^2 -spectrum of the Laplacian on a k -regular tree, by a classic result of Kesten [Kes59]. Thus, Ramanujan graphs are those whose nontrivial spectrum is contained in that of their universal cover: this relation is a discrete analogue of the original Ramanujan conjecture (which regards quotients of the hyperbolic plane), and this is what led Lubotzky, Phillips and Sarnak to the name *Ramanujan graphs*. For the complete story of Ramanujan graphs we refer the reader to the monographs [Sar90, Lub94].

In recent years, several authors turned to study higher dimensional Ramanujan complexes, which are quotients of the Bruhat-Tits buildings associated with PGL_d . This includes the works [Bal00, CSŽ03, Li04, LSV05a, LSV05b, Sar07, KLW10, KL14], and a recent survey of the field appears in [Lub14]. This surge of interest was prompted by advances in the theory of automorphic representations, especially the resolution of the generalized Ramanujan conjecture for GL_d in positive characteristic by Lafforgue [Laf02]. While Ramanujan complexes form natural generalizations of Ramanujan graphs, it is not clear what are the best generalizations of graph theoretic notions such as Laplace spectrum, Cheeger constant, pseudo-randomness, girth and so on. Until now, most of the research has focused on the spectral theory of the so-called *Hecke operators*, which act again on the vertices of the complex, and sum up to the graph adjacency operator. Our interest lies in the the spectrum of the *simplicial Laplace operators*, also known as combinatorial Laplacians. These originate in [Eck44], and form natural analogues of the Hodge-Laplace operators acting on the spaces of differential forms on a Riemannian manifold. The i -th simplicial Laplacian, denoted Δ_i^+ , acts on the simplicial i -forms of the complex, which are skew-symmetric functions on oriented cells of dimension i . The 0-forms are just the functions on vertices, and Δ_0^+ is the standard graph Laplacian.

An Alon-Boppana theorem holds here as well:

Theorem 1 ([Li04, PR12]). *If $\{X_i\}$ is a family of quotients of the Bruhat-Tits building \mathcal{B}_d with unbounded injectivity radius^(†), then the closure of $\bigcup_i \mathrm{Spec} \Delta_j^+(X_i)$ contains the L^2 -spectrum of $\Delta_j^+(\mathcal{B}_d)$.*

(The statement in [Li04] addresses the simultaneous spectrum of the aforementioned Hecke operators, and this implies the theorem for Δ_0^+ ; [PR12] handles the Laplacians in higher dimensions. For a different generalization of the Alon-Boppana theorem see also [Fir14].)

As in graphs, the Ramanujan quotients of \mathcal{B}_d should be those whose Laplace spectrum, in every dimension, is contained within the corresponding L^2 -spectrum of \mathcal{B}_d . But what

^(†)Note that the injectivity radius of a quotient of a tree is half its girth.

is the spectrum of Δ_i^+ on \mathcal{B}_d and its Ramanujan quotients? It is well known that the spectrum of Δ_0^+ (i.e., the underlying graph spectrum) is concentrated, though not as good as in Ramanujan graphs [Mac79, Li04, LSV05a]. For higher dimensions, a celebrated local-to-global argument of Garland [Gar73] establishes lower bounds on the Δ_i^+ -spectra (for any complex, not necessarily \mathcal{B}_d and its quotients). While Garland's argument can be adjusted to give an upper bound as well (see [Pap08, GW12]), the local picture of \mathcal{B}_d gives trivial bounds for the higher end of the spectra of its quotients.

The purpose of this paper is twofold: to compute the higher Laplace spectrum of Ramanujan triangle complexes, and to relate the simplicial Laplace spectra to combinatorial properties of the complex. The first half of the paper (§3-§5) is dedicated to Theorem 5, which determines explicitly the spectrum of Δ_1^+ for Ramanujan triangle complexes. We find that the picture for Δ_1^+ is more intricate than that in dimension zero: most of the spectrum is concentrated in the strip $\mathcal{I}_{k_1} = [k_1 - 2\sqrt{k_1 - 1}, k_1 + 2\sqrt{k_1 - 1}]$, where k_1 is the degree of edges in the complex. However, a small fraction of the spectrum is concentrated within the narrow strip $[2k_1 - 1, 2k_1 + 8]$ (see Theorem 5). This gives, in particular, an explanation for the failure of Garland's method to achieve spectral concentration.

Our computational approach is inspired by [Lub94, LSV05a], and especially [KLW10], but it is different: rather than computing the Laplacians directly, we show (Proposition 6) that certain elements of the Iwahori-Hecke algebra of PGL_3 act on appropriate representations as simplicial boundary and coboundary maps (see §2). We then invoke the classification of unitary, Iwahori-spherical representations of PGL_3 which is computed explicitly in [KLW10] (using results from [Bor76, Zel80, Cas80, Tad86]), to obtain the complete spectrum. While much of the work should be useful for higher dimensions as well, the relevant classification problem becomes harder and we do not pursue it here.

As in the case of Ramanujan graphs, we would like to infer combinatorial properties from the spectral theory of the complex, and the second half of the paper (§6 and §7) is devoted to this purpose. It can be read independently from §3-§5, as the theorems therein apply to arbitrary complexes and not only to quotients of Bruhat-Tits buildings.

In §6 we prove a Cheeger-type inequality, which uses the concentration of the vertex spectrum, and the fact that the edge spectrum is bounded away from zero:

Theorem 2. *Let X be a triangle complex on n vertices, with nontrivial Δ_0^+ -spectrum contained in $[k_0 - \mu_0, k_0 + \mu_0]$, and nontrivial Δ_1^+ -spectrum bounded below by λ_1 . Then for any partition of the vertices of X into nonempty sets A, B, C , one has*

$$\frac{|T(A, B, C)| n^2}{|A| |B| |C|} \geq \lambda_1 \left(k_0 - \mu_0 \left(1 + \frac{10 n^3}{9 |A| |B| |C|} \right) \right),$$

where $T(A, B, C)$ is the set of triangles with one vertex in each of A, B and C .

For complexes with a complete skeleton (underlying graph) this inequality reduces to the one which appears in [PRT12]. It can also be generalized to higher dimensions - this is done in Theorem 8. For an interpretation in terms of a Cheeger-type constant, see (6.1) and (6.2).

In §7 we turn to pseudo-randomness, and use the full strength of Theorem 5. In a previous paper [Par13] we have shown that simultaneous concentration of both vertex and edge spectra implies pseudo-randomness, in the sense that $|T(A, B, C)|$ is close to its expected

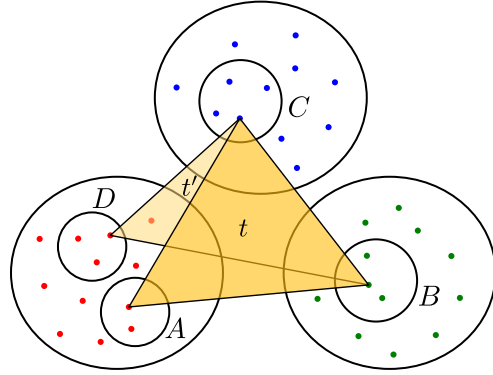
value for any large disjoint sets A, B, C (not necessarily partitions). Since in Ramanujan triangle complexes the edge spectrum is not concentrated, having eigenvalues near both k_1 and $2k_1$, this result cannot be applied directly. Instead, we use the concentration around these two values to deduce a pseudo-random behavior of 2-galleries, namely, pairs of triangles which intersect in an edge. We denote by $F^2(A, B, C, D)$ the galleries (t, t') such that $t \in T(A, B, C)$ and $t' \in T(B, C, D)$ (see Figure 1.1).

Theorem 3. *Let X be a tripartite Ramanujan triangle complex on n vertices with edge degrees $k_1 = q + 1$. Let A, B, C, D be disjoint sets of vertices such that each of $A \cup D$, B and C is contained in a different block of the three-partition of X . If A, B, C and D are of sizes at most ϑn , then*

$$\left| F^2(A, B, C, D) - \frac{27q^4 |A| |B| |C| |D|}{n^3} \right| \leq (65q^{3.5}\vartheta + 244q^{2.5}) \vartheta n.$$

A version which applies to general triangle complexes appears in Theorem 9. As an application of the theorem we show that the weak chromatic number of non-tripartite Ramanujan triangle complexes is $\Omega(\sqrt[3]{q})$, improving on the bound $\Omega(\sqrt[6]{q})$ which is obtained in [EGL14].

FIGURE 1.1. A 2-gallery through A, B, C, D in a tripartite triangle complex.



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2. COMPLEXES AND BUILDINGS

In this section we recall the basic elements of simplicial Hodge theory, and the definition of affine Bruhat-Tits buildings. For a more relaxed exposition of the former we refer to [PRT12, §2], and for the latter to [Li04, LSV05a, Lub14].

Let X be a finite simplicial complex of dimension d , and denote by X^i the set of (unoriented) cells of dimension i in X (i -cells). The *degree* of an i -cell is the number of $(i + 1)$ -cells which contain it. We denote by $\Omega^i = \Omega^i(X)$ the space of i -forms, namely skew-symmetric complex functions on the oriented i -cells, equipped with the

inner product $\langle f, g \rangle = \sum_{\sigma \in X^i} f(\sigma) \overline{g(\sigma)}$. The i -th boundary map $\partial_i : \Omega^i \rightarrow \Omega^{i-1}$ is defined by $(\partial_i f)(\sigma) = \sum_{v: v\sigma \in X^i} f(v\sigma)$, and its dual is the i -th coboundary map $\delta_i = \partial_i^* : \Omega^{i-1} \rightarrow \Omega^i$, given by $(\delta_i f)(\sigma) = \sum_{j=0}^i (-1)^j f(\sigma \setminus \sigma_j)$. We use the standard notation $Z_i = \ker \partial_i$, $Z^i = \ker \delta_{i+1}$, $B_i = \text{im } \partial_{i+1}$ and $B^i = \text{im } \delta_i$ for cycles, cocycles, boundaries and coboundaries.

The *upper*, *lower* and *full i -Laplacians* are $\Delta_i^+ = \partial_{i+1} \delta_{i+1}$, $\Delta_i^- = \delta_i \partial_i$ and $\Delta_i = \Delta_i^+ + \Delta_i^-$, respectively. For our purposes the upper Laplacian is the most important one, and as we have remarked, Δ_0^+ is the classical Laplacian of graph theory:

$$(\Delta_0^+ f)(v) = \deg(v) f(v) - \sum_{w \sim v} f(w). \quad (2.1)$$

Since (Ω^*, δ_*) is a cochain complex, $\text{im } \delta_i$ is always in $\ker \Delta_i^+$, and the corresponding zeros in $\text{Spec } \Delta_i^+$ are considered the *trivial spectrum* (of Δ_i^+). The nontrivial i -spectrum is thus

$$\text{Spec } \Delta_i^+|_{(\text{im } \delta_i)^\perp} = \text{Spec } \Delta_i^+|_{\ker \partial_i} = \text{Spec } \Delta_i^+|_{Z_i}, \quad (2.2)$$

and zero is in the nontrivial spectrum if and only if the i -th cohomology of X (over \mathbb{C}) is nontrivial. We should remark that in the theory of Ramanujan graphs, the Laplacian eigenvalue $2k$ (for a k -regular graph) is also considered trivial. We will return to this point later.

Next, let F be a nonarchimedean local field with ring of integers \mathcal{O} , uniformizer π , and residue field $\mathcal{O}/\pi\mathcal{O}$ of size q , which we shall identify with \mathbb{F}_q . The canonical examples are $F = \mathbb{Q}_p$ with $\mathcal{O} = \mathbb{Z}_p$ and $\pi = q = p$, and $F = \mathbb{F}_{p^e}((t))$ with $\mathcal{O} = \mathbb{F}_{p^e}[[t]]$, $\pi = t$ and $q = p^e$. The affine Bruhat-Tits building $\mathcal{B} = \mathcal{B}_d(F)$ associated with $\text{PGL}_d(F)$ is an infinite contractible complex of dimension $d - 1$. Denoting $G = \text{PGL}_d(F)$ and $K = \text{PGL}_d(\mathcal{O})$, the vertices of \mathcal{B} are in correspondence with the left K -cosets in G . To every vertex gK we associate the \mathcal{O} -lattice $g\mathcal{O}^d$, which is only defined up to scaling by F^\times , since $g \in \text{PGL}_d(F) = \text{GL}_d(F)/F^\times$. A collection of vertices $\{g_i K\}_{i=0..r}$ forms an r -cell if, possibly after reordering, there exist representatives $g'_i \in \text{GL}_d(F)$ for g_i , such that the corresponding lattices satisfy

$$\pi g'_0 \mathcal{O}^d < g'_r \mathcal{O}^d < g'_{r-1} \mathcal{O}^d < \dots < g'_1 \mathcal{O}^d < g'_0 \mathcal{O}^d. \quad (2.3)$$

This relation turns out to be equivalent to having an edge between every $g_i K$ and $g_j K$ (i.e., representatives $g'_i, g'_j \in \text{GL}_d(F)$ with $\pi g'_i \mathcal{O}^d < g'_j \mathcal{O}^d < g'_i \mathcal{O}^d$). In other words, \mathcal{B} is a clique (or “flag”) complex.

The action of G on G/K defines an action on the complex \mathcal{B} , as it clearly preserves (2.3). Given a cocompact, torsion-free lattice $\Gamma \leq G$, the quotient $\Gamma \backslash \mathcal{B}$ is a finite complex. The building \mathcal{B}_2 is a $(q + 1)$ -regular tree, and its quotients by lattices in G are $(q + 1)$ -regular graphs. Certain lattices give rise to Ramanujan quotients:

Theorem ([LPS88, Mar88], cf. [Sar90, Lub94]). *If Γ is a congruence subgroup of a torsion-free arithmetic lattice in G , then $\Gamma \backslash \mathcal{B}_2$ is a Ramanujan graph.*

As we have remarked, some of these graphs have the eigenvalue $2k = 2(q + 1)$ in their spectrum (with the rest of the nontrivial spectrum in \mathcal{I}_k). This is the origin of this eigenvalue: the vertices of \mathcal{B}_d admit a coloring in $\mathbb{Z}/d\mathbb{Z}$, defined by $\text{col}(gK) = \text{ord}_\pi(\det g) + d\mathbb{Z}$, but the action of G does not respect the coloring, since $\text{col}(gv) = \text{col}(v) + \text{ord}_\pi(\det g)$. The following are equivalent:

- $2k = 2(q + 1)$ appears in $\text{Spec } \Delta_0^+(\Gamma \backslash \mathcal{B}_2)$.

- The action of Γ preserves the color of vertices.
- $\text{col } \Gamma \equiv 0$ (i.e. $\text{ord}_\pi(\det \gamma) \in 2\mathbb{Z}$ for all $\gamma \in \Gamma$).
- $\Gamma \backslash \mathcal{B}_2$ is bipartite.

Together with [Mor94], the papers [LPS88, Mar88] establish that if $k-1$ is a prime power, then there exist infinitely many non-bipartite k -regular Ramanujan graphs, and infinitely many bipartite ones. For general k , the existence of infinitely many bipartite Ramanujan graphs was only settled in [MSS13] (by a non-constructive argument), and it is still open for the non-bipartite case.

We now continue to dimension two. The Bruhat-Tits building \mathcal{B}_3 is a regular triangle complex, with vertex and edge degrees

$$k_0 = 2(q^2 + q + 1) \quad \text{and} \quad k_1 = q + 1$$

respectively. There are several plausible ways to define what are the Ramanujan quotients of \mathcal{B}_d , and these are discussed in [CSŽ03, Li04, LSV05a, KLV10, Fir14]. Luckily, they all coincide for PGL_3 (see [KLW10]), though not in general (see [FLW13]). The definition we use is the following (the notions in the definition are explained throughout §3):

Definition 4. A quotient of \mathcal{B}_3 by a torsion-free cocompact lattice Γ in $G = \text{PGL}_3(F)$ is *Ramanujan* if every Iwahori-spherical representation of G which appears in $L^2(\Gamma \backslash G)$ is either finite-dimensional or tempered.

The papers [Li04, LSV05b, Sar07] give several constructions of Ramanujan complexes, some of which are the clique complexes of Cayley graphs.

We begin the study of these complexes at the high end of the spectrum. It is not hard to see that a d -complex whose j -cells have degrees bounded by k_j satisfies $\text{Spec } \Delta_j^+ \subseteq [0, (j+2)k_j]$ (see e.g. [PRT12]). We say that the complex is $(d+1)$ -partite if there exists a $(d+1)$ -coloring of its vertices such that no two vertices in a d -cell are of the same color. If X is a $(d+1)$ -partite d -complex whose $(d-1)$ -cells have degree k_{d-1} , then $(d+1)k_{d-1} \in \text{Spec } \Delta_{d-1}^+$. Unlike the graph case, the converse may fail: rather than $(d+1)$ -partiteness, the $(d+1)k_{d-1}$ eigenvalue indicates *disorientability* (see [PR12]), the existence of an orientation of the d -cells so that they *agree* on the orientation of their codimension one intersections.

In our case, all quotients of \mathcal{B}_3 by a lattice in G are disorientable, i.e. have $3k_1 \in \text{Spec } \Delta_1^+$, and a similar situation holds in higher dimensions (see [Pap08]). The reason is the following: if we define the color of a directed edge to be $\text{col}([v, w]) = \text{col}(w) - \text{col}(v)$ (in $\mathbb{Z}/d\mathbb{Z}$), then the action of G does preserve edge colors, since $\text{col}(gv) = \text{col}(v) + \text{ord}_\pi(\det g)$. Thus, the disorientation eigenform $f(e) = (-1)^{\text{col } e}$ always factors modulo Γ , giving $3k_1 \in \text{Spec } \Delta_1^+$.

If Γ also preserves vertex coloring, i.e. $\text{col } \Gamma \equiv 0$, then $X = \Gamma \backslash \mathcal{B}_3$ is naturally tripartite. The converse also holds: if $\text{col } \Gamma \not\equiv 0$ then

$$\widehat{\Gamma} = \{\gamma \in \Gamma \mid \text{col } \gamma K = 0\} = \Gamma \cap \ker(\text{ord}_\pi \det : G \rightarrow \mathbb{Z}/3\mathbb{Z}) \quad (2.4)$$

is a normal subgroup of Γ of index three, and $\widehat{X} = \widehat{\Gamma} \backslash \mathcal{B}_3$ is a tripartite three-cover of X , with the coloring $\text{col} = \text{ord}_\pi \det$. But any two vertices in \mathcal{B}_3 , and thus in \widehat{X} , can be connected by a 2-gallery, that is, a sequence of triangles in which every consecutive pair intersect in an edge. This shows that \widehat{X} admits a unique 3-coloring (up to renaming of

colors). Thus, any three-coloring of X lifts to col , which must therefore be Γ -invariant, contradicting $\text{col} \Gamma \neq 0$.

Having understood the “colored” part of the spectrum^(†), we can move on to its interesting parts:

Theorem 5. *Let X be a non-tripartite Ramanujan triangle complex on n vertices.*

- (1) *The nontrivial spectrum of Δ_0^+ is contained within $[k_0 - 6q, k_0 + 3q]$.*
- (2) *The nontrivial spectrum of Δ_1^+ consists of:*
 - (a) *$n(q^2 + q - 2) + 2$ eigenvalues in the strip*

$$\mathcal{I} = [k_1 - 2\sqrt{q}, k_1 + 2\sqrt{q}]. \quad (2.5)$$

- (b) *For every nontrivial $\lambda \in \text{Spec } \Delta_0^+$, the eigenvalues $\frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda}$.
This amounts to $n - 1$ eigenvalues in each of the strips*

$$\begin{aligned} \mathcal{I}_- &= \left[\frac{3k_1}{2} - \sqrt{\left(\frac{k_1}{2}\right)^2 + 8q}, k_1 + 1 \right] \\ \mathcal{I}_+ &= \left[2k_1 - 1, \frac{3k_1}{2} + \sqrt{\left(\frac{k_1}{2}\right)^2 + 8q} \right]. \end{aligned} \quad (2.6)$$

- (c) *The disorientation eigenvalue $3k_1$, corresponding to $f(e) = (-1)^{\text{col } e}$.*

If X is tripartite (and Ramanujan):

- (1) *The nontrivial spectrum of Δ_0^+ consists of:*
 - (a) *$n - 3$ eigenvalues of Δ_0^+ in $[k_0 - 6q, k_0 + 3q]$.*
 - (b) *The eigenvalue $\frac{3k_0}{2}$, twice, corresponding to $f(v) = \exp(\pm \frac{2\pi i}{3} \text{col}(v))$.*
- (2) *The nontrivial spectrum of Δ_1^+ consists of:*
 - (a) *$n(q^2 + q - 2) + 6$ eigenvalues in \mathcal{I} .*
 - (b) *$n - 3$ eigenvalues in each of \mathcal{I}_\pm , corresponding to $\frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda}$ for the eigenvalues of Δ_0^+ in (a) above.*
 - (c) *As before, $3k_1$.*

As we have mentioned, the spectrum of Δ_0^+ is well-known [Mac79, Li04, LSV05a], but our computation methods are different, and give the spectrum in both dimensions in a unified manner. Let us make a few remarks:

- (1) These bounds cannot be improved: by Theorem 1, a sequence of quotients with unbounded injectivity radius (as constructed in [LM07]) has Laplace spectra which accumulate to any point in these intervals.
- (2) The strip \mathcal{I}_- is contained in \mathcal{I} , and both of the strips \mathcal{I}_\pm are highly concentrated:

$$\mathcal{I}_- \subseteq [k_1 - 8, k_1 + 1], \quad \mathcal{I}_+ \subseteq [2k_1 - 1, 2k_1 + 8],$$

so that in particular

$$\text{Spec } \Delta_1^+|_{Z_1} \subseteq [k_1 - 2\sqrt{k_1 - 1}, k_1 + 2\sqrt{k_1 - 1}] \cup [2k_1 - 1, 2k_1 + 8] \cup \{3k_1\}.$$

- (3) There are no nontrivial zeros in the spectra of Δ_0^+ and Δ_1^+ , so that the zeroth and first (torsion-free) homologies of X vanish, in accordance with [Gar73, Cas74].

^(†)We reserve the term “trivial” for the homological meaning - zeros obtained on coboundaries.

It is interesting to compare the theorem with Garland's (extended) method, which relates the high-dimensional Laplace spectrum to the graph spectrum of links of cells in codimension two. We denote by $\text{lk } \sigma$ the link of a cell σ .

Theorem ([Gar73, Pap08, GW12]^(†)). *If X is a finite complex such that $\text{Spec } \Delta_0^+(\text{lk } \sigma)|_{Z_0(\text{lk } \sigma)} \subseteq [\lambda, \Lambda]$ for every $\sigma \in X^{j-2}$ and $k \leq \deg \sigma \leq K$ for every $\sigma \in X^{j-1}$, then*

$$\text{Spec } \Delta_j^+(X)|_{Z_j(X)} \subseteq [(j+1)\lambda - jK, (j+1)\Lambda - jk].$$

The link of every vertex in \mathcal{B}_3 is the incidence graph of the projective plane over \mathbb{F}_q . This is a bipartite k_1 -regular graph with

$$\text{Spec } \Delta_0^+|_{Z_0} = \{k_1 - \sqrt{q}, k_1 + \sqrt{q}, 2k_1\},$$

and therefore by Garland any quotient of \mathcal{B}_3 satisfies

$$\text{Spec } \Delta_1^+|_{Z_1} \subseteq [k_1 - 2\sqrt{q}, 3k_1].$$

Theorem 5 shows that both ends are tight! (for the high end this follows from disorientability, and was already observed in [Pap08].) The problem with Garland's method is that it completely misses the sparse picture within this interval, which is crucial for our combinatorial purposes. We return to the concentration of the spectrum in §6 and §7. The proof of Theorem 5 occupies the next three sections.

3. IWAHORI-HECKE BOUNDARY MAPS

Let us fix the fundamental vertex $v_0 = K$ in $\mathcal{B}^0 = G/K$. A torsion-free lattice $\Gamma \leq G$ acts freely on \mathcal{B}^0 , and for a quotient $X = \Gamma \backslash \mathcal{B}$ we denote by v the image of $v \in \mathcal{B}^0$ in X^0 as well. The vertices of X are in correspondence with the double cosets in $\Gamma \backslash G/K$. If g_1, \dots, g_n are representatives for $\Gamma \backslash G/K$, then $D = \coprod_{i=1}^n g_i K$ is a fundamental domain for Γ in G ; indeed, $\gamma D \cap D \neq \emptyset$ implies $\gamma g_i K \cap g_j K \neq \emptyset$ for some i, j , forcing $i = j$, and thus $\gamma \in g_i K g_i^{-1}$; but Γ acts freely on \mathcal{B}^0 and $g_i K g_i^{-1}$ fixes $g_i v_0$, so that $\gamma = 1$. Thus, normalizing the Haar measure on G so that $\mu(K) = 1$, we have $\mu(\Gamma \backslash G) = \mu(D) = n$. Since Γ does not intersect any conjugate of K , the measure induced by μ on $\Gamma \backslash G/K$ is the counting measure, so that there is a linear isometry $\Omega^0(X) \cong L^2(\Gamma \backslash G/K)$, given explicitly by $f(gv_0) = f(\Gamma gK)$. It will also be useful for us to identify $L^2(\Gamma \backslash G/K)$ with $L^2(\Gamma \backslash G)^K$, the space of K -fixed vectors in the G -representation $L^2(\Gamma \backslash G)$, in the natural manner.

The element $\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & 0 & 0 \end{pmatrix} \in G$ acts on \mathcal{B} by rotation on the triangle consisting of the vertices v_0 , σv_0 , and $\sigma^2 v_0$. We choose as a fundamental oriented edge $e_0 = [v_0, \sigma v_0]$, and denote

$$E = \text{stab}_G e_0 = K \cap \sigma K \sigma^{-1} = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ x & y & * \end{pmatrix} \in K \mid x, y \in \pi \mathcal{O} \right\} \quad (3.1)$$

(E is sometimes called a “parahoric subgroup”.) Recall the coloring $\text{col} : \mathcal{B}^1 \rightarrow \mathbb{Z}/3\mathbb{Z}$ given by $\text{col}([gK, g'K]) = \text{ord}_\pi \det(g'g^{-1})$. It is invariant under G , and thus well defined on $X^1 = \Gamma \backslash \mathcal{B}^1$. Every edge in X (and \mathcal{B}) has one orientation with color 1, and one with color 2 (there are no edges of color 0). Furthermore, G acts transitively on the non-oriented

^(†)The cited papers incorporate normalized versions of the Laplacian. The version we use follows from [PRT12, Lem. 4.2].

edges of \mathcal{B} , so that the oriented edges of color 1 in \mathcal{B} can be identified with G/E (as $\text{col } e_0 = 1$), and those of X with $\Gamma \backslash G/E$, giving

$$\mu(E) = \frac{\mu(\Gamma \backslash G)}{|\Gamma \backslash G/E|} = \frac{n}{|X^1|} = \frac{1}{q^2 + q + 1},$$

and we can identify $\Omega^1(X)$ with $L^2(\Gamma \backslash G)^E$, by

$$f(ge_0) = f([gv_0, g\sigma v_0]) = \sqrt{\mu(E)}f(\Gamma g), \quad f([g\sigma v_0, gv_0]) = -\sqrt{\mu(E)}f(\Gamma g). \quad (3.2)$$

The scaling by $\sqrt{\mu(E)}$ is needed to make the isomorphism $\Omega^1(X) \cong L^2(\Gamma \backslash G)^E$ an isometry: if $g_1, \dots, g_{nk_0/2}$ is a set of representatives for $\Gamma \backslash G/E$, then for $f \in L^2(\Gamma \backslash G)^E$ we have

$$\begin{aligned} \|f\|_{\Omega^1(X)}^2 &= \sum_{i=1}^{nk_0/2} |f(g_i e_0)|^2 = \sum_{i=1}^{nk_0/2} \mu(E) |f(\Gamma g_i)|^2 = \sum_{i=1}^{nk_0/2} \int_E |f(\Gamma g_i)|^2 de \\ &= \sum_{i=1}^{nk_0/2} \int_E |f(\Gamma g_i e)|^2 de = \int_{\Gamma \backslash G} |f(\Gamma g)|^2 dg = \|f\|_{L^2(\Gamma \backslash G)}^2. \end{aligned}$$

(Let us remark that this is simpler than the picture in higher dimensions: for example, $G = \text{PGL}_4(F)$ acts transitively on the edges of color 1, and on the edges of color 3 in \mathcal{B}_4 . It also preserves their orientations, as it preserves edge coloring, and $\text{col}(vw) = -\text{col}(wv)$. However, G acts transitively on the *oriented* edges of color 2, i.e. flipping their orientations as well, which is possible since $2 \equiv -2 \pmod{4}$. If E_i is the stabilizer (as a cell) of some fundamental edge e_i of color i , and $\xi : E_2 \rightarrow \pm 1$ is the character which indicates whether an element of E_2 fixes e_2 pointwise or flips it, then

$$\Omega^1(\Gamma \backslash \mathcal{B}_4) \cong L^2(\Gamma \backslash G)^{E_1} \oplus L^2(\Gamma \backslash G)^{E_2, \xi},$$

where $V^{E_2, \xi}$ denotes the ξ -isotypic component of a representation V (this can be thought of as a higher “ E -type”).)

Coming back to \mathcal{B}_3 , we fix $t_0 = [v_0, \sigma v_0, \sigma^2 v_0]$ as a fundamental triangle. The pointwise stabilizer of t_0 is the Iwahori subgroup

$$I = K \cap \sigma K \sigma^{-1} \cap \sigma^2 K \sigma^{-2} = \left\{ \begin{pmatrix} * & * & * \\ x & * & * \\ y & z & * \end{pmatrix} \in K \mid x, y, z \in \pi \mathcal{O} \right\}.$$

As for edges, G acts transitively on non-oriented triangles, and preserves triangle orientation, since a flip of a triangle would imply a flip of some edge. Thus, the stabilizer of t_0 as a cell (both oriented and non-oriented) is

$$T \stackrel{\text{def}}{=} \text{stab}_G t_0 = \langle \sigma \rangle I = I \amalg \sigma I \amalg \sigma^2 I,$$

and in particular $\langle \sigma \rangle$ and I commute. Again, $f(gt_0) = \sqrt{\mu(T)}f(\Gamma g)$ gives a linear isometry $\Omega^2(X) \cong L^2(\Gamma \backslash G)^T$, and

$$\mu(T) = \frac{\mu(\Gamma \backslash G)}{|\Gamma \backslash G/T|} = \frac{n}{|X^2|} = \frac{3!}{k_0 k_1} = \frac{3}{(q^2 + q + 1)(q + 1)}.$$

Let us denote $K_0 = K$, $K_1 = E$, and $K_2 = T$, so that $\Omega^i(X) \cong L^2(\Gamma \backslash G)^{K_i}$. As $I \leq E \leq K$ and $I \leq T$, the three spaces $L^2(\Gamma \backslash G)^{K_i}$ are contained in $L^2(\Gamma \backslash G)^I$ - the subspace of Iwahori-fixed vectors. The Iwahori-Hecke algebra of G is $\mathcal{H} = C_c(I \backslash G / I)$, which stands for compactly supported, bi- I -invariant complex functions on G . Multiplication in \mathcal{H} is

given by convolution, and if (ρ, V) is a representation of G , then $(\bar{\rho}, V^I)$ is a representation of \mathcal{H} , by $\bar{\rho}(\eta)v = \int_G \eta(g)\rho(g)v dg$.

We proceed to construct elements in \mathcal{H} whose actions on the subspaces $L^2(\Gamma \backslash G)^{K_i}$ of $L^2(\Gamma \backslash G)$ coincide with the boundary and coboundary operators in the chain complex of simplicial forms on X . For a bi- I -invariant set $S \subseteq G$, let $\mathbb{1}_S$ denote the corresponding characteristic function, as an element of \mathcal{H} .

Proposition 6. *Let*

$$\begin{aligned} \partial_1 &= \frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_{K\sigma^2} - \mathbb{1}_K) & \delta_1 &= \frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_{\sigma K} - \mathbb{1}_K) \\ \partial_2 &= \frac{1}{\sqrt{\mu(E)\mu(T)}} \cdot \mathbb{1}_{ET} & \delta_2 &= \frac{1}{\sqrt{\mu(E)\mu(T)}} \cdot \mathbb{1}_{TE}. \end{aligned} \quad (3.3)$$

Then each $\partial_i \in \mathcal{H}$ takes $L^2(\Gamma \backslash G)^{K_i}$ to $L^2(\Gamma \backslash G)^{K_{i-1}}$, and acts as the boundary operator $\partial_i : \Omega^i(X) \rightarrow \Omega^{i-1}(X)$ with respect to the identifications of $\Omega^i(X)$ with $L^2(\Gamma \backslash G)^{K_i}$. The parallel statements hold for $\delta_i \in \mathcal{H}$ and $\delta_i : \Omega^{i-1} \rightarrow \Omega^i$ as well.

Proof. As ∂_i is constant on right K_{i-1} cosets, the image of its action on any G -representation V is in $V^{K_{i-1}}$, and similarly for δ_i and K_i (note that $\sigma K = E\sigma K$ as $\sigma^{-1}E\sigma$ fixes v_0). Observe $V = L^2(\Gamma \backslash G)$ and $f \in V^E \cong \Omega^1(E)$, and denote by $\{k\}_{kE \in K/E}$ an arbitrary choice of a left transversal for E in K . For any $gv_0 \in X^0$ we have

$$\begin{aligned} (\mathbb{1}_K f)(gv_0) &= (\mathbb{1}_K f)(\Gamma g) = \int_G \mathbb{1}_K(x)(xf)(\Gamma g) dx = \int_K (xf)(\Gamma g) dx \\ &= \int_K f(\Gamma gx) dx = \sum_{kE \in K/E} \int_E f(\Gamma gke) de = \sum_{kE \in K/E} \int_E f(\Gamma gk) de \\ &= \mu(E) \sum_{kE \in K/E} f(\Gamma gk) = \sqrt{\mu(E)} \sum_{kE \in K/E} f(gke_0), \end{aligned}$$

where the first and last equalities are the identifications of $L^2(\Gamma \backslash G)^{K_i}$ with $\Omega^i(X)$. The group K acts transitively on the $q^2 + q + 1$ edges in \mathcal{B} with origin v_0 and color 1, which correspond to the projective points in $\mathcal{O}^3/\pi \cdot \mathcal{O}^3 \cong \mathbb{P}_q^3$. Thus, the color-1 edges with origin gv_0 (for any $g \in G$) are precisely $\{gke_0\}_{kE \in K/E}$, giving

$$\frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_K f)(gv_0) = \sum_{kE \in K/E} f(gke_0) = \sum_{\substack{\text{orig } e = gv_0 \\ \text{col } e = 1}} f(e) = - \sum_{\substack{\text{term } e = gv_0 \\ \text{col } e = 2}} f(e). \quad (3.4)$$

In a similar manner, $\sigma^2 e_0 = [\sigma^2 v_0, v_0]$ has color 1 and terminates in v_0 , and K act transitively on such edges (which correspond to projective lines in $\mathcal{O}^3/\pi \cdot \mathcal{O}^3$). Thus, the color 1 edges which terminate in gv_0 are $\{gk\sigma^2 e_0\}_{k \in K}$. We have $K\sigma^2 = (\sigma K)^{-1} = (E\sigma K)^{-1} = K\sigma^2 E$, and if $\{k\sigma^2\}_{k\sigma^2 E \in K\sigma^2 E/E}$ is a left transversal of E in $K\sigma^2 E$ then

$$\begin{aligned} (\mathbb{1}_{K\sigma^2} f)(gv_0) &= \int_{K\sigma^2 E} f(\Gamma gx) dx = \sum_{k\sigma^2 E \in K\sigma^2 E/E} \int_E f(\Gamma gk\sigma^2 e) de \\ &= \sqrt{\mu(E)} \sum_{k\sigma^2 E \in K\sigma^2 E/E} f(gk\sigma^2 e_0) = \sqrt{\mu(E)} \sum_{\substack{\text{term } e = gv_0 \\ \text{col } e = 1}} f(e). \end{aligned} \quad (3.5)$$

Together with (3.4), this implies that the diagram

$$\begin{array}{ccc} L^2(\Gamma \backslash G)^K & \xleftarrow{\partial_1 \in \mathcal{H}} & L^2(\Gamma \backslash G)^E \\ \downarrow \cong & & \downarrow \cong \\ \Omega^0(X) & \xleftarrow{\partial_1} & \Omega^1(X) \end{array}$$

commutes (which should justify the abuse of notation). The reasoning for ∂_2 is similar, save for the fact that $T \not\leq E$ (in fact, $E \cap T = I$, as a common stabilizer of t_0 and e_0 must fix t_0 pointwise.) We observe that E acts transitively on $(q+1)$ triangles containing e_0 , which correspond to incidences of projective lines and points in $\mathcal{O}^3/\pi \cdot \mathcal{O}^3$. Therefore, for $f \in V = L^2(\Gamma \backslash G)^T$ and $ge_0 \in X^1$,

$$\begin{aligned} \frac{1}{\sqrt{\mu(E)\mu(T)}} (\mathbb{1}_{ET} f)(ge_0) &= \frac{1}{\sqrt{\mu(T)}} (\mathbb{1}_{ET} f)(\Gamma g) = \sqrt{\mu(T)} \sum_{eT \in ET/T} f(\Gamma ge) \\ &= \sum_{eT \in ET/T} f(ge t_0) = \sum_{\tau \in X^2 : e_0 \in \partial \tau} f(g\tau) = \sum_{\tau \in X^2 : ge_0 \in \partial \tau} f(\tau), \end{aligned}$$

agreeing with $\partial_2 : \Omega^2 \rightarrow \Omega^1$.

The coboundary operators can be analyzed in a similar manner, but they also follow from the unitary structure: \mathcal{H} has a $*$ -algebra structure induced by the group inversion, namely $\eta^*(g) = \overline{\eta(g^{-1})}$. For any unitary representation (ρ, V) of G , the induced representation $(\bar{\rho}, V^I)$ is unitary, i.e. $\bar{\rho}(\eta)^* = \bar{\rho}(\eta^*)$ (this uses unimodularity of G). For $V = L^2(\Gamma \backslash G)$ this gives

$$\partial_1^* = \frac{1}{\sqrt{\mu(E)}} \left(\overline{\mathbb{1}_{(K\sigma^2)^{-1}}} - \overline{\mathbb{1}_{K^{-1}}} \right) = \frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_{\sigma K} - \mathbb{1}_K)$$

and similarly for ∂_2^* . \square

Since Γ is cocompact, $L^2(\Gamma \backslash G)$ decomposes as a sum of irreducible unitary representations, $L^2(\Gamma \backslash G) = \bigoplus_{\alpha} W_{\alpha}$, and $\Omega^i(X) \cong L^2(\Gamma \backslash G)^{K_i} = \bigoplus_{\alpha} W_{\alpha}^{K_i} \leq \bigoplus_{\alpha} W_{\alpha}^I$. Each W_{α}^I is a sub- \mathcal{H} -representation, so that the operators ∂_i, δ_i decompose with respect to this sum, and thus the Laplacians as well, giving $\text{Spec} \Delta_i^{\pm} = \bigcup_{\alpha} \text{Spec} \Delta_i^{\pm}|_{W_{\alpha}^{K_i}}$, with the correct multiplicities. To understand the spectra it is enough look at the W_{α} which are *Iwahori-spherical*, namely, contain I -fixed vectors (for otherwise $W_{\alpha}^{K_i} = 0$). Furthermore, as Δ_i^{\pm} act by elements in \mathcal{H} , the isomorphism type of W_{α}^I as an \mathcal{H} -module already determines the spectrum of Δ_i^{\pm} on $W_{\alpha}^{K_i}$. By [Cas80, Prop. 2.6], every Iwahori-spherical representation is isomorphic to a subrepresentation of a *principal series representation*. The principal series representation $V_{\mathfrak{z}}$ with *Satake parameters* $\mathfrak{z} = (z_1, z_2, z_3)$, where $z_i \in \mathbb{C}$ and $z_1 z_2 z_3 = 1$, is obtained as follows: The standard Borel group $B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}$ admits a character $\chi_{\mathfrak{z}}(b) = \prod_{i=1}^3 z_i^{\text{ord}_{\pi} b_{ii}}$ (here $z_1 z_2 z_3 = 1$ ensures that $\chi_{\mathfrak{z}}$ is well defined on PGL), and $V_{\mathfrak{z}}$ is the unitary induction of $\chi_{\mathfrak{z}}$ from B to G , namely

$$V_{\mathfrak{z}} = \text{UInd}_B^G \chi_{\mathfrak{z}} = \left\{ (\text{smooth}) f : G \rightarrow \mathbb{C} \mid f(bg) = \delta^{-\frac{1}{2}}(b) \chi_{\mathfrak{z}}(b) f(g) \ \forall b \in B \right\}, \quad (3.6)$$

where $\delta(b) = |b_{33}|^2 / |b_{11}|^2$ is the modular character of B . For obvious reasons, it is convenient to introduce the notation

$$\tilde{\chi}_{\mathfrak{z}}(b) = \delta^{-\frac{1}{2}}(b) \chi_{\mathfrak{z}}(b) = \frac{|b_{11}|}{|b_{33}|} \prod_{i=1}^3 z_i^{\text{ord}_{\pi} b_{ii}} = \left(\frac{z_1}{q} \right)^{\text{ord}_{\pi} b_{11}} z_2^{\text{ord}_{\pi} b_{22}} (q z_3)^{\text{ord}_{\pi} b_{33}}.$$

Let us stress the following point: having decomposed $L^2(\Gamma \backslash G) = \bigoplus_{\alpha} W_{\alpha}$, and found some Δ_i^{\pm} -eigenform $f \in W_{\alpha}^{K_i} \leq \Omega^i(X)$, we can lift it to a Γ -periodic form $\tilde{f} \in {}^{\Gamma}\Omega^i(\mathcal{B})$, which is still a Δ_i^{\pm} -eigenform (as Δ_i^{\pm} act in a local manner). For some Satake parameters \mathfrak{z} there is an embedding of G -representations $\Psi : W_{\alpha} \rightarrow V_{\mathfrak{z}}$, and naturally $\Psi f \in V_{\mathfrak{z}}^{K_i}$. Since the elements in the explicit construction of $V_{\mathfrak{z}}$ in (3.6) are complex functions on G , the ones in $V_{\mathfrak{z}}^{K_i}$ can also be seen as i -forms on \mathcal{B} . Thus, both \tilde{f} and Ψf are in $\Omega^i(\mathcal{B})$, and they are Δ_i^{\pm} -eigenforms with the same eigenvalue (as Ψ is \mathcal{H} -equivariant). However, they are not the same, as \tilde{f} attains finitely many values and Ψf infinitely many, in general.

While the forms \tilde{f} and Ψf are different, their averages on K_i -orbits are the same: these averages correspond to the associated matrix coefficient of $W_{\alpha} \cong \Psi W_{\alpha}$. When these matrix coefficient are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, the representation W_{α} is said to be *tempered*. For any irreducible unitary representation V and $v \in V^{K_i}$, the matrix coefficient $\phi(g) = \langle gv, v \rangle$ is bi- K_i -invariant, and if V is tempered then ϕ is an approximate L^2 -eigenform in $\Omega^i(\mathcal{B})$, so that $\text{Spec } \Delta_i^{\pm}|_{V^{K_i}}$ is contained in the L^2 -spectrum of Δ_i^{\pm} on \mathcal{B} .

4. ANALYSIS OF THE PRINCIPAL SERIES

While in general an irreducible Iwahori-spherical representation W is only isomorphic to a subrepresentation of $V_{\mathfrak{z}}$, it is simpler to consider the action of ∂_i, δ_i and Δ_i^{\pm} on $V_{\mathfrak{z}}$, and later find which eigenvalues are attained on W . By the Iwasawa decomposition $G = BK$, for $f \in V_{\mathfrak{z}}^K$ and $g = bk \in G$ one has $f(g) = f(bk) = \tilde{\chi}_{\mathfrak{z}}(b) f(1)$, so that $V_{\mathfrak{z}}^K$ is at most one-dimensional. In fact, it is one-dimensional, since $b \in B \cap K$ implies $b_{11}, b_{22}, b_{33} \in \mathcal{O}^{\times}$, so that $f^K(bk) \stackrel{\text{def}}{=} \tilde{\chi}_{\mathfrak{z}}(b)$ is well defined and spans $V_{\mathfrak{z}}^K$. Similarly, identifying S_3 with the permutation matrices in G , the so-called Iwahori-Bruhat decomposition $G = \coprod_{w \in S_3} BwI$ shows that $\dim V_{\mathfrak{z}}^I = 6$, with basis $\{f_w^I\}_{w \in S_3}$ defined by $f_w^I(w') = \delta_{w, w'}$ (note that $f^K = \sum_{w \in S_3} f_w^I$). Also, $(12) \in E$ (see (3.1)) implies $G = \coprod_{w \in A_3} BwE$, and $\dim V_{\mathfrak{z}}^E = 3$ with basis $\{f_w^E\}_{w \in A_3}$, where $f_w^E \stackrel{\text{def}}{=} f_w^I + f_{w \cdot (12)}^I$ satisfy $f_w^E(w') = \delta_{w, w'}$ for $w, w' \in A_3$. For T things are slightly more complicated, since $T \not\leq K$: here $G = BT \amalg B(12)T$, and $\dim V_{\mathfrak{z}}^T = 2$. One can then define $f_w^T(w') \stackrel{\text{def}}{=} \delta_{w, w'}$ for $w, w' \in \{(), (12)\}$, and this gives

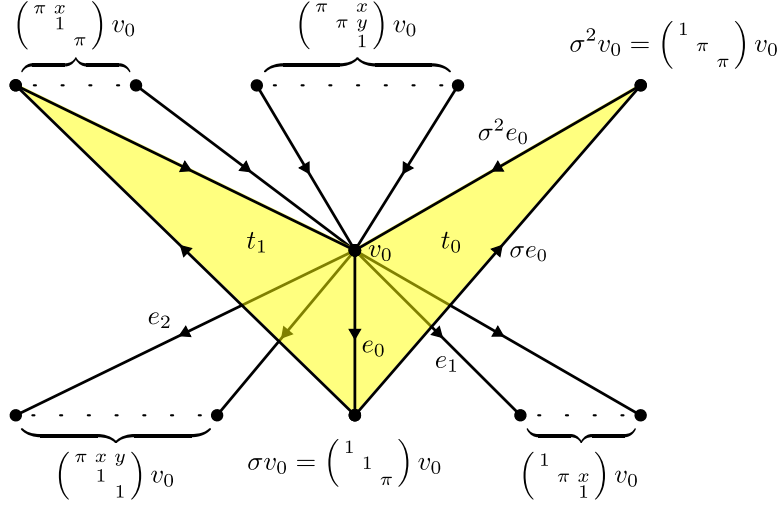
$$\begin{aligned} f_{()}^T &= f_{()}^I + \frac{1}{qz_3} f_{(321)}^I + \frac{z_1}{q} f_{(123)}^I \\ f_{(12)}^T &= f_{(12)}^I + z_2 f_{(23)}^I + \frac{1}{qz_3} f_{(13)}^I; \end{aligned} \tag{4.1}$$

indeed, if c_w is the coefficient of f_w^I in $f_{()}^T$, then

$$c_{(123)} = f_{()}^T((123)) = f_{()}^T\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1 \end{pmatrix} \sigma^2\right) = f_{()}^T\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1 \end{pmatrix}\right) = \tilde{\chi}_{\mathfrak{z}}\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1 \end{pmatrix}\right) = \frac{z_1}{q},$$

and the other coefficients in (4.1) are obtained similarly.

Now, let $\Omega_{\mathfrak{z}}^i(\mathcal{B})$ be the embedding of $V_{\mathfrak{z}}^{K_i}$ in $\Omega^i(\mathcal{B})$ given by the explicit construction (3.6). Any $f \in \Omega_{\mathfrak{z}}^0(\mathcal{B})$ is determined by its value on v_0 , namely $f = f(v_0) f^K$. Similarly, the value on $e_0, e_1 = (321)e_0$ and $e_2 = (123)e_0$ determine a unique element in $\Omega_{\mathfrak{z}}^1(\mathcal{B})$, and likewise for $t_0, t_1 = (12)t_0$ and $\Omega_{\mathfrak{z}}^2(\mathcal{B})$. As $S_3 \leq K$, all of $v_0, e_0, e_1, e_2, t_0, t_1$ are contained in the star around v_0 in \mathcal{B} , and we can understand $\partial_i|_{\Omega_{\mathfrak{z}}^i(\mathcal{B})}$ and $\delta_i|_{\Omega_{\mathfrak{z}}^{i-1}(\mathcal{B})}$ by evaluation on star(v_0) alone (see Figure 4.1).

FIGURE 4.1. The star of v_0 in \mathcal{B} .

Thus,

$$\begin{aligned} (\delta_1 f^K)(e_0) &= f^K(\sigma v_0) - f^K(v_0) = f^K\left(\begin{pmatrix} 1 & x \\ \pi & \end{pmatrix} v_0\right) - 1 = \tilde{\chi}_3\left(\begin{pmatrix} 1 & x \\ \pi & \end{pmatrix}\right) - 1 = qz_3 - 1 \\ (\delta_1 f^K)(e_1) &= f^K((3\ 2\ 1)\sigma v_0) - f^K(v_0) = f^K\left(\begin{pmatrix} 1 & \pi \\ \pi & 1 \end{pmatrix} v_0\right) - 1 = z_2 - 1 \\ (\delta_1 f^K)(e_2) &= f^K((1\ 2\ 3)\sigma v_0) - f^K(v_0) = f^K\left(\begin{pmatrix} \pi & 1 \\ 1 & \end{pmatrix} v_0\right) - 1 = \frac{z_1}{q} - 1 \end{aligned}$$

gives

$$\left[\delta_1|_{\Omega_3^0(\mathcal{B})}\right]_{\mathfrak{B}^E}^{\{f^K\}} = \begin{pmatrix} qz_3 - 1 \\ z_2 - 1 \\ \frac{z_1}{q} - 1 \end{pmatrix}, \quad (4.2)$$

where \mathfrak{B}^E denotes the ordered basis $f_{(\cdot)}^E, f_{(3\ 2\ 1)}^E, f_{(1\ 2\ 3)}^E$.

The edges of color 1 with origin v_0 are $e_0, \begin{pmatrix} 1 & x \\ \pi & 1 \end{pmatrix} e_1$ for $x \in \mathbb{F}_q = \mathcal{O}/\pi\mathcal{O}$, and $\begin{pmatrix} 1 & x & y \\ \pi & 1 & \end{pmatrix} e_2$ with $x, y \in \mathbb{F}_q$. As $\tilde{\chi}_3$ is trivial on upper-triangular unipotent matrices, every $f \in \Omega_3^1(\mathcal{B})$ is constant on these q translations of e_1 , and on the q^2 translations of e_2 (in particular this implies that $\oplus_3 \Omega_3^1(\mathcal{B})$ is a proper subspace of $\Omega^1(\mathcal{B})$). Thus, (3.4) implies that for $f \in \Omega_3^1(\mathcal{B})$ we have $(\mu(E)^{-\frac{1}{2}} \mathbb{1}_K f)(v_0) = f(e_0) + qf(e_1) + q^2f(e_2)$. The edges of color 1 which terminate in v_0 are

$$[\sigma^2 v_0, v_0] = \sigma^2 e_0 = \begin{pmatrix} \pi & 1 \\ \pi & \end{pmatrix} e_0 = \begin{pmatrix} 1 & \pi \\ \pi & \pi \end{pmatrix} (1\ 2\ 3) e_0 = \begin{pmatrix} 1 & \pi \\ \pi & \pi \end{pmatrix} e_2,$$

and similarly $\begin{pmatrix} \pi & x \\ \pi & 1 \end{pmatrix} e_1$ and $\begin{pmatrix} \pi & x & y \\ \pi & \pi & 1 \end{pmatrix} e_0$ ($x, y \in \mathbb{F}_q$). By (3.5),

$$\begin{aligned} \left(\frac{1}{\sqrt{\mu(E)}} \mathbb{1}_{K\sigma^2 E} f\right)(v_0) &= f\left(\begin{pmatrix} 1 & \pi \\ \pi & \pi \end{pmatrix} e_2\right) + \sum_{x \in \mathbb{F}_q} f\left(\begin{pmatrix} \pi & x \\ \pi & 1 \end{pmatrix} e_1\right) + \sum_{x, y \in \mathbb{F}_q} f\left(\begin{pmatrix} \pi & x & y \\ \pi & \pi & 1 \end{pmatrix} e_0\right) \\ &= z_2 q z_3 f(e_2) + q \cdot z_1 z_3 f(e_1) + q^2 \cdot \frac{z_1}{q} z_2 f(e_0), \end{aligned}$$

and in total (see (3.3))

$$\left[\partial_1|_{\Omega_3^1(\mathcal{B})} \right]_{\{f^K\}}^{\mathfrak{B}^E} = \begin{pmatrix} \frac{q}{z_3} - 1 & \frac{q}{z_2} - q & \frac{q}{z_1} - q^2 \end{pmatrix}. \quad (4.3)$$

