SHELAH'S REVISED GCH THEOREM AND A QUESTION BY ALON ON INFINITE GRAPHS COLORINGS

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Abstract. For every graph G, the coloring number of G does not exceed the least strong limit cardinal above the graph's list-chromatic number.

1. introduction

Let $G = (V, E)$ always denote an undirected, simple graph. Two well-known variations on the chromatic number $\chi(G)$ are the *list-chromatic number* $\chi_\ell(G)$ and the *coloring-number* $\mathrm{Col}(G)$. For every G,

$$\chi(G) \le \chi_\ell(G) \le \mathrm{Col}(G).$$

For every cardinal λ the list-chromatic number of the complete bipartite graph $K(\lambda, \lambda^\lambda)$ is larger than λ, hence no function in $\chi(G)$ bounds $\chi_\ell(G)$. Alon proved that for every finite G,

$$\mathrm{Col}(G) \le (4 + o(1))^{\chi_\ell(G)} \tag{1}$$

and that this bound is tight up to a constant (see [2]).

Alon's question was whether the coloring number of an infinite graph is absolutely bounded by some function of the list-chromatic number. A theorem of Erdős and Hajnal [3] implies that the double successor is such a function if the additional axiom of the GCH is assumed.

Below it is proved that for every graph G,

$$\mathrm{Col}(G) \le \beth_\omega(\chi_\ell(G)). \tag{2}$$

The function $\beth_\omega(\lambda)$ is the ω-th iterated exponent, defined by $\beth_0(\lambda) = \lambda$, $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$ and $\beth_\omega(\lambda) = \lim_{n\in\omega} \beth_n(\lambda)$. We do not know if this bound is tight. Komjáth [7] proved that for every infinite cardinal λ it is consistent that $2^{\aleph_0} = \lambda^+$ and $\chi_\ell(K(\lambda, \lambda)) = \aleph_0$, hence, as $\mathrm{Col}(K(\lambda, \lambda)) = \lambda$, at least one exponent in $\chi_\ell(G)$ is required to bound $\mathrm{Col}(G)$ for all infinite G. Below we show that a cardinal arithmetic assumption considerably weaker than the SCH implies

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$$\mathrm{Col}(G) \le \max\{2^{\chi_\ell(G)}, (\chi_\ell(G))^{++}\}. \tag{3}$$

1.1. **Notation.** The *coloring number* $\mathrm{Col}(G)$ of a graph $G = (V, E)$ is the least cardinal κ for which there exists a well-ordering $\prec$ of V with the property that for every vertex $v \in V$ the set $G_\prec[v] := \{u : \{u, v\} \in E \text{ and } u \prec v\}$ satisfies $|G_\prec[v]| < \kappa$. The *list-chromatic number* of G is the least cardinal κ such that for every assignment of lists of colors $L(v)$ to all vertices of G such that $|L(v)| \le \kappa$ there is a proper coloring c of G such that $c(v) \in L(v)$ for all $v \in V$.

For a set A and cardinal κ we denote by $[A]^\kappa$ the collection of all κ-subsets of A, that is, $[A]^\kappa = \{X : X \subseteq A \text{ and } |X| = \kappa\}$. If $\kappa \le \lambda$ are cardinals, then the cardinality of $[\lambda]^\kappa$ is λ^κ.

2. PROOFS

Theorem 2.1. *For every graph G with $\nu := \chi_\ell(G)$ it holds that*

$$\mathrm{Col}(G) \le \beth_\omega(\nu).$$

For every cardinal ν the list chromatic number of $K(\nu, \nu^\nu)$ exceeds ν by a direct counting argument (see [7]), hence a graph with list-chromatic number ν omits $K(\nu, \nu^\nu)$. Therefore, in the case of finite $\nu = n$, Theorem 2.1 follows from the case $\rho = \aleph_0$ in:

Theorem 2.2 (Erdős and Hajnