Advances in Applied Mathematics 49 (2012) 375–390

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Descent pattern avoidance

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ARTICLE INFO

Article history:
Received 12 July 2012
Accepted 12 August 2012
Available online 5 September 2012

MSC:
primary 05A05
secondary 05A16, 45C05

Keywords:
Weighted consecutive pattern avoidance
Integral operators
Spectrum
Asymptotic expansions

ABSTRACT

We extend the notion of consecutive pattern avoidance to considering sums over all permutations where each term is a product of weights depending on each consecutive pattern of a fixed length. We study the problem of finding the asymptotics of these sums. Our technique is to extend the spectral method of Ehrenborg, Kitaev and Perry. When the weight depends on the descent pattern we show how to find the equation determining the spectrum. We give two length 4 applications. First, we find the asymptotics of the number of permutations with no triple ascents and no triple descents. Second, we give the asymptotics of the number of permutations with no isolated ascents or descents. Our next result is a weighted pattern of length 3 where the associated operator only has one non-zero eigenvalue. Using generating functions we show that the error term in the asymptotic expression is the smallest possible.

1. Introduction

Ehrenborg, Kitaev and Perry [3] used the spectrum of linear operators on the space $L^2([0,1]^m)$ to study the asymptotics of consecutive pattern avoidance. We extend their techniques to study asymptotics of sums over all permutations where each term is a product of weights which depend on the consecutive patterns of a fixed length $m+1$. When the weights are all zero or one, this reduces to studying consecutive pattern avoidance. Furthermore, when the weights depend on the descent pattern, we show how to obtain the equation whose roots are the spectrum of the associated linear operator. In general this is a transcendental equation.

We give two length 4 examples. First we study the number of permutations with no triple ascents and no triple descents. This is equivalent to $\{1234, 4321\}$-avoiding permutations. We determine the

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0196-8858/$ – see front matter
http://dx.doi.org/10.1016/j.aam.2012.08.004

transcendental eigenvalue equation and a numerical approximation to the largest root, which gives the asymptotics of the number such permutations.

The second example is permutations that avoid the ten alternating patterns 1324, 1423, 2314, 2413, 3412 and 2143, 3142, 3241, 4132, 4231. This is the class of permutations with no isolated ascents or descents. Yet again, we obtain the transcendental eigenvalue equation satisfied by the spectrum and give numerical approximation to its largest root.

We next turn to a weighted length 3 example. We are interested in the sum over all 123-avoiding permutations where the term is 2 to the power of the number of double descents. Here we also consider the extra conditions if the permutation begins/ends with an ascent or a descent. The associated operator only has one non-zero eigenvalue, namely 1. Hence the asymptotics is a constant c times n factorial and the error term is bounded by $n! \cdot r^n$ where r is an arbitrary small positive number.

It remains to understand the error term. We are able to find the associated generating functions. Furthermore, we show that the error term is the smallest possible! The asymptotics is $c \cdot n!$ (where the constant c is irrational, in fact, transcendental) and the explicit expression is the nearest integer to $c \cdot n!$ for large enough n. This behavior also occurs with the derangement numbers. This classical sequence makes its appearance as one of the sequences that we study.

We end the paper with concluding remarks and open problems.

2. Weighted consecutive pattern avoidance

For $x_1, x_2, \ldots, x_k$ distinct real values, define $\Pi(x_1, x_2, \ldots, x_k)$ to be the unique permutation σ in the symmetric group $\mathfrak{S}_k$ such that $x_i < x_j$ if and only if $\sigma_i < \sigma_j$ for all indices $1 \leqslant i < j \leqslant k$. We say that a permutation π in $\mathfrak{S}_n$ consecutively avoid a permutation σ in $\mathfrak{S}_m$ if there is no index i such that $\Pi(\pi_i, \pi_{i+1}, \ldots, \pi_{i+m-1}) = \sigma$.

Let wt be a real-valued weight function on the symmetric group $\mathfrak{S}_{m+1}$. Similarly, let wt_1, wt_2 be two real-valued weight functions on the symmetric group $\mathfrak{S}_m$. We call wt_1 and wt_2 the initial, respectively, the final weight function. We extend these three weight functions to the symmetric group $\mathfrak{S}_n$ for $n \geqslant m$ by defining

$$\begin{aligned} \mathrm{Wt}(\pi) = {} & \mathrm{wt}_1\big(\Pi(\pi_1, \pi_2, \ldots, \pi_m)\big) \\ & \cdot \prod_{i=1}^{n-m} \mathrm{wt}\big(\Pi(\pi_i, \pi_{i+1}, \ldots, \pi_{i+m})\big) \\ & \cdot \mathrm{wt}_2\big(\Pi(\pi_{n-m+1}, \pi_{n-m+2}, \ldots, \pi_n)\big). \end{aligned}$$

In other words, the weight of a permutation π in $\mathfrak{S}_n$ is the product of the initial weight function wt_1 applied to the m first entries of π with the product of the weight function wt applied to every segment of π of length $m+1$ with the final weight function wt_2 applied to the m last entries of π. The question is what can one say about the quantity

$$\alpha_n = \sum_{\pi \in \mathfrak{S}_n} \mathrm{Wt}(\pi).$$

Consecutive pattern avoidance can be studied this way by using the weight functions $\mathrm{wt}_1(\sigma) = \mathrm{wt}_2(\sigma) = 1$ for all σ in $\mathfrak{S}_m$ and $\mathrm{wt}(\sigma) = 1$ if $\sigma \notin S$ and $\mathrm{wt}(\sigma) = 0$ otherwise, where $S \subseteq \mathfrak{S}_{m+1}$ is the set of forbidden patterns. Observe then that a permutation $\pi \in \mathfrak{S}_n$ avoids the patterns in S if and only if $\mathrm{Wt}(\pi) = 1$. Note that by letting the initial weight function wt_1 and the final weight function wt_2 be $0, 1$-functions, we are studying consecutive pattern avoidance with forbidden initial and final configurations.

The methods of Ehrenborg, Kitaev and Perry [3] to study the asymptotics of consecutive pattern avoidance by considering the spectrum of operators on $L^2([0,1]^m)$ naturally extend to this more general setting of weights on permutations.

Define the function χ on the $(m+1)$-dimensional unit cube $[0,1]^{m+1}$ by $\chi(x) = \mathrm{wt}(\Pi(x))$. Note that χ is undefined on a point with two equal coordinates. However, this situation occurs on a set of measure zero and hence can be ignored. Next define the operator T on the space $L^2([0,1]^m)$ by

$$T\big(f(x_1,\ldots,x_m)\big) = \int_0^1 \chi(t,x_1,\ldots,x_m)\cdot f(t,x_1,\ldots,x_{m-1})\,dt. \tag{2.1}$$

Note that $L^2([0,1]^m)$ is a Hilbert space with the inner product defined by

$$(f,g) = \int_{[0,1]^m} f(x_1,\ldots,x_m)\cdot\overline{g(x_1,\ldots,x_m)}\,dx_1\cdots dx_m.$$

The adjoint operator T^* is defined by the relation $(f, T^*(g)) = (T(f), g)$. For the operator T defined in Eq. (2.1) we have that

$$T^*\big(f(x_1,\ldots,x_m)\big) = \int_0^1 \chi(x_1,\ldots,x_m,u)\cdot f(x_2,\ldots,x_m,u)\,du.$$

Finally, the spectrum of an operator T is all the values λ such that $T - \lambda\cdot I$ is not an invertible operator.

Similarly to the function χ, define the two functions κ and μ on the m-dimensional unit cube $[0,1]^m$ by $\kappa(x) = \mathrm{wt}_1(\Pi(x))$ and $\mu(x) = \mathrm{wt}_2(\Pi(x))$.

Generalizing the main result in [3], we have the following theorem.

Theorem 2.1. *The non-zero spectrum of the associated operator T consists of discrete eigenvalues of finite multiplicity which may accumulate only at* 0. *Furthermore, let r be a positive real number such that there is no eigenvalue of T with modulus r and let $\lambda_1,\ldots,\lambda_k$ be the eigenvalues of T greater in modulus than r. Assume that $\lambda_1,\ldots,\lambda_k$ are simple eigenvalues with associated eigenfunctions φ_i and that the adjoint operator T^* has eigenfunctions ψ_i corresponding the eigenvalues λ_i. Then we have the expansion*

$$\alpha_n/n! = \big(T^{n-m}(\kappa),\mu\big) = \sum_{i=1}^{k} \frac{(\varphi_i,\mu)\cdot(\kappa,\overline{\psi_i})}{(\varphi_i,\overline{\psi_i})}\cdot\lambda_i^{n-m} + O\big(r^n\big). \tag{2.2}$$

The proof is the same as in [3, Section 2.2] and hence omitted.

Theorem 2.1 requires us to determine both the eigenfunction φ and the adjoint eigenfunction ψ for each eigenvalue in order to compute the constant in each term. However, when the weight function has symmetry in the sense described below then the adjoint eigenfunction can be determined from the eigenfunction.

Let J be the involution on $L^2([0,1]^m)$ given by $J(f(x_1,x_2,\ldots,x_m)) = f(1-x_m,\ldots,1-x_2,1-x_1)$. Note that J is a self-adjoint operator on $L^2([0,1]^m)$, that is, $(Jf,g) = (f,Jg)$. Similar to [3, Lemma 4.7] we have that

Lemma 2.2. *Assume that the weight function* wt *is real-valued and satisfies the symmetry*

$$\mathrm{wt}(\sigma) = \mathrm{wt}(m+2-\sigma_{m+1}, m+2-\sigma_m,\ldots,m+2-\sigma_1)$$

for all $\sigma\in\mathfrak{S}_{m+1}$. If φ is an eigenfunction of the operator T with eigenvalue λ then $\psi = J\varphi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ. Furthermore, we have the equality $(f,\overline{\psi}) = (\varphi, Jf)$ for a real-valued function f.

To prove Lemma 2.2, the only part that differs from the proof in [3, Lemma 4.7] is the line $(f,\overline{\psi}) = (f,\overline{J\varphi}) = (f,J\overline{\varphi}) = (Jf,\overline{\varphi}) = (\varphi,\overline{Jf}) = (\varphi,Jf)$.

3. Weighted descent pattern avoidance

We now introduce weighted descent pattern avoidance and the connection with consecutive pattern avoidance. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ define its *descent word* (see for instance [4,8]) to be $u(\pi) = u_1u_2\cdots u_{n-1}$ where $u_i = \mathbf{a}$ if $\pi_i < \pi_{i+1}$ and $u_i = \mathbf{b}$ if $\pi_i > \pi_{i+1}$, that is, an $\mathbf{a}$ at position i encodes that π has an ascent at position i and a $\mathbf{b}$ encodes a descent.

Let wt be a weight function on $\mathbf{ab}$-words of length m, that is, the set $\{\mathbf{a},\mathbf{b}\}^m$. Similarly, let wt_1 and wt_2 be weight functions on $\mathbf{ab}$-words of length $m-1$. We extend this weight function to words of length n greater than $m-1$ by letting

$$\mathrm{Wt}(v_1\cdots v_n) = \mathrm{wt}_1(v_1\cdots v_{m-1})\cdot\prod_{i=1}^{n-m+1}\mathrm{wt}(v_i\cdots v_{i+m-1})\cdot\mathrm{wt}_2(v_{n-m+2}\cdots v_n).$$

Finally, we extend the weight to permutations by letting $\mathrm{Wt}(\pi) = \mathrm{Wt}(u(\pi))$.

Recall that the word x has the word w as a *factor* if we can write $x = v\cdot w\cdot z$, where v and z are also words and the dot denotes concatenation. Let U be a collection of $\mathbf{ab}$-words of length m, that is, U is a subset of $\{\mathbf{a},\mathbf{b}\}^m$. Define $S(U)$ by

$$S(U) = \bigl\{\sigma\in\mathfrak{S}_{m+1}\colon u(\sigma)\in U\bigr\}.$$

It is clear that a permutation π that avoids the descent patterns in U is equivalent to that the permutation avoids the consecutive patterns in $S(U)$. Hence descent pattern avoidance is a special case of consecutive pattern avoidance.

A few examples are in order.

Example 3.1. $m = 1$ and $U = \{\mathbf{b}\}$. There is only one permutation without any descents, namely $12\cdots n$, and hence $\alpha_n = 1$.

Example 3.2. $m = 2$ and $U = \{\mathbf{ab}\}$. This forces the permutation to have no peaks. Hence $\alpha_n = 2^{n-1}$ for $n \geqslant 1$.

Example 3.3. $m = 2$ and $U = \{\mathbf{aa},\mathbf{bb}\}$. This forces the permutation to be alternating. Alternating permutations are enumerated by the Euler numbers, that is, $\alpha_n = 2\cdot E_n$ for $n\geqslant 2$ and $\alpha_n = 1$ for $n\leqslant 1$. See for instance [8, Section 1.6.1] or [3, Example 1.11].

For an $\mathbf{ab}$-word u of length $m-1$ define the descent polytope P_u to be the subset of the unit cube $[0,1]^m$ corresponding to all vectors with descent word u, that is,

$$P_u = \bigl\{(x_1,\ldots,x_m)\in[0,1]^m\colon x_i\leqslant x_{i+1} \text{ if } u_i = \mathbf{a} \text{ and } x_i\geqslant x_{i+1} \text{ if } u_i = \mathbf{b}\bigr\}.$$

Observe that the unit cube $[0,1]^m$ is the union of the 2^{m-1} descent polytopes. Similar to [3, Proposition 4.3 and Corollary 4.4] we have the next proposition. Furthermore, the proof is also similar and hence omitted.

Proposition 3.4. *Let T be the operator associated with a weighted descent pattern avoidance and k an integer such that $0\leqslant k\leqslant m-1$. Let u be an* $\mathbf{ab}$*-word of length $m-1$ and f a function in $L^2([0,1]^m)$. Then the function $T^k(f)$ restricted to the descent polytope P_u only depends on the variables x_1 through x_{m-k}.*

A direct consequence of Proposition 3.4 is that the eigenfunctions have a special form:

Corollary 3.5. *If φ is an eigenfunction of T associated to a non-zero eigenvalue, then the eigenfunction φ restricted to any descent polytope P_u only depends on the variable x_1.*

Let V be the subspace of $L^2([0,1]^m)$ consisting of all functions f such that the restriction $f|_{P_u}$ only depends on the variable x_1 for all words u of length $m-1$. Let f be a function in the subspace V. Then the function $T(f)$ is described as follows. For an **ab**-word u of length $m-2$ and $y \in \{\mathbf{a}, \mathbf{b}\}$ we have

$$T(f)|_{P_{uy}} = \int_0^{x_1} \mathrm{wt}(\mathbf{a}uy) \cdot f(t)|_{P_{\mathbf{a}u}}\, dt + \int_{x_1}^{1} \mathrm{wt}(\mathbf{b}uy) \cdot f(t)|_{P_{\mathbf{b}u}}\, dt. \tag{3.1}$$

In light of Corollary 3.5 to solve the eigenvalue problem for the operator $T : L^2([0,1]^m) \longrightarrow L^2([0,1]^m)$, it is enough to solve the eigenvalue problem for the restricted operator $T|_V : V \longrightarrow V$. The restricted operator is of a particular form, which we describe in the next section.

4. A general operator and its spectrum

Recall that for a square matrix M the exponential matrix of M is defined by the converging power series

$$e^M = \sum_{k \geqslant 0} M^k/k! = I + M + M^2/2 + M^3/3! + \cdots.$$

The general solution of the system of first order linear equations $\frac{d}{dx}\vec{p}(x) = M \cdot \vec{p}(x)$ is given by $\vec{p}(x) = e^{M \cdot x} \cdot \vec{c}$ where $\vec{c}$ is the initial condition $\vec{p}(0)$.

Let $\gamma(M)$ denote the matrix

$$\gamma(M) = \int_0^1 e^{M \cdot t}\, dt,$$

where the integration is entrywise. Observe that

$$M \cdot \gamma(M) = \int_0^1 M \cdot e^{M \cdot t}\, dt = \left[e^{M \cdot t}\right]_0^1 = e^M - I. \tag{4.1}$$

Hence when M is non-singular we can write $\gamma(M) = M^{-1} \cdot (e^M - I)$. Also note that by integrating the power series of $e^{M \cdot t}$ term by term we obtain that

$$\gamma(M) = \sum_{k \geqslant 0} M^k/(k+1)! = I + M/2 + M^2/3! + M^3/4! + \cdots.$$

Lemma 4.1. *The two following indefinite integrals hold*:

$$\int e^{M\cdot t}\,dt = \gamma(M\cdot t)\cdot t + \vec{C},$$

$$\int M\cdot t\cdot e^{M\cdot t}\,dt = t\cdot e^{M\cdot t} - \gamma(M\cdot t)\cdot t + \vec{C}.$$

Proof. The first identity follows by integrating the power series termwise. The second identity follows from integrating the equality $M\cdot t\cdot e^{M\cdot t} + e^{M\cdot t} = \frac{d}{dt}(t\cdot e^{M\cdot t})$. $\square$

Let A and B be two $k\times k$ matrices. Consider the integral operator T defined on vector-valued functions by

$$T\big(\vec{p}(x)\big) = A\cdot\int_0^x \vec{p}(t)\,dt + B\cdot\int_x^1 \vec{p}(t)\,dt, \tag{4.2}$$

where the integration is componentwise.

Observe that the restricted operator described in Eq. (3.1) is of the form (4.2) by letting A and B be matrices indexed by **ab**-words of length $m-1$ and the entries be given by

$$A_{uy,\mathbf{a}u} = \mathrm{wt}(\mathbf{a}uy) \quad \text{and} \quad B_{uy,\mathbf{b}u} = \mathrm{wt}(\mathbf{b}uy)$$

where $y\in\{\mathbf{a},\mathbf{b}\}$ and u is an **ab**-word of length $m-2$, and the remaining entries of the matrices are 0.

The following theorem concerns the eigenvalues and eigenfunctions of the operator in (4.2).

Theorem 4.2. *The non-zero spectrum of the operator T is given by the set of non-zero roots of the equation* $\det(P)=0$, *where the matrix P is given by*

$$P = -\lambda\cdot I + B\cdot\gamma\big((A-B)/\lambda\big), \tag{4.3}$$

and the eigenfunctions are of the form $\vec{p}(x) = e^{(A-B)/\lambda\cdot x}\cdot\vec{c}$, where the vector $\vec{c}$ satisfies the equation $P\cdot\vec{c}=0$.

Proof. Differentiate the eigenfunction equation $\lambda\cdot\vec{p} = T(\vec{p})$ with respect to x to obtain the differential equation

$$\frac{d}{dx}\vec{p}(x) = M\cdot\vec{p}(x),$$

where we let M denote the matrix $1/\lambda\cdot(A-B)$. This equation has the solution

$$\vec{p}(x) = e^{M\cdot x}\cdot\vec{c},$$

where $\vec{c}$ is the initial condition. Substituting the solution for the differential equation back into the eigenfunction equation, we obtain

$$\begin{aligned}
\lambda \cdot e^{M\cdot x} \cdot \vec{c} &= A \cdot \int_0^x e^{M\cdot t} \cdot \vec{c}\, dt + B \cdot \int_x^1 e^{M\cdot t} \cdot \vec{c}\, dt \\
&= A \cdot \big[\gamma(M \cdot t) \cdot t\big]_0^x \cdot \vec{c} + B \cdot \big[\gamma(M \cdot t) \cdot t\big]_x^1 \cdot \vec{c} \\
&= \big((A - B) \cdot \gamma(M \cdot x) \cdot x + B \cdot \gamma(M)\big) \cdot \vec{c} \\
&= \big(\lambda \cdot \big(e^{M\cdot x} - I\big) + B \cdot \gamma(M)\big) \cdot \vec{c}.
\end{aligned}$$

Canceling terms we obtain $P \cdot \vec{c} = 0$. We can only find the non-zero vector $\vec{c}$ if the matrix P is singular, that is, has a zero determinant. $\square$

In the case when $A - B$ is non-singular the condition in Theorem 4.2 can be expressed as

$$\begin{aligned}
0 &= \det(P) \cdot \det(M) \\
&= \det\big(-A + B \cdot e^{(A-B)/\lambda}\big).
\end{aligned}$$

Theorem 4.3. *An eigenvalue λ of the operator T is simple if its associated eigenfunction $\vec{p}(x)$ satisfies the vector identity*

$$B \cdot e^{(A-B)/\lambda} \cdot \vec{p}(0) \neq 0. \tag{4.4}$$

Proof. Assume that the eigenvalue λ is not simple, that is, it satisfies the generalized eigenvalue equation $\lambda \cdot \vec{q} = T(\vec{q}) + \vec{p}$. Differentiate this equation to obtain

$$\lambda \cdot \frac{d}{dx}\vec{q}(x) = (A - B) \cdot \vec{q}(x) + \frac{d}{dx}\vec{p}(x).$$

Again let $M = (A - B)/\lambda$. Multiply both sides with $1/\lambda \cdot e^{-M\cdot x}$ to obtain

$$e^{-M\cdot x} \cdot \frac{d}{dx}\vec{q}(x) - M \cdot e^{-M\cdot x} \cdot \vec{q}(x) = 1/\lambda \cdot e^{-M\cdot x} \cdot \frac{d}{dx}\vec{p}(x).$$

This equation is equivalent to

$$\frac{d}{dx}\big(e^{-M\cdot x} \cdot \vec{q}(x)\big) = 1/\lambda \cdot M \cdot \vec{c}.$$

Hence we have the general solution

$$\vec{q}(x) = 1/\lambda \cdot e^{M\cdot x} \cdot M \cdot \vec{c} \cdot x + e^{M\cdot x} \cdot \vec{d},$$

where $\vec{d}$ is a constant vector. Without loss of generality we can set $\vec{d} = 0$ since we are looking for a particular solution. Inserting the particular solution $1/\lambda \cdot e^{M\cdot x} \cdot M \cdot \vec{c} \cdot x$ into the generalized eigenvalue equation, we obtain

$$\begin{aligned} M \cdot x \cdot e^{M \cdot x} \cdot \vec{c} &= A/\lambda \cdot \int_0^x M \cdot t \cdot e^{M \cdot t} dt \cdot \vec{c} + B/\lambda \cdot \int_x^1 M \cdot t \cdot e^{M \cdot t} dt \cdot \vec{c} + e^{M \cdot x} \cdot \vec{c} \\ &= A/\lambda \cdot \left[t \cdot e^{M \cdot t} - \gamma(M \cdot t) \cdot t\right]_0^x \cdot \vec{c} + B/\lambda \cdot \left[t \cdot e^{M \cdot t} - \gamma(M \cdot t) \cdot t\right]_x^1 \cdot \vec{c} + e^{M \cdot x} \cdot \vec{c} \\ &= M \cdot \left(x \cdot e^{M \cdot x} - \gamma(M \cdot x) \cdot x\right) \cdot \vec{c} + B/\lambda \cdot \left(e^M - \gamma(M)\right) \cdot \vec{c} + e^{M \cdot x} \cdot \vec{c}. \end{aligned}$$

Canceling terms using the identity (4.1) and multiplying by λ we have

$$0 = B \cdot \left(e^M - \gamma(M)\right) \cdot \vec{c} + \lambda \cdot \vec{c}.$$

Adding the equation $P \cdot \vec{c} = 0$ to this identity gives us the conclusion of the theorem. □

5. Two length 4 examples

5.1. No triple ascents, no triple descents

Let us consider the case when we avoid the two words **aaa** and **bbb**. This is equivalent to avoiding the consecutive patterns 1234 and 4321. In this case we have the two matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note the matrix $A - B$ is invertible and diagonalizable. To simplify calculations let

$$\tau = \sqrt{\frac{1 + \sqrt{5}}{2}} \quad \text{and} \quad \sigma = \sqrt{\frac{-1 + \sqrt{5}}{2}}.$$

That is, the four eigenvalues of the matrix $A - B$ are $\pm\sigma$ and $\pm\tau \cdot i$.

Using a computer algebra package as Maple, we obtain that the determinant of the matrix P from Theorem 4.2 expands as

$$\begin{aligned} \frac{20}{\lambda^4} \cdot \det(P) = 8 &+ \left(3 + i + \sqrt{5} \cdot (\tau + \sigma \cdot i)\right) \cdot e^{(\sigma + \tau \cdot i)/\lambda} \\ &+ \left(3 - i + \sqrt{5} \cdot (\tau - \sigma \cdot i)\right) \cdot e^{(\sigma - \tau \cdot i)/\lambda} \\ &+ \left(3 - i + \sqrt{5} \cdot (-\tau + \sigma \cdot i)\right) \cdot e^{(-\sigma + \tau \cdot i)/\lambda} \\ &+ \left(3 + i + \sqrt{5} \cdot (-\tau - \sigma \cdot i)\right) \cdot e^{(-\sigma - \tau \cdot i)/\lambda}. \end{aligned}$$

Thus we obtain

Proposition 5.1. *Let λ_0 be the largest real positive root of the equation*

$$\begin{aligned} -8 &= \left(3 + i + \sqrt{5} \cdot (\tau + \sigma \cdot i)\right) \cdot e^{(\sigma + \tau \cdot i)/\lambda} \\ &\quad + \left(3 - i + \sqrt{5} \cdot (\tau - \sigma \cdot i)\right) \cdot e^{(\sigma - \tau \cdot i)/\lambda} \end{aligned}$$

$$+\left(3-i+\sqrt{5}\cdot(-\tau+\sigma\cdot i)\right)\cdot e^{(-\sigma+\tau\cdot i)/\lambda}$$
$$+\left(3+i+\sqrt{5}\cdot(-\tau-\sigma\cdot i)\right)\cdot e^{(-\sigma-\tau\cdot i)/\lambda}. \tag{5.1}$$

Then λ_0 is the largest eigenvalue (in modulus) of the associated operator T and the asymptotics of the number of permutations without triple ascents and triple descents is given by

$$\alpha_n/n! = c\cdot\lambda_0^{n-3} + O\left(r^n\right),$$

where c and r are two positive constants such that $r < \lambda_0$.

Proof. It remains to show that the eigenvalue λ_0 is simple. Observe that the de Bruijn graph with the two directed edges $\mathbf{aa} \xrightarrow{\mathbf{aaa}} \mathbf{aa}$ and $\mathbf{bb} \xrightarrow{\mathbf{bbb}} \mathbf{bb}$ removed is ergodic. Now the conclusion follows from combining Theorems 1.7 and 4.2 in [3]. $\square$

Solving Eq. (5.1) numerically we obtain the three largest roots:

$$\lambda_0 = 0.9240358576\ldots,$$
$$\lambda_{1,2} = -0.2875224461\ldots \pm 0.4015233122\ldots\cdot i.$$

Hence we have that r is bounded below by $|\lambda_{1,2}| = 0.4938523335\ldots.$

For the eigenvalue $\lambda = 0.9240358576\ldots$ we can solve for the vector $\vec{c}$ and we have

$$\vec{c} = \begin{pmatrix} 0.6536190979\ldots \\ 0.6536190979\ldots \\ 0.3815287011\ldots \\ 0 \end{pmatrix}.$$

Thus we have the eigenfunction $\varphi = e^{(A-B)/\lambda\cdot x}\cdot\vec{c}$ and adjoint eigenfunction $\psi = J\varphi$. Note that when we restrict the adjoint eigenfunction ψ to a descent polytope we obtain a function only depending on the last variable x_3. For these two functions we calculate

$$(\varphi,\mathbf{1}) = (\mathbf{1},\overline{\psi}) = 0.6020376937\ldots,$$
$$(\varphi,\overline{\psi}) = 0.3647767214\ldots.$$

Combining this we have the constant

$$\frac{(\varphi,\mathbf{1})\cdot(\mathbf{1},\overline{\psi})}{(\varphi,\overline{\psi})} = 0.9936198319\ldots.$$

Thus in numerical terms we have that the asymptotics for the number of permutations with no triple ascents and triple descent is given by

$$0.9936198319\ldots\cdot(0.9240358576\ldots)^{n-3}\cdot n!.$$

5.2. Avoiding isolated ascents and descents

We next consider the case when we avoid the two words **aba** and **bab**. This is equivalent to avoiding the ten alternating permutations 1324, 1423, 2314, 2413, 3412 and 2143, 3142, 3241, 4132, 4231. In this case we have the two matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Yet again the matrix $A - B$ is invertible and diagonalizable. The eigenvalues are $\pm\tau$ and $\pm\sigma \cdot i$. Similar to Proposition 5.1 we have:

Proposition 5.2. *Let λ_0 be the largest real positive root of the equation*

$$\begin{aligned} -8 = {} & \bigl(3 - i + \sqrt{5} \cdot (-\tau + \sigma \cdot i)\bigr) \cdot e^{(\tau+\sigma\cdot i)/\lambda} \\ & + \bigl(3 + i + \sqrt{5} \cdot (-\tau - \sigma \cdot i)\bigr) \cdot e^{(\tau-\sigma\cdot i)/\lambda} \\ & + \bigl(3 + i + \sqrt{5} \cdot (\tau + \sigma \cdot i)\bigr) \cdot e^{(-\tau+\sigma\cdot i)/\lambda} \\ & + \bigl(3 - i + \sqrt{5} \cdot (\tau - \sigma \cdot i)\bigr) \cdot e^{(-\tau-\sigma\cdot i)/\lambda}. \end{aligned} \tag{5.2}$$

Then λ_0 is the largest eigenvalue (in modulus) of the associated operator T and the asymptotics of the number of permutations not having any isolated ascents or descents is given by

$$\alpha_n/n! = c \cdot \lambda_0^{n-3} + O\bigl(r^n\bigr),$$

where c and r are two positive constants such that $r < \lambda_0$.

The same argument as in Proposition 5.1 yields that the largest eigenvalue λ is simple. The only difference is that we consider the de Bruijn graph with the two edges $\mathbf{ab} \xrightarrow{\mathbf{aba}} \mathbf{ba}$ and $\mathbf{ba} \xrightarrow{\mathbf{bab}} \mathbf{ab}$ removed.

Numerically, we find the following three largest roots to Eq. (5.2):

$$\lambda_0 = 0.6869765032\ldots,$$

$$\lambda_{1,2} = 0.1559951131\ldots \pm 0.5317098371\ldots \cdot i.$$

The next largest root $\lambda_{1,2}$ bounds r from below by $|\lambda_{1,2}| = 0.5541207686\ldots.$

Similar to Section 5.1 we can obtain the numerical asymptotic expression for the quantity α_n. The numerical data is as follows:

$$\vec{c} = \begin{pmatrix} 0.4315640876\ldots \\ 0 \\ 0.6378684967\ldots \\ 0.6378684967\ldots \end{pmatrix},$$

and

$$(\varphi, \mathbf{1}) = (\mathbf{1}, \overline{\psi}) = 0.2798342976\ldots,$$
$$(\varphi, \overline{\psi}) = 0.0878970625\ldots.$$

Combining this we have the constant

$$\frac{(\varphi, \mathbf{1}) \cdot (\mathbf{1}, \overline{\psi})}{(\varphi, \overline{\psi})} = 0.8908970548\ldots.$$

Finally, we conclude that the asymptotics for the number of permutations with no isolated ascents and no isolated descents is given by

$$0.8908970548\ldots \cdot (0.6869765032\ldots)^{n-3} \cdot n!.$$

6. A weighted example of length 3

Define a weight function on the set of **ab**-words of length 2 such that $\mathrm{wt}(\mathbf{aa}) = 0$, $\mathrm{wt}(\mathbf{bb}) = 2$ and $\mathrm{wt}(\mathbf{ab}) = \mathrm{wt}(\mathbf{ba}) = 1$ and the initial and final weight functions wt_1 and wt_2 are identical to 1. We are interested in understanding the sum

$$\alpha_n = \sum_{\pi \in \mathfrak{S}_n} \mathrm{Wt}(\pi).$$

A more explicit way to write this sum is as follows

$$\alpha_n = \sum_{\pi} 2^{\mathbf{bb}(\pi)},$$

where the sum is over all 123-avoiding permutations of length n and $\mathbf{bb}(\pi)$ denotes the number of double descents of π.

Let us refine the number α_n by considering if the permutation begins with an ascent or a descent, and similarly how the permutation ends, that is, we define $\alpha_n(\mathbf{a}, \mathbf{a})$, $\alpha_n(\mathbf{a}, \mathbf{b})$, $\alpha_n(\mathbf{b}, \mathbf{a})$ and $\alpha_n(\mathbf{b}, \mathbf{b})$ for $n \geqslant 2$ by

$$\alpha_n(x, y) = \sum \mathrm{Wt}(\pi),$$

where the sum is over all permutations π in $\mathfrak{S}_n$ whose descent word $u(\pi)$ begins with the letter x and ends with the letter y. Note that $\alpha_2(x, y)$ is given by the Kronecker delta $\delta_{x,y}$. These quantities can also be expressed by changing the initial and final weight functions.

By the symmetry $\pi_1, \pi_2, \ldots, \pi_n \longmapsto n+1-\pi_n, \ldots, n+1-\pi_2, n+1-\pi_1$ we have that $\alpha_n(\mathbf{a}, \mathbf{b}) = \alpha_n(\mathbf{b}, \mathbf{a})$.

First we consider the spectrum of the associated operator.

Theorem 6.1. *The only non-zero eigenvalue of the operator T is $\lambda = 1$. This is a simple eigenvalue. Furthermore, the eigenfunction φ and the adjoint eigenfunction ψ associated with this eigenvalue are given by*

$$\varphi = e^{-x} \cdot \begin{cases} 1 - x & \text{if } 0 \leqslant x \leqslant y \leqslant 1, \\ 2 - x & \text{if } 0 \leqslant y \leqslant x \leqslant 1, \end{cases} \quad \text{and} \quad \psi = e^{y-1} \cdot \begin{cases} y & \text{if } 0 \leqslant x \leqslant y \leqslant 1, \\ y + 1 & \text{if } 0 \leqslant y \leqslant x \leqslant 1. \end{cases}$$

Proof. The associated operator T can be written in the form (4.2) using the matrices

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Note that $A - B$ has eigenvalue -1 of algebraic multiplicity 2, but geometric multiplicity 1, that is, the Jordan form of $A - B$ consists of one Jordan block of size 2. Computing the matrix P we obtain

$$0 = \det(P) = \exp(-1/\lambda) \cdot \lambda \cdot (\lambda - 1),$$

which only has the non-zero root $\lambda = 1$. Furthermore for this root, the null space of the matrix P is spanned by the vector

$$\vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Finally, it is straightforward to verify $B \cdot e^M \cdot \vec{c} \neq \vec{0}$, hence $\lambda = 1$ is a simple eigenvalue by Theorem 4.3. Moreover the eigenfunction φ is given by

$$\varphi = \exp\left(\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \cdot x\right) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{-x} \cdot \begin{pmatrix} 1 - x \\ 2 - x \end{pmatrix}.$$

Since the weight function wt satisfies the symmetry in Lemma 2.2, we obtain that the adjoint eigenfunction is given by $\psi = J(\varphi)$. $\square$

Theorem 6.2. *The asymptotics of the sequences $\alpha_n(\mathbf{a}, \mathbf{a})$, $\alpha_n(\mathbf{a}, \mathbf{b})$, $\alpha_n(\mathbf{b}, \mathbf{b})$ and α_n are given by*

$$\begin{aligned}
\alpha_n(\mathbf{a}, \mathbf{a})/n! &= e - 4 + 4/e + O\left(r^n\right), \\
\alpha_n(\mathbf{a}, \mathbf{b})/n! &= 1 - 2/e + O\left(r^n\right), \\
\alpha_n(\mathbf{b}, \mathbf{b})/n! &= 1/e + O\left(r^n\right), \\
\alpha_n/n! &= e - 2 + 1/e + O\left(r^n\right),
\end{aligned}$$

where r is an arbitrary small positive real number.

Proof. Let $\mathbf{1_a}$ denote the function encoding an ascent, that is, $\mathbf{1_a}(x, y) = 1$ if $x < y$ and 0 otherwise. Similarly, let $\mathbf{1_b}$ be the function encoding a descent, that is, $\mathbf{1_b}(x, y) = 1$ if $x > y$ and 0 otherwise. Note that we have that $J\mathbf{1_a} = \mathbf{1_a}$ and $J\mathbf{1_b} = \mathbf{1_b}$. By letting the initial function κ and the final function μ vary over the two functions $\mathbf{1_a}$ and $\mathbf{1_b}$, we obtain the constant term in the asymptotic expression in Theorem 2.1. First we compute the inner products

$$\begin{aligned}
(\varphi, \mathbf{1_a}) = (\mathbf{1_a}, \overline{\psi}) &= 1 - 2 \cdot 1/e, \\
(\varphi, \mathbf{1_b}) = (\mathbf{1_b}, \overline{\psi}) &= 1/e, \\
(\varphi, \overline{\psi}) &= 1/e,
\end{aligned}$$

where we used Lemma 2.2 for two of the five equalities. Hence the constants are:

$$\frac{(\varphi, \mathbf{1_a}) \cdot (\mathbf{1_a}, \overline{\psi})}{(\varphi, \overline{\psi})} = e - 4 + 4 \cdot 1/e,$$
$$\frac{(\varphi, \mathbf{1_b}) \cdot (\mathbf{1_a}, \overline{\psi})}{(\varphi, \overline{\psi})} = 1 - 2 \cdot 1/e,$$
$$\frac{(\varphi, \mathbf{1_b}) \cdot (\mathbf{1_b}, \overline{\psi})}{(\varphi, \overline{\psi})} = 1/e.$$

This proves the three first results of the theorem. The fourth result is obtained by adding the asymptotic expressions for $\alpha_n(\mathbf{a}, \mathbf{a})$, $\alpha_n(\mathbf{a}, \mathbf{b})$, $\alpha_n(\mathbf{b}, \mathbf{a})$ and $\alpha_n(\mathbf{b}, \mathbf{b})$. □

In order to study these sequences further, we introduce the associated exponential generating functions. Let $F_{x,y}(z)$ denote the generating function

$$F_{x,y}(z) = \sum_{n \geqslant 2} \alpha_n(x, y) \cdot \frac{z^n}{n!}.$$

Similarly, let $F(z)$ be the generating function for the sequence α_n.

Proposition 6.3. *The generating function $F_{x,y}(z)$ satisfies the following equation*:

$$\begin{aligned} F_{x,y}(z) = {} & \delta_{x,y} \cdot \frac{z^2}{2!} + \delta_{x,\mathbf{b}} \cdot \delta_{y,\mathbf{a}} \cdot 2 \cdot \frac{z^3}{3!} \\ & + \int_0^z \big(F_{x,\mathbf{a}}(w) + 2 \cdot F_{x,\mathbf{b}}(w)\big) \cdot F_{\mathbf{b},y}(w)\, dw \\ & + \delta_{x,\mathbf{a}} \cdot \int_0^z F_{\mathbf{b},y}(w)\, dw \\ & + \delta_{x,\mathbf{b}} \cdot \int_0^z w \cdot F_{\mathbf{b},y}(w)\, dw \\ & + \delta_{y,\mathbf{b}} \cdot \int_0^z \big(F_{x,\mathbf{a}}(w) + 2 \cdot F_{x,\mathbf{b}}(w)\big)\, dw \\ & + \delta_{y,\mathbf{a}} \cdot \int_0^z \big(F_{x,\mathbf{a}}(w) + 2 \cdot F_{x,\mathbf{b}}(w)\big) \cdot w\, dw. \end{aligned} \tag{6.1}$$

Proof. We demonstrate that all the terms on the right-hand side are in fact counting permutations. The first term corresponds to permutations of length 2. The second term corresponds to permutations of length 3 with the element 1 in the middle position, that is, the two permutations 213 and 312.

For the remaining permutations we break a permutation at the position where the element 1 occurs. We obtain two smaller permutations σ and τ of lengths k, respectively, r, where $k + r = n - 1$. The elements are distributed in $\binom{n-1}{k}$ ways between these two permutations. This is encoded by multiplication of exponential generating functions. Finally, the integral shifts the coefficient from $w^{n-1}/(n-1)!$ to $z^n/n!$.

We continue to describe the terms. The third term corresponds to $2 \leqslant k, r$, that is, at least two elements precede the element 1 and at least two elements follow the element 1. Note that τ must begin with a descent to avoid creating a double ascent. Also when σ ends with a descent, we create a double descent when concatenating σ with the element 1. This explains the factor 2 in front of the term $F_{x,b}$.

The fourth term corresponds to $k = 0$ and $r \geqslant 2$. The Kronecker delta states that the permutation starts with an ascent. The fifth term corresponds to $k = 1$ and $r \geqslant 2$, in which the permutation starts with a consecutive descent and ascent. Similarly, the sixth and seventh terms correspond to the two cases $r = 0$ and $k \geqslant 2$, respectively, $r = 1$ and $k \geqslant 2$.

Since each permutation has been accounted for, the equality holds. $\square$

Note that Proposition 6.3 is similar in spirit to the equations obtained by Elizalde and Noy [5] for the generating functions for certain classes of pattern avoidance permutations.

Theorem 6.4. *The generating functions $F_{x,y}(z)$ and $F(z)$ are given by*

$$\begin{aligned} F_{\mathbf{a},\mathbf{a}}(z) &= \frac{1}{1-z} \cdot \left(e^{z} - 4 + 4 \cdot e^{-z}\right) - 1 + 2 \cdot z, \\ F_{\mathbf{a},\mathbf{b}}(z) &= \frac{1}{1-z} \cdot \left(1 - 2 \cdot e^{-z}\right) + 1 - z, \\ F_{\mathbf{b},\mathbf{b}}(z) &= \frac{1}{1-z} \cdot e^{-z} - 1, \\ F(z) &= \frac{1}{1-z} \cdot \left(e^{z} - 2 + e^{-z}\right). \end{aligned}$$

Proof. Proposition 6.3 can be viewed as a recursion for the coefficient $\alpha_n(x, y)$. Hence the equation in this proposition has a unique solution and it is enough to verify the theorem by showing that the proposed generating functions satisfy Eq. (6.1).