As $\Delta_0^+ = \partial_1 \delta_1$ and $\Delta_1^- = \delta_1 \partial_1$, we can now compute explicitly their action on the \mathfrak{z} -principal series. Denoting $\tilde{\mathfrak{z}} = \sum_{i=1}^3 (z_i + z_i^{-1})$, we have by (4.2) and (4.3)

$$\begin{aligned} \Delta_0^+|_{\Omega_3^0(\mathcal{B})} &= (\lambda^K)^{\text{def}} (k_0 - \tilde{\mathfrak{z}}), \\ \left[\Delta_1^-|_{\Omega_3^1(\mathcal{B})} \right]_{\mathfrak{B}^E} &= \begin{pmatrix} q^2 - qz_3 - \frac{q}{z_3} + 1 & -q^2 z_3 + \frac{q^2 z_3}{z_2} + q - \frac{q}{z_2} & -q^3 z_3 + \frac{q^2 z_3}{z_1} + q^2 - \frac{q}{z_1} \\ \frac{qz_2}{z_3} - z_2 - \frac{q}{z_3} + 1 & -qz_2 + 2q - \frac{q}{z_2} & -q^2 z_2 + q^2 + \frac{qz_2}{z_1} - \frac{q}{z_1} \\ -\frac{q}{z_3} - \frac{z_1}{q} + \frac{z_1}{z_3} + 1 & q - z_1 - \frac{q}{z_2} + \frac{z_1}{z_2} & q^2 - qz_1 - \frac{q}{z_1} + 1 \end{pmatrix}. \end{aligned} \quad (4.4)$$

Note that Δ_0^+ agrees with the computation of the spectrum of the Hecke operators in [Mac79, Li04, LSV05a], as $\Delta_0^+ = k_0 \cdot I - \sum_{i=1}^{d-1} A_i$ (here A_i is the i -th Hecke operator on \mathcal{B}_d , loc. cit.). We also understand Δ_1^- well enough, as it has the eigenvalue λ^K corresponding to $\delta_1 \Omega_3^0(\mathcal{B})$ (since $\Delta_1^- \delta_1 f^K = \delta_1 \Delta_0^+ f^K = \lambda^K \delta_1 f^K$), and two zeros (which come from $\partial_2 \Omega_3^2(\mathcal{B})$). However, we can now proceed to compute as easily the edge/triangle spectrum: for $f \in \Omega_3^1(\mathcal{B})$,

$$\begin{aligned} (\delta_2 f)(t_0) &= \sum_{i=0}^2 f(\sigma^i e_0) = f(e_0) + f\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} (3\ 2\ 1) e_0\right) + f\left(\begin{pmatrix} 1 & & \\ & \pi & \\ & & \pi \end{pmatrix} (1\ 2\ 3) e_0\right) \\ &= f(e_0) + qz_3 f(e_1) + z_2 qz_3 f(e_2) \\ (\delta_2 f)(t_1) &= \sum_{i=0}^2 f((1\ 2) \sigma^i e_0) = f(e_0) + f\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} (1\ 2\ 3) e_0\right) + f\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & \pi \end{pmatrix} (3\ 2\ 1) e_0\right) \\ &= f(e_0) + qz_3 f(e_2) + z_1 z_3 f(e_1) \end{aligned}$$

show that $\left[\delta_2|_{\Omega_3^1(\mathcal{B})} \right]_{\mathfrak{B}^T}^{\mathfrak{B}^E} = \begin{pmatrix} 1 & qz_3 & qz_2 z_3 \\ 1 & z_1 z_3 & qz_3 \end{pmatrix}$, where \mathfrak{B}^T is the ordered basis $f_{()}^T, f_{(1\ 2)}^T$ (and at this point one can verify directly that $\delta_2 \delta_1|_{\Omega_3^1(\mathcal{B})} = 0$).

The triangles containing e_0 are obtained by adjoining $\sigma^2 v_0$ (which gives t_0) and $\begin{pmatrix} \pi & x & \\ & 1 & \\ & & \pi \end{pmatrix} v_0$ ($x \in \mathbb{F}_q$), giving $\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} t_1$. This yields $(\partial_2 f)(e_0) = f(t_0) + qf(t_1)$, but for e_1, e_2 we need to work a little harder, and use (4.1):

$$\begin{aligned} (\partial_2 f)(e_1) &= (\partial_2 f)((3\ 2\ 1) e_0) = f((3\ 2\ 1) t_0) + \sum_{x \in \mathbb{F}_q} f\left((3\ 2\ 1) \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} t_1\right) \\ &= \frac{1}{qz_3} f(t_0) + f((3\ 2\ 1) t_1) + \sum_{x \in \mathbb{F}_q^\times} f\left((3\ 2\ 1) \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} (1\ 2) t_0\right) \\ &= \frac{1}{qz_3} f(t_0) + f((2\ 3) t_0) + \sum_{x \in \mathbb{F}_q^\times} f\left(\begin{pmatrix} -\frac{1}{x} & & \\ & 1 & \\ & & x \end{pmatrix} (3\ 2\ 1) \begin{pmatrix} 1 & \frac{1}{x} & \\ & 1 & \\ & & 1 \end{pmatrix} t_0\right) \\ &= \frac{1}{qz_3} f(t_0) + z_2 f(t_1) + \sum_{x \in \mathbb{F}_q^\times} f((3\ 2\ 1) t_0) = \frac{1}{z_3} f(t_0) + z_2 f(t_1) \end{aligned}$$

$$\begin{aligned}
(\partial_2 f)(e_2) &= (\partial_2 f)((123)e_0) = f((123)t_0) + \sum_{x \in \mathbb{F}_q} f\left((123)\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} t_1\right) \\
&= \frac{z_1}{q} f(t_0) + \sum_{x \in \mathbb{F}_q} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} (13)t_0\right) = \frac{z_1}{q} f(t_0) + \frac{1}{z_3} f(t_1),
\end{aligned}$$

so that $\left[\partial_2|_{\Omega_3^2(B)}\right]_{\mathfrak{B}^E}^{\mathfrak{B}^T} = \begin{pmatrix} 1 & q \\ 1/z_3 & z_2 \\ z_1/q & 1/z_3 \end{pmatrix}$, giving

$$\begin{aligned}
\left[\Delta_1^+|_{\Omega_3^1(B)}\right]_{\mathfrak{B}^E} &= \begin{pmatrix} q+1 & \frac{q}{z_2} + qz_3 & q^2 z_3 + \frac{q}{z_1} \\ z_2 + \frac{1}{z_3} & q+1 & \frac{q}{z_1} + qz_2 \\ \frac{z_1}{q} + \frac{1}{z_3} & \frac{1}{z_2} + z_1 & q+1 \end{pmatrix} \\
\left[\Delta_2^-|_{\Omega_3^2(B)}\right]_{\mathfrak{B}^T} &= \begin{pmatrix} q+2 & \frac{q}{z_1} + qz_2 + q \\ \frac{1}{z_2} + z_1 + 1 & 2q+1 \end{pmatrix}.
\end{aligned}$$

Recalling that $\lambda^K = k_0 - q\tilde{\mathfrak{z}} = 2(q^2 + q + 1) - q(\sum z_i + \frac{1}{z_i})$, one obtains

$$\begin{aligned}
\text{Spec } \Delta_1^+|_{\Omega_3^1(B)} &= \{\lambda_0^E, \lambda_{\pm}^E\} \stackrel{\text{def}}{=} \left\{0, \frac{3}{2}(q+1) \pm \frac{1}{2}\sqrt{(q+1)^2 + 4q(2+\tilde{\mathfrak{z}})}\right\} \\
&= \left\{0, \frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda^K}\right\},
\end{aligned} \tag{4.5}$$

and again $\text{Spec } \Delta_2^-|_{\Omega_3^2(B)} = \{\lambda_{\pm}^E\}$ as we have argued for Δ_1^- . For Δ_1^+ , $\lambda_0^E = 0$ is obtained on $\delta_1 f^K$ (whose f_w^E coefficients were computed in (4.2)), and λ_{\pm}^E are obtained on

$$\begin{aligned}
f_{\pm}^E &= \begin{pmatrix} 2\left(z_2^{-2} + \frac{z_1}{z_2}\right)q^2 - 2(z_3+1)q \\ 1 - q^2 z_1 + q\left(z_1 + \frac{2}{z_2} - \frac{2}{z_3} - 1\right) \pm (qz_1 - 1)\sqrt{9k_1^2 - 4\lambda^K} \\ qz_1(z_2^{-1} + 2z_1 + 1) - \frac{z_1}{z_2} - z_1 - \frac{2}{z_2} \pm \left(-z_1 + \frac{z_1}{z_2}\right)\sqrt{9k_1^2 - 4\lambda^K} \end{pmatrix}^T \cdot \begin{pmatrix} f_{()}^E \\ f_{(321)}^E \\ f_{(123)}^E \end{pmatrix} \\
&= 2q\left(1 + z_2 + \frac{1}{z_1}\right)\partial_2 f_{()}^T + \left(q - 1 \pm \sqrt{9k_1^2 - 4\lambda^K}\right)\partial_2 f_{(12)}^T.
\end{aligned}$$

5. UNITARY IWAHORI-SPHERICAL REPRESENTATIONS

In general, an irreducible Iwahori-spherical representation is only a subrepresentation of $V_{\mathfrak{z}}$. Let us denote by $W_{\mathfrak{z}}$ this subrepresentation (there is only one such). Tadic [Tad86] has classified the Satake parameters for which the representation $W_{\mathfrak{z}}$ admits a unitary structure. In [KLW10] the possible \mathfrak{z} for $\text{PGL}_3(F)$ are listed, and a basis for $W_{\mathfrak{z}} \leq V_{\mathfrak{z}}$ is computed explicitly, using results from [Bor76, Zel80]. It turns out that a unitary E -spherical $W_{\mathfrak{z}}$ is of one of the following types:

- (a) $|z_i| = 1$ for $i = 1, 2, 3$. In this case $V_{\mathfrak{z}}$ is irreducible, hence $W_{\mathfrak{z}} = V_{\mathfrak{z}}$, and $\tilde{\mathfrak{z}} \in [-3, 6]$ gives $\lambda^K \in [k_0 - 6q, k_0 + 3q]$ and $\lambda_{\pm}^E \in \mathcal{I}_{\pm}$ (see (4.5) and (2.6)).
- (b) $\mathfrak{z} = (c^{-2}, cq^a, cq^{-a})$ for some $|c| = 1$ and $0 < a < \frac{1}{2}$. Here too $V_{\mathfrak{z}}$ is irreducible.

- (c) $\mathfrak{z} = \left(\frac{c}{\sqrt{q}}, c\sqrt{q}, c^{-2}\right)$ for some $|c| = 1$. In this case $W_{\mathfrak{z}}^E$ is one-dimensional, and spanned by f_-^E , which is proportional to $qf_{(3\ 2\ 1)}^E - f_{(1\ 2\ 3)}^E$. It corresponds to

$$\begin{aligned}\lambda_-^E &= \frac{1}{2} \left(3k_1 - \sqrt{k_1^2 + 8q + 4q \left(\frac{c}{\sqrt{q}} + \bar{c}\sqrt{q} + c\sqrt{q} + \frac{\bar{c}}{q} + c^{-2} + c^2 \right)} \right) \\ &= \frac{1}{2} \left(3k_1 - \sqrt{q^2 + 8q\sqrt{q}\Re(c) + 2q + 16q\Re(c)^2 + 1 + 8\sqrt{q}\Re(c)} \right) \\ &= \frac{1}{2} (3k_1 - (q + 4\sqrt{q}\Re(c) + 1)) = k_1 - 2\sqrt{q}\Re(c)\end{aligned}$$

which lies in \mathcal{I} (see (2.5)). As f_-^E is not K -fixed, $W_{\mathfrak{z}}^K = 0$.

- (d) $\mathfrak{z} = \left(c\sqrt{q}, \frac{c}{\sqrt{q}}, c^{-2}\right)$ for some $|c| = 1$. Here $W_{\mathfrak{z}}^E = \langle f_0^E, f_+^E \rangle$, where f_+^E is proportional to $(q+1)f_{(\)}^E + \left(c^2 + \frac{c}{\sqrt{q}}\right)(f_{(3\ 2\ 1)}^E + f_{(1\ 2\ 3)}^E)$, and $\lambda_+^E = 2k_1 + 2\sqrt{q}\Re(c)$ similarly to the computation in type (c). This time f^K is in $W_{\mathfrak{z}}^E$, and corresponds to $\lambda^K = k_0 - 2q\Re\left(\frac{(q+1)}{\sqrt{q}}c + c^2\right)$.
- (e) $\mathfrak{z} = (q, 1, \frac{1}{q})$. In this case $W_{\mathfrak{z}}$ is the trivial representation $\rho : G \rightarrow \mathbb{C}^\times$, and $W_{\mathfrak{z}}^E = W_{\mathfrak{z}}^K$ are spanned by $f^K = f_+^E$. Since f^K is constant and f_+^E is a disorientation we have $\lambda^K = 0$ and $\lambda_+^E = 3k_1$ (alternatively, use (4.4) and (4.5)).
- (f) $\mathfrak{z} = (\omega q, \omega, \frac{\omega}{q})$ where $\omega = e^{\frac{2\pi i}{3}}$ or $\omega = e^{-\frac{2\pi i}{3}}$. Here $W_{\mathfrak{z}}$ is the one-dimensional representation $\rho(g) = \omega^{\text{col}(g)}$, and $W_{\mathfrak{z}}^K = W_{\mathfrak{z}}^E = \langle f^K \rangle = \langle f_0^E \rangle$, giving $\lambda^K = \frac{3k_0}{2}$.

Apart from these there is the Steinberg (Stn) representation $\mathfrak{z} = (\frac{1}{q}, 1, q)$. It is not E -spherical, and W^T is spanned by $f_0^T = qf_{(\)}^T - f_{(1\ 2)}^T$, which is always in $\ker \partial_2 = \ker \Delta_2^-$. (In [KLW10] the twisted Steinberg representations $\mathfrak{z} = (\frac{\omega}{q}, \omega, \omega q)$ are considered as well, but these do not contribute to the simplicial theory as they have no K , E or T -fixed vectors.)

Let $X = \Gamma \backslash \mathcal{B}$ be a non-tripartite Ramanujan complex with $L^2(\Gamma \backslash G) \cong \bigoplus_i W_{\mathfrak{z}_i}$, and denote by $N_{(t)}$ the number of $W_{\mathfrak{z}_i}$ of type (t) . These are computed in [KLW10] for the tripartite case, and our arguments are very similar. By the Ramanujan assumption every Iwahori-spherical $W_{\mathfrak{z}_i}$ is either tempered, which are the types (a), (c), and (Stn), or finite-dimensional (types (e), (f)), so that $N_{(b)} = N_{(d)} = 0$. The trivial representation (e) always appears once in $L^2(\Gamma \backslash G)$ as the constant functions, so that $N_{(e)} = 1$.^(†) The color representations (type (f)) correspond to $f \in L^2(\Gamma \backslash G)$ satisfying $f(\Gamma g) = (gf)(\Gamma) = \omega^{\text{col } g} f(\Gamma)$, which is unique up to a scalar, and well defined iff $\text{col } \Gamma \equiv 0$. As we assume X to be non-tripartite, $N_{(f)} = 0$ (and otherwise we would have had $N_{(f)} = 2$). Next,

$$\begin{aligned}n &= \dim \Omega^0(X) = \sum_i \dim W_{\mathfrak{z}_i}^K = N_{(a)} + N_{(e)} + N_{(f)} \\ \frac{nk_0}{2} &= \dim \Omega^1(X) = \sum_i \dim W_{\mathfrak{z}_i}^E = 3N_{(a)} + N_{(c)} + N_{(e)} + N_{(f)}\end{aligned}$$

give $N_{(a)} = n - 1$ and $N_{(c)} = n(q^2 + q - 2) + 2$. This is summarized in Table 5.1, together with the tripartite case, and this also completes the proof of Theorem 5.

^(†)This gives an algebraic explanation to the fact that $3k_1$ always appear in $\text{Spec } \Delta_1^+$, unlike the graph case, where $2k_0 \in \text{Spec } \Delta_0^+$ only for bipartite quotients of \mathcal{B}_2 .

For completeness, we list in Table 5.1 what is W^T for each type. This gives $\frac{nk_0k_1}{6} = \dim \Omega^2(X) = 2N_{(a)} + N_{(c)} + N_{(e)} + N_{(Stn)}$, from which it follows that $N_{(Stn)} = \sum_{i=-1}^2 (-1)^i |X^i| = \tilde{\chi}(X)$, the reduced Euler characteristic of X (as is shown in [KLW10] for the tripartite case).

Type		W^K	Δ_0^+ e.v.	W^E	Δ_1^+ e.v.	W^T	mult. col $\Gamma=0$	mult. col $\Gamma \neq 0$
(a)	tempered	f^K	$k_0 + \tilde{q}\tilde{3}$	f_0^E, f_{\pm}^E	$0, \frac{3k_1}{2} \pm \sqrt{\frac{9k_1^2}{4} - \lambda^K}$	$\delta_1 f_{\pm}^E$	$n - 3$	$n - 1$
(b)		f^K	$k_0 + \tilde{q}\tilde{3}$	f_0^E, f_{\pm}^E	$0, \frac{3k_1}{2} \pm \sqrt{\frac{9k_1^2}{4} - \lambda^K}$	$\delta_1 f_{\pm}^E$	0	0
(c)	tempered	0	-	f_-^E	$k_1 - 2\sqrt{q}\Re(c)$	$\delta_1 f_-^E$	$\frac{nq^2+nq}{-2n+6}$	$\frac{nq^2+nq}{-2n+2}$
(d)		f^K	$k_0 + \tilde{q}\tilde{3}$	f_0^E, f_+^E	$0, 2k_1 + 2\sqrt{q}\Re(c)$	$\delta_1 f_+^E$	0	0
(e)	trivial	f^K	0	f_+^E	$3k_1$	$\delta_1 f_+^E$	1	1
(f)	fin. dim.	f^K	$\frac{3k_0}{2}$	f_0^E	0	0	2	0
(Stn)	tempered	0	-	0	-	f_0^T	$\tilde{\chi}(X)$	$\tilde{\chi}(X)$

TABLE 5.1. The representations appearing in $L^2(\Gamma \backslash G)$, with the corresponding Laplacian eigenvalues, and the multiplicity of appearance in the tripartite and non-tripartite Ramanujan cases.

6. COMBINATORIAL EXPANSION

We continue focusing on the non-tripartite Ramanujan case. We now know that the nontrivial vertex spectrum ($\text{Spec } \Delta_0^+|_{Z_0}$) is highly concentrated, lying in a $k_0 \pm O(\sqrt{k_0})$ strip. The nontrivial edge spectrum is “almost concentrated”: there are $\approx nq^2$ eigenvalues in a $k_1 \pm O(\sqrt{k_1})$ strip, but also $n - 1$ eigenvalues at $2k_1 \pm O(1)$ (and the eigenvalue $3k_1$). Nevertheless, having a concentrated vertex spectrum, and edge spectrum bounded away from zero is enough to prove the Cheeger-type inequality stated in Theorem 2. It can be rephrased in terms of a Cheeger-type constant: for a fixed constant $0 \leq \vartheta \leq \frac{1}{3}$, define

$$h_{\vartheta}(X) = \min \left\{ \frac{|T(A, B, C)| n^2}{|A| |B| |C|} \mid V = A \amalg B \amalg C, |A|, |B|, |C| \geq \vartheta n \right\} \quad (6.1)$$

Theorem 2 then implies that

$$h_{\vartheta}(X) \geq \lambda_1 \left(k_0 - \mu_0 \left(1 + \frac{10}{9\vartheta^3} \right) \right). \quad (6.2)$$

For Ramanujan triangle complexes, Theorem 5 gives $\mu_0 = 6q$ and $\lambda_1 = q + 1 - 2\sqrt{q}$, and therefore we have the following:

Corollary 7. *Fix $\vartheta > 0$, and let X be a non-tripartite Ramanujan triangle complex with n vertices, vertex degree $k_0 = 2(q^2 + q + 1)$ and edge degree $k_1 = q + 1$. For any partition of the vertices of X into sets A, B, C of sizes at least ϑn ,*

$$\frac{|T(A, B, C)| n^2}{|A| |B| |C|} \geq (q + 1 - 2\sqrt{q}) \left(2q^2 + 2q + 2 - 6q \left(1 + \frac{10}{9\vartheta^3} \right) \right).$$

In particular,

$$h_{\vartheta}(X) \geq 2q^3 - O_{\vartheta}(q^{2.5}). \quad (6.3)$$

We make a few remarks:

- (1) The inequality (6.3) corresponds to the pseudo-random expectation: X has $\frac{1}{3!}nk_0k_1$ triangles, so its triangle density is indeed $\frac{\frac{1}{3}n(q^2 + q + 1)(q + 1)}{\binom{n}{3}} \approx \frac{2q^3}{n^2}$.
- (2) If a triangle complex X has a complete skeleton, then $\text{Spec } \Delta_0^+|_{Z_0} = \{n\}$, so that $k_0 = n$ and $\mu_0 = 0$. Therefore, Theorem 2 reads $\frac{|T(A,B,C)|n^2}{|A||B||C|} \geq \lambda_1 \cdot n$, which is precisely the Cheeger inequality which appears in [PRT12] for complexes with a complete skeleton.
- (3) The reason to consider only partitions into “large blocks” in (6.1) is the following: If $f(n)$ is any sub-linear function, one can take $A \subseteq X^0$ to be any set of size $f(n)$, B to be $\partial A = \{v \mid \text{dist}(v, A) = 1\}$ (if $|B| < f(n)$ enlarge it by adding any vertices), and C the rest of the vertices. Assuming n is large enough one has $|A|, |B|, |C| \geq f(n)$, and $T(A, B, C) = \emptyset$ since all triangles with a vertex in A have their other vertices in either A or B . Thus, we are led to partitions into blocks of linear size.
- (4) Another Cheeger constant for complexes with non-complete skeleton was suggested in [PRT12], and related to the spectrum in [GS14]:

$$\tilde{h}(X) = \min_{V=A \amalg B \amalg C} \frac{|T(A, B, C)| \cdot n}{|T^\partial(A, B, C)|},$$

where $T^\partial(A, B, C)$ is the set of triangle-boundaries with one vertex in each of A , B and C . This constant, however, is not interesting for Ramanujan complexes - these are clique complexes, so that $\tilde{h}(X) = n$ trivially.

We now proceed to the proof:

Proof of Theorem 2. Denote $|A|, |B|, |C|$ by a, b, c , respectively, and define $f \in \Omega^1$ by

$$f(vw) = \begin{cases} c & v \in A, w \in B & -c & w \in A, v \in B \\ a & v \in B, w \in C & -a & w \in B, v \in C \\ b & v \in C, w \in A & -b & w \in C, v \in A \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_B = \mathbb{P}_{B^1} f$ and $f_Z = \mathbb{P}_{Z_1} f$. Then

$$\begin{aligned} |T(A, B, C)|n^2 &= \sum_{t \in T} (\delta f)^2(t) = \|\delta f\|^2 = \|\delta f_Z\|^2 \\ &= \langle \Delta^+ f_Z, f_Z \rangle \geq \lambda_1 \|f_Z\|^2 = \lambda_1 (\|f\|^2 - \|f_B\|^2). \end{aligned}$$

Let us denote

$$\mathfrak{D} = k_0 \mathbb{P}_{B^1} - \Delta_1^-. \quad (6.4)$$

Any linear maps $T : V \rightarrow W$ and $S : W \rightarrow V$ satisfy $\text{Spec } TS \setminus \{0\} = \text{Spec } ST \setminus \{0\}$, and thus

$$\begin{aligned} \text{Spec } \Delta_1^-|_{B^1} &= \text{Spec } \Delta_1^- \setminus \{0\} = \text{Spec } \Delta_0^+ \setminus \{0\} = \text{Spec } \Delta_0^+|_{B_0} \\ &\subseteq \text{Spec } \Delta_0^+|_{Z_0} \subseteq [k_0 - \mu_0, k_0 + \mu_0]. \end{aligned}$$

Together with $\Delta_1^-|_{Z_1} = 0$ this implies $\|\mathfrak{D}\| \leq \mu_0$, so that

$$\|f_B\|^2 = \langle \mathbb{P}_{B^1} f, \mathbb{P}_{B^1} f \rangle = \langle \mathbb{P}_{B^1} f, f \rangle \leq \frac{|\langle \mathfrak{D} f, f \rangle| + |\langle \Delta_1^- f, f \rangle|}{k_0} \leq \frac{\mu_0 \|f\|^2 + \|\partial f\|^2}{k_0}.$$

Using the expander mixing lemma^(†) for $E(A, B)$, $E(B, C)$ and $E(C, A)$ we have

$$\begin{aligned}\|f\|^2 &= |E(A, B)|c^2 + |E(B, C)|a^2 + |E(C, A)|b^2 \\ &\geq \left(\frac{k_0 ab}{n} - \mu_0 \sqrt{ab}\right)c^2 + \left(\frac{k_0 bc}{n} - \mu_0 \sqrt{bc}\right)a^2 + \left(\frac{k_0 ca}{n} - \mu_0 \sqrt{ca}\right)b^2 \\ &= k_0 abc - \mu_0 \left[\sqrt{abc}^2 + \sqrt{bca}^2 + \sqrt{acb}^2\right] \geq k_0 abc - \frac{\mu_0 n^3}{9},\end{aligned}$$

and we are left with the task of bounding $\|\partial f\|^2$. Let us begin with

$$\begin{aligned}\sum_{\alpha \in A} (\partial f)^2(\alpha) &= \sum_{\alpha \in A} \left(c \sum_{\beta \in B} \delta_{\alpha\beta} - b \sum_{\gamma \in C} \delta_{\alpha\gamma}\right)^2 \\ &= c^2 \sum_{\substack{\alpha \in A \\ \beta, \beta' \in B}} \delta_{\alpha\beta} \delta_{\alpha\beta'} - 2bc \sum_{\alpha, \beta, \gamma} \delta_{\alpha\beta} \delta_{\alpha\gamma} + b^2 \sum_{\alpha, \gamma, \gamma'} \delta_{\alpha\gamma} \delta_{\alpha\gamma'} \\ &= c^2 |P(B, A, B)| - 2bc |P(B, A, C)| + b^2 |P(C, A, C)|,\end{aligned}\tag{6.5}$$

where $P(S, T, R)$ is the sets of paths of length two going from S to R through T . By [Par13, Lem. 1.3] with $\ell = 2$ and $j = 0$ (and also $k_{-1} = n$ and $\varepsilon_{-1} = 0$), we have

$$\begin{aligned}\left| |P(B, A, B)| - \left(\frac{k_0}{n}\right)^2 b^2 a \right| &\leq 2k_0 \mu_0 b \\ \left| |P(B, A, C)| - \left(\frac{k_0}{n}\right)^2 bac \right| &\leq 2k_0 \mu_0 \sqrt{bc} \\ \left| |P(C, A, C)| - \left(\frac{k_0}{n}\right)^2 c^2 a \right| &\leq 2k_0 \mu_0 c.\end{aligned}$$

Therefore,

$$\sum_{\alpha \in A} (\partial f)^2(\alpha) \leq 2k_0 \mu_0 \left[c^2 b + 2(bc)^{\frac{3}{2}} + b^2 c \right] = 2k_0 \mu_0 bc \left(\sqrt{b} + \sqrt{c} \right)^2,$$

and repeating this for $\sum_{\beta \in B}$ and $\sum_{\gamma \in C}$ gives

$$\|\partial f\|^2 \leq 2k_0 \mu_0 \left[bc \left(\sqrt{b} + \sqrt{c} \right)^2 + ac \left(\sqrt{a} + \sqrt{c} \right)^2 + ab \left(\sqrt{a} + \sqrt{b} \right)^2 \right] \leq k_0 \mu_0 n^3.$$

Combining everything together,

$$\begin{aligned}\frac{|T(A, B, C)| n^2}{abc} &\geq \frac{\lambda_1}{abc} \left(\|f\|^2 - \|f_B\|^2 \right) \geq \frac{\lambda_1}{abc} \left(\|f\|^2 \left(1 - \frac{\mu_0}{k_0} \right) - \mu_0 n^3 \right) \\ &\geq \frac{\lambda_1}{abc} \left(\left(k_0 abc - \frac{\mu_0 n^3}{9} \right) \left(1 - \frac{\mu_0}{k_0} \right) - \mu_0 n^3 \right) \geq \lambda_1 \left(k_0 - \mu_0 \left(1 + \frac{10n^3}{9abc} \right) \right).\end{aligned}$$

□

This can be generalized to higher dimension. For a partition A_0, \dots, A_d of the vertices of a d -complex, let $F(A_0, \dots, A_d)$ be the set of d -cells with one vertex in each A_i . We then have:

^(†)See e.g. [Par13] for a statement which does not assume regularity.