Finally, the generating function $F(z)$ is obtained by adding the four generating functions $F_{\mathbf{a},\mathbf{a}}(z)$, $F_{\mathbf{a},\mathbf{b}}(z)$, $F_{\mathbf{b},\mathbf{a}}(z)$ and $F_{\mathbf{b},\mathbf{b}}(z)$. $\square$

Since $e^{-z}/(1 - z)$ is the generating function for the number of derangements, we obtain

Corollary 6.5. *For $n \geqslant 2$, the number of derangements on n elements, D_n, is given by $\alpha_n(\mathbf{b}, \mathbf{b})$, that is,*

$$D_n = \sum_{\pi} 2^{\mathbf{bb}(\pi)},$$

where the sum is over all permutations π on n elements with no double ascents and starting and ending with a descent.

Muldoon Brown and Readdy [7, Theorem 6.4] have essentially given a bijective proof of this corollary. For more details see [6, Section 3.7].

As a corollary to Theorem 6.4 we have the following recursions:

Corollary 6.6. *Recursions for the sequences $\alpha_n(\mathbf{a}, \mathbf{a})$, $\alpha_n(\mathbf{a}, \mathbf{b})$, $\alpha_n(\mathbf{b}, \mathbf{b})$ and α_n are given by, where $n \geqslant 3$,*

$$\begin{aligned} \alpha_n(\mathbf{a}, \mathbf{a}) &= n \cdot \alpha_{n-1}(\mathbf{a}, \mathbf{a}) + 1 + 4 \cdot (-1)^n, \\ \alpha_n(\mathbf{a}, \mathbf{b}) &= n \cdot \alpha_{n-1}(\mathbf{a}, \mathbf{b}) - 2 \cdot (-1)^n, \end{aligned}$$

$$\alpha_n(\mathbf{b},\mathbf{b}) = n\cdot\alpha_{n-1}(\mathbf{b},\mathbf{b}) + (-1)^n,$$
$$\alpha_n = n\cdot\alpha_{n-1} + 1 + (-1)^n.$$

Using the generating functions in Theorem 6.4 we now obtain that the error terms are the smallest possible. We express the result as explicit expressions using the nearest integer function, which we denote by $\lfloor x\rceil$.

Theorem 6.7. *The quantities $\alpha_n(\mathbf{a},\mathbf{a})$, $\alpha_n(\mathbf{a},\mathbf{b})$, $\alpha_n(\mathbf{b},\mathbf{b})$ and α_n are given by the explicit expressions*

$$\begin{aligned}
\alpha_n(\mathbf{a},\mathbf{a}) &= \big\lfloor (e-4+4/e)\cdot n!\big\rceil \quad \textit{for } n\geqslant 8,\\
\alpha_n(\mathbf{a},\mathbf{b}) &= \big\lfloor (1-2/e)\cdot n!\big\rceil \quad \textit{for } n\geqslant 3,\\
\alpha_n(\mathbf{b},\mathbf{b}) &= \lfloor 1/e\cdot n!\rceil \quad \textit{for } n\geqslant 2,\\
\alpha_n &= \big\lfloor (e-2+1/e)\cdot n!\big\rceil \quad \textit{for } n\geqslant 4.
\end{aligned}$$

Proof. The third equality is classical. We show the first equality. The coefficient of $z^n/n!$ in the generating function $F_{\mathbf{a},\mathbf{a}}(z)$, for $n\geqslant 2$, is given by

$$\alpha_n(\mathbf{a},\mathbf{a}) = n!\cdot\sum_{k=0}^{n}\frac{1^k - 4\cdot 0^k + 4\cdot(-1)^k}{k!}.$$

Hence the difference

$$n!\cdot(e-4+4/e) - \alpha_n(\mathbf{a},\mathbf{a}) = n!\cdot\sum_{k\geqslant n+1}\frac{1^k+4\cdot(-1)^k}{k!},$$

is bounded in absolute value by

$$n!\cdot\sum_{k\geqslant n+1}\frac{5}{k!} = \frac{5}{n+1} + \frac{5}{(n+1)\cdot(n+2)} + \cdots.$$

Note that this is a decreasing function in n. For $n=10$ this function dips below $1/2$, showing the first equality for $n\geqslant 10$. The two cases $n=8,9$ can be done by hand. The second and fourth equalities follow by similar arguments. □

7. Concluding remarks

Are there other operators of the form (2.1) which only have a finite number of non-zero eigenvalues? Furthermore, if the associated sequences are integer sequences would the corresponding error term be the smallest possible, as in Theorem 6.7?

The operators of the form (2.1) have so far yielded four types of behavior:

(i) The operator has an infinite number of eigenvalues and the asymptotic expansion converges. An example of this is alternating permutations. See [3, Example 1.11] and [4]. Another example is $\{123, 231, 312\}$-avoiding permutations. See [3, Section 7].

(ii) The operator has an infinite number of eigenvalues and the asymptotic expansion does not give an expression that converges. This occurs with 123-avoiding permutations and 213-avoiding permutations. See [3, Sections 5 and 6].

(iii) The operator has a finite, but positive, number of non-zero eigenvalues. See Section 6. For instance, is there such an operator with exactly two non-zero eigenvalues? What behavior does the error term of the asymptotic expansion have? Are there other examples with the smallest possible error term?
(iv) The operator has no non-zero eigenvalues. Here the behavior can vary a lot. Compare **ba**-avoiding permutations in Example 3.2 with $\{312, 321\}$-avoiding permutations in [2]. Also see [3, Example 3.9].

The two Eqs. (5.1) and (5.2) in Section 5 have an interesting pattern in their roots. Consider the two equations in terms of the variable $z = 1/\lambda$. Then the roots lie on the real axis and close to a vertical line in the complex plane. Is there an explanation for this behavior? Switching back to the variable λ it says that the roots lie on the real axis and close to a circle in the complex plane.

Baxter, Nakamura and Zeilberger [1] have developed efficient methods to compute the number of permutations avoiding certain patterns. Their methods use umbral techniques and have been implemented in Maple. Their techniques can be extended to compute the weighted problem introduced in this paper.

Acknowledgments

The authors thank Margaret Readdy and the two referees for their comments on an earlier draft of this paper. The authors are partially funded by the National Science Foundation grant DMS-0902063. The first author also thanks the Institute for Advanced Study and is also partially supported by the National Science Foundation grants DMS-0835373 and CCF-0832797.

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