Theorem 8. *Let X be a d -complex on n vertices, with $\text{Spec } \Delta_i^+|_{Z_i} \subseteq [k_i - \mu_i, k_i + \mu_i]$ for $0 \leq i \leq d-2$ and $\text{Spec } \Delta_{d-1}^+|_{Z_{d-1}} \subseteq [\lambda_{d-1}, \infty)$. Then for any partition $V = \coprod_{i=0}^d A_i$*

$$\frac{|F(A_0, \dots, A_d)| n^d}{|A_0| \cdots |A_d|} \geq k_0 \cdots k_{d-2} \cdot \lambda_{d-1} \left(1 - \frac{\mu_{d-2}}{k_{d-2}} - C_d \left(\frac{\mu_0}{k_0} + \dots + \frac{\mu_{d-2}}{k_{d-2}} \right) \frac{n^{d+1}}{\prod_{i=0}^d |A_i|} \right),$$

where C_d depends only on d .

Thus, if we define

$$h_{\vartheta}(X) = \min \left\{ \frac{|F(A_0, \dots, A_d)| n^d}{|A_0| \cdots |A_d|} \mid \begin{array}{l} V = \coprod_{i=0}^d A_i \\ |A_i| \geq \vartheta n \end{array} \right\}$$

(where $0 \leq \vartheta \leq \frac{1}{d+1}$), then

$$h_{\vartheta}(X) \geq k_0 \cdots k_{d-2} \cdot \lambda_{d-1} \left(1 - \frac{\mu_{d-2}}{k_{d-2}} - \frac{C_d}{\vartheta^{d+1}} \left(\frac{\mu_0}{k_0} + \dots + \frac{\mu_{d-2}}{k_{d-2}} \right) \right).$$

For a complex with a complete skeleton, this reduces again to [PRT12, Thm. 1.2], as $k_i = n$ and $\mu_i = 0$ for $0 \leq i \leq d-2$.

Proof. For $f \in \Omega^{d-1}(X)$ defined by

$$f([\sigma_0 \sigma_1 \dots \sigma_{d-1}]) = \begin{cases} \text{sgn } \pi |A_{\pi(d)}| & \exists \pi \in \text{Sym}_{\{0 \dots d\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \leq i \leq d-1 \\ 0 & \text{else, i.e. } \exists k, i \neq j \text{ with } \sigma_i, \sigma_j \in A_k, \end{cases}$$

it is shown in [PRT12, §4.1] that $\|\delta f\|^2 = |F(A_0, \dots, A_d)| n^2$ (this part does not use the complete skeleton assumption.) For $f_B = \mathbb{P}_{B^{d-1}} f$, this gives again $|F(A_0, \dots, A_d)| n^2 \geq \lambda_{d-1} (\|f\|^2 - \|f_B\|^2)$. Denoting $\mathcal{K} = k_0 \cdots k_{d-2}$ and $\mathcal{E} = \frac{\mu_0}{k_0} + \dots + \frac{\mu_{d-2}}{k_{d-2}}$, we have by [Par13, Thm. 1.1]

$$\begin{aligned} \|f\|^2 &= \sum_{i=0}^d \left| F(A_0, \dots, \widehat{A_i}, \dots, A_d) \right| |A_i|^2 \geq \sum_{i=0}^d \left[\frac{\mathcal{K}}{n^{d-1}} \prod_{j \neq i} |A_j| - c_{d-1} \mathcal{K} \mathcal{E} \max_{j \neq i} |A_j| \right] |A_i|^2 \\ &\geq \frac{\mathcal{K}}{n^{d-2}} \prod_{i=0}^d |A_i| - (d+1) c_{d-1} \mathcal{K} \mathcal{E} n^3 \geq \mathcal{K} \left(n^{2-d} \prod_{i=0}^d |A_i| - (d+1) c_{d-1} \mathcal{E} n^3 \right). \end{aligned}$$

Turning to f_B , we have $\|f_B\|^2 \leq \frac{\mu_{d-2}}{k_{d-2}} \|f\|^2 + \frac{1}{k_{d-2}} \|\partial f\|^2$ in a similar way to the triangle case, and we note that ∂f is supported on $(d-2)$ -cells with vertices in distinct blocks of the partition $\{A_i\}$. For a sequence of sets B_0, \dots, B_ℓ , denote by $F^j(B_0, \dots, B_\ell)$ the set of j -galleries in B_0, \dots, B_ℓ , namely, sequences of j -cells $\tau_i \in F(B_i, \dots, B_{i+j})$ such that

τ_i and τ_{i+1} intersect in a $(j-1)$ -cell. Arguing similarly to (6.5),

$$\begin{aligned} \|\partial f\|^2 &= \sum_{0 \leq i < j \leq d} \sum_{\tau \in F(A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_d)} (\partial f)(\tau)^2 \\ &= \sum_{0 \leq i < j \leq d} |A_j|^2 F^{d-1} \left(A_i, A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_d, A_i \right) \\ &\quad - 2 |A_i| |A_j| F^{d-1} \left(A_i, A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_d, A_j \right) \\ &\quad + |A_i|^2 F^{d-1} \left(A_j, A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_d, A_j \right). \end{aligned} \quad (6.6)$$

Proposition 3.1 in [Par13] estimates of the number of j -galleries in B_0, \dots, B_ℓ when each $j+1$ tuple $B_i, B_{i+1}, \dots, B_{i+j+1}$ consists of disjoint sets, giving

$$\left| F^{d-1} \left(A_i, A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_d, A_i \right) - \frac{\mathcal{K} k_{d-2}}{n^d} |A_i| \prod_{k \neq j} |A_k| \right| \leq c_{d-2,d} \mathcal{K} \mathcal{E} k_{d-2} \max_{k \neq j} |A_k|,$$

and similarly for the other terms in (6.6). Again, the main terms cancel out and the error terms result in $\|\partial f\|^2 \leq 4 \binom{d+1}{2} c_{d-2,d} \mathcal{K} \mathcal{E} k_{d-2} n^3$. In total,

$$\begin{aligned} \frac{|F(A_0, \dots, A_d)| n^d}{|A_0| \dots |A_d|} &\geq \frac{\lambda_{d-1} n^{d-2}}{|A_0| \dots |A_d|} \left(\left(1 - \frac{\mu_{d-2}}{k_{d-2}} \right) \|f\|^2 - \frac{1}{k_{d-2}} \|\partial f\|^2 \right) \\ &\geq \mathcal{K} \lambda_{d-1} \left(\left(1 - \frac{\mu_{d-2}}{k_{d-2}} \right) \left(1 - (d+1) c_{d-1} \mathcal{E} \frac{n^{d+1}}{\prod |A_i|} \right) - 4 \binom{d+1}{2} c_{d-2,d} \mathcal{E} \frac{n^{d+1}}{\prod |A_i|} \right) \\ &\geq \mathcal{K} \lambda_{d-1} \left(1 - \frac{\mu_{d-2}}{k_{d-2}} - \left((d+1) c_{d-1} + 4 \binom{d+1}{2} c_{d-2,d} \right) \frac{\mathcal{E} n^{d+1}}{\prod_{i=0}^d |A_i|} \right) \end{aligned}$$

and the theorem follows. \square

7. PSEUDO-RANDOMNESS

In this section we use not only the lower bound on the edge spectrum, but the fact that it is concentrated in two narrow stripes, to show pseudo-random behavior for 2-galleries. We first state a general theorem, and then specialize to Ramanujan complexes.

Theorem 9. *Let X be a regular tripartite triangle complex with vertex and edge degrees k_0 and k_1 , respectively, such that*

$$\begin{aligned} \text{Spec } \Delta_0^+|_{Z_0} &\subseteq [k_0 - \mu_0, k_0 + \mu_0] \cup \left\{ \frac{3k_0}{2} \right\} \quad \text{and} \\ \text{Spec } \Delta_1^+|_{Z_1} &\subseteq [k_1 - \mu_1, k_1 + \mu_1] \cup [2k_1 - \mu_1, 2k_1 + \mu_1] \cup \{3k_1\}. \end{aligned}$$

If A, B, C, D are disjoint sets of vertices of sizes a, b, c, d , respectively, and each of $A \cup D$, B and C is contained in a different block of the three-partition of X (see Figure 1.1), then

$$\begin{aligned} \left| F^2(A, B, C, D) - \frac{27k_0 k_1^2 abcd}{2n^3} \right| &\leq \frac{6\mu_0 k_1^2 \sqrt{abcd}}{k_0 n} \left(\frac{3k_0 (\sqrt{ab} + \sqrt{cd})}{2n} + \mu_0 \right) \\ &\quad + \left[\frac{2k_1^2 \mu_0}{k_0} + (k_1 + \mu_1) \mu_1 \right] \sqrt[4]{abcd} \sqrt{\left(\frac{3k_0 \sqrt{ab}}{2n} + \mu_0 \right) \left(\frac{3k_0 \sqrt{cd}}{2n} + \mu_0 \right)}. \end{aligned} \quad (7.1)$$

The main term agrees with the pseudo-random expectation: given vertices $\alpha, \beta, \gamma, \delta$ in A, B, C, D respectively, the probability that $\beta\gamma$ is an edge in X is $\frac{3k_0}{2n}$, since there are $\frac{k_0}{2}$ edges connecting β with a vertex from the block containing C , and $\frac{n}{3}$ such vertices. Given that $\beta\gamma$ is in X , there are k_1 triangle containing it and all have their third vertex in the block containing $A \cup D$. The probability that α and δ are two of these is $\frac{k_1(k_1-1)}{n/3(n/3-1)}$, so that

$$\mathbb{E}(|F^2(A, B, C, D)|) = \frac{3k_0}{2n} \cdot \frac{k_1(k_1-1)}{n/3(n/3-1)} \cdot abcd \approx \frac{27k_0k_1^2abcd}{2n^3}$$

If $a, b, c, d \leq \vartheta n$ (where $\vartheta \leq \frac{1}{3}$) then the l.h.s. in (7.1) is bounded by

$$\begin{aligned} \left[\left(\frac{(6\vartheta+2)\mu_0}{k_0} + 18\vartheta^2 + 3\vartheta \right) k_1^2\mu_0 + \mu_1 \left(\frac{3}{2}\vartheta k_0 + \mu_0 \right) (k_1 + \mu_1) \right] \vartheta n \\ \leq \vartheta \left(9\vartheta + \frac{4\mu_0}{k_0} \right) (k_1\mu_0 + k_0\mu_1) k_1 n, \end{aligned}$$

where in the second line we assume also $\mu_1 \leq k_1$.

Ramanujan triangle complexes have $k_0 = 2(q^2 + q + 1)$, $k_1 = q + 1$, $\mu_0 = 6q$ and $\mu_1 = 2\sqrt{q}$, and Theorem 3 is obtained by applying this to (7.1) and using the trivial bounds $q \geq 2$ and $\vartheta \leq \frac{1}{3}$.

During the proof of the theorem we shall need a mixing lemma for colored graphs, which we state now in broader generality: say that a graph is c -colored k -regular if its vertices can be partitioned into disjoint sets V_0, \dots, V_{c-1} so that every $v \in V_i$ has no neighbors in V_i , and precisely $\frac{k}{c-1}$ neighbors in each other V_j . The functions $f_j(V_\ell) \equiv \exp\left(\frac{2\pi i j \ell}{c}\right) / \sqrt{n}$ are orthonormal eigenfunctions of Δ_0^+ with corresponding eigenvalues

$$\lambda_j = \begin{cases} 0 & j = 0 \\ \left(\frac{c}{c-1}\right)k & 0 < j < c, \end{cases} \quad (7.2)$$

and we call $\{\lambda_0, \dots, \lambda_{c-1}\}$ the colored spectrum.

Lemma 10. *If the non-colored spectrum of a c -colored k -regular graph is contained in $[k - \mu, k + \mu]$, and $A \subseteq V_i$, $B \subseteq V_j$ for $i \neq j$, then*

$$\left| |E(A, B)| - \frac{ck|A||B|}{(c-1)n} \right| \leq \mu \sqrt{|A||B|}. \quad (7.3)$$

Proof. This is a straightforward generalization of the standard lemma. Assuming that $A \subseteq V_0$ and $B \subseteq V_1$, and denoting by \mathbb{P}_W the orthogonal projection on $W = \langle f_0, \dots, f_{c-1} \rangle^\perp$, we have

$$\begin{aligned} |E(A, B)| &= \langle (kI - \Delta_0^+) \mathbb{1}_A, \mathbb{1}_B \rangle \\ &= \sum_{j=0}^{c-1} (k - \lambda_j) \langle \mathbb{1}_A, f_j \rangle \langle \mathbb{1}_B, f_j \rangle + \langle (kI - \Delta_0^+) \mathbb{P}_W \mathbb{1}_A, \mathbb{1}_B \rangle \\ &= \frac{|A||B|}{n} \sum_{j=0}^{c-1} (k - \lambda_j) \exp\left(\frac{2\pi i j}{c}\right) + \langle (kI - \Delta_0^+) \mathbb{P}_W \mathbb{1}_A, \mathbb{1}_B \rangle, \end{aligned}$$

and (7.3) follows by (7.2) and $\|(kI - \Delta_0^+) \mathbb{1}_W\| \leq \mu$. \square

Proof of Theorem 9. Denote by U^+ the space spanned by the Δ_1^+ -eigenforms with eigenvalue in $[k_1 - \mu_1, k_1 + \mu_1] \cup [2k_1 - \mu_1, 2k_1 + \mu_1]$, and by η the normalized disorientation

$$\eta(e) = \sqrt{\frac{2}{nk_0}} (-1)^{\text{col } e},$$

so that $\Omega^1(X) = B^1 \oplus U^+ \oplus \langle \eta \rangle$. If $p(x) = (x - k_1)(x - 2k_1)$, then $p(\Delta_1^+)$ acts on $B^1 \oplus \langle \eta \rangle$ as the scalar $2k_1^2$, and

$$\|p(\Delta_1^+) \mid_{U^+}\| \leq \max \left\{ |p(\lambda)| \mid \lambda \in \frac{[k_1 - \mu_1, k_1 + \mu_1] \cup [2k_1 - \mu_1, 2k_1 + \mu_1]}{2} \right\} = (k_1 + \mu_1) \mu_1.$$

Let us express $p(\Delta_1^+)$ in combinatorial terms: we say that two directed edges e, e' are neighbors if they have a common origin or a common terminus, and their union (as an unoriented cell) is in X^2 . We denote this by $e \sim e'$, and by \mathcal{A} the corresponding adjacency operator on $\Omega^1(X)$, namely $(\mathcal{A}f)(e) = \sum_{e' \sim e} f(e')$. The upper Laplacian satisfies

$$\Delta_1^+ = k_1 \cdot I - \mathcal{A},$$

(see [PRT12, Par13]), and it follows that

$$p(\Delta_1^+) = \mathcal{A}^2 + k_1 \mathcal{A}.$$

Define $\mathbb{1}_{AB} \in \Omega^1(X)$ by

$$\mathbb{1}_{AB}(vw) = \begin{cases} 1 & v \in A, w \in B \\ -1 & v \in B, w \in A \\ 0 & \text{otherwise} \end{cases}$$

and similarly $\mathbb{1}_{CD}$. We claim that

$$|F^2(A, B, C, D)| = \langle p(\Delta_1^+) \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle. \quad (7.4)$$

Indeed, edges in $E(A, B)$ have no neighbors in $E(C, D)$, so that $\langle \mathcal{A} \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle = 0$, and \mathcal{A} is self-adjoint (since Δ_1^+ is), giving

$$\begin{aligned} \langle p(\Delta_1^+) \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle &= \langle \mathcal{A}^2 \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle = \langle \mathcal{A} \mathbb{1}_{AB}, \mathcal{A} \mathbb{1}_{CD} \rangle \\ &= \sum_{\substack{e, e', e'' \in X^1 \\ e' \sim e \sim e''}} \mathbb{1}_{AB}(e') \mathbb{1}_{CD}(e''). \end{aligned} \quad (7.5)$$

The nonzero terms in this sum come from edges e which have neighbors $e' \in E(A, B)$ and $e'' \in E(C, D)$, so they must be in one of $E(A, C)$, $E(A, D)$, $E(B, C)$ or $E(C, D)$. In fact, only $E(B, C)$ is possible: $E(A, D)$ is empty as A and D are of the same color; if $e \in E(A, C)$ then $e \sim e''$ implies that $e \cup e''$ is a triangle in $T(A, C, D)$ which is again impossible, and similarly for $E(B, D)$ using e' . Thus, $(e' \cup e, e \cup e'')$ is a 2-gallery in $F^2(A, B, C, D)$, and its contribution to (7.5) is always one: if e goes from B to C , it shares its origin with e' and its terminus with e'' , so that $\mathbb{1}_{AB}(e') = \mathbb{1}_{CD}(e'') = 1$, and if e goes from C to B then $\mathbb{1}_{AB}(e') = \mathbb{1}_{CD}(e'') = -1$. On the other hand, for every gallery $(t, t') \in F^2(A, B, C, D)$, the edges $e' = t \setminus C$, $e = t \cap t'$, $e'' = t' \setminus B$ form such a triplet, and we obtain (7.4).

On the spectral side we have

$$\langle p(\Delta_1^+) \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle = 2k_1^2 \langle \mathbb{P}_{B^1 \oplus \langle \eta \rangle} \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle + \langle p(\Delta_1^+) \mathbb{P}_{U^+} \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle,$$

and the last term is bounded by

$$\|p(\Delta_1^+) \mid_{U^+}\| \|\mathbb{1}_{AB}\| \|\mathbb{1}_{CD}\| \leq (k_1 + \mu_1) \mu_1 \sqrt{E_{AB} E_{CD}}, \quad (7.6)$$

where $E_{AB} = |E(A, B)|$ and $E_{CD} = |E(C, D)|$. As η has constant sign on $V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_0$,

$$2k_1^2 \langle \mathbb{P}_{\langle \eta \rangle} \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle = 2k_1^2 \langle \mathbb{1}_{AB}, \eta \rangle \langle \eta, \mathbb{1}_{CD} \rangle = \frac{4k_1^2 E_{AB} E_{CD}}{k_0 n}, \quad (7.7)$$

and we are left to analyze $\mathbb{P}_{B^1} \mathbb{1}_{AB}$. As in the non-tripartite case, one has $\text{Spec } \Delta_1^-|_{B^1} = \text{Spec } \Delta_+^0|_{B_0}$, but now the latter comprises not only eigenvalues in $[k_0 - \mu_0, k_0 + \mu_0]$, but also $\frac{3k_0}{2}$ (twice, see Theorem 5). If $\omega = \exp\left(\frac{2\pi i}{3}\right)$ and

$$\xi(vw) = \begin{cases} \sqrt{2/k_0 n} & v \in V_0, w \in V_1 & -\sqrt{2/k_0 n} & w \in V_0, v \in V_1 \\ \omega\sqrt{2/k_0 n} & v \in V_1, w \in V_2 & -\omega\sqrt{2/k_0 n} & w \in V_1, v \in V_2 \\ \bar{\omega}\sqrt{2/k_0 n} & v \in V_2, w \in V_0 & -\bar{\omega}\sqrt{2/k_0 n} & w \in V_2, v \in V_0, \end{cases}$$

then $\{\xi, \bar{\xi}\}$ is an orthonormal basis for the $\frac{3k_0}{2}$ -eigenspace of Δ_1^- . Denote by U^- the space spanned by the Δ_1^- -eigenforms with eigenvalue in $[k_0 - \mu_0, k_0 + \mu_0]$. By the action of each summand in

$$\mathfrak{D}' \stackrel{\text{def}}{=} k_0 \mathbb{P}_{B^1} + \frac{k_0}{2} \mathbb{P}_{\langle \xi, \bar{\xi} \rangle} - \Delta_1^-$$

on each of the terms in the orthogonal decomposition $\Omega^1(X) = Z_1 \oplus U^- \oplus \langle \xi, \bar{\xi} \rangle$ we see that $\|\mathfrak{D}'\| \leq \mu_0$. Due to the fact that $\partial_1 \mathbb{1}_{AB}$ and $\partial_1 \mathbb{1}_{CD}$ are supported on different vertices, $\langle \Delta_1^- \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle$ vanishes, and together with

$$\left\langle \mathbb{P}_{\langle \xi, \bar{\xi} \rangle} \mathbb{1}_{AB}, \mathbb{1}_{CD} \right\rangle = 2\Re(\langle \mathbb{1}_{AB}, \xi \rangle \langle \xi, \mathbb{1}_{CD} \rangle) = -\frac{2E_{AB}E_{CD}}{k_0 n}$$

(and $\mathbb{P}_{B^1} = \frac{1}{k_0} \Delta_1^- - \frac{1}{2} \mathbb{P}_{\langle \xi, \bar{\xi} \rangle} + \frac{1}{k_0} \mathfrak{D}'$) this gives

$$\left| 2k_1^2 \langle \mathbb{P}_{B^1} \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle - \frac{2k_1^2 E_{AB} E_{CD}}{k_0 n} \right| \leq \frac{2k_1^2}{k_0} \langle \mathfrak{D}' \mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle \leq \frac{2\mu_0 k_1^2 \sqrt{E_{AB} E_{CD}}}{k_0}.$$

Combining this with (7.6) and (7.7) we conclude that

$$\left| F^2(A, B, C, D) - \frac{6k_1^2 E_{AB} E_{CD}}{k_0 n} \right| \leq \left(\frac{2\mu_0 k_1^2}{k_0} + (k_1 + \mu_1) \mu_1 \right) \sqrt{E_{AB} E_{CD}}.$$

This estimates $|F^2(A, B, C, D)|$ in terms of E_{AB} and E_{CD} . To have an estimate in terms of a, b, c and d we use Lemma 10, which gives $|E_{AB} - \frac{3k_0 ab}{2n}| \leq \mu_0 \sqrt{ab}$ and similarly for E_{CD} , and the theorem follows. \square

As an application of our “2-gallery mixing lemma” we give a lower bound for the weak chromatic number of non-tripartite Ramanujan complexes. This is the minimal number of colors needed to color X^0 so that every triangle has vertices of at least two different colors. For example, the weak chromatic number of a tripartite triangle complex is two, as one can merge two of the blocks in a three-partition into a single color. If $X = \Gamma \backslash \mathcal{B}_3$ is non-tripartite, recall that $\widehat{\Gamma}$ (see (2.4)) is a normal subgroup of Γ of index three, and $\widehat{X} = \widehat{\Gamma} \backslash \mathcal{B}_3$ is a tripartite three-cover of X . Let $V_0 \amalg V_1 \amalg V_2$ be the partition of \widehat{X}^0 according to $\text{col} = \text{ord}_\pi \det$. If the chromatic number of X is χ , we can find a set $N \subseteq X^0$ of size $\frac{n}{\chi}$ with $T(N, N, N) = \emptyset$ (any monochromatic set would satisfy this). Let N_0, N_1, N_2 be the preimages of N in V_0, V_1 and V_2 , respectively, and take $A = N_0$,

$B = N_1$, $C = N_2$ and $D = V_0 \setminus N_0$. Theorem 9 then estimates $|F^2(A, B, C, D)|$ with main term

$$\frac{27k_0k_1^2abcd}{2(3n)^3} \geq \frac{q^4abcd}{n^3} = \frac{q^4n(\chi-1)}{\chi^4} \geq \frac{q^4n}{2\chi^3}$$

and an error term bounded by

$$\left(\frac{84q^{3.5}}{\chi^{1.5}} + \frac{119q^{2.5}}{\sqrt{\chi}} + 139q^{1.5} \right) n.$$

However, $F^2(A, B, C, D)$ must be empty, since any triangle in $T(A, B, C)$ would cover one in $T(N, N, N)$, hence the error term is at least $\frac{q^4n}{2\chi^3}$. This implies $\chi \geq \frac{\sqrt[3]{q}}{30}$: otherwise, $\chi \leq \frac{\sqrt[3]{q}}{30}$ leads to

$$\frac{\sqrt{q}}{2\chi^{1.5}} \leq \left(84 + \frac{119\chi}{q} + \frac{139\chi^{1.5}}{q^2} \right) \leq 87$$

which implies again $\chi \geq \frac{\sqrt[3]{q}}{30}$. This bound is better than $\frac{\sqrt[6]{q}}{2}$ which is obtained in [EGL14] using only the action of Hecke operators on vertices. Note however that the results in [EGL14] do not assume the complex to be Ramanujan, and apply to higher dimensions as well.

Other applications of a triangle mixing lemma can be adjusted, in a similar manner, to use our gallery mixing lemma. This includes, for example, Gromov's overlap property, along the lines of [FGL⁺12, Par13], and the crossing numbers of complexes, as discussed in [GW13, §8.1]. Nevertheless, the question of triangle pseudorandomness remains interesting, and should give better results if it does hold. The fact that most of the spectrum is concentrated in the strip \mathcal{I} (see Theorem 5) gives hope that this can be done by analyzing the combinatorics of eigenforms which occur in the principal series (type (a) in §5), and showing that their contribution is negligible.

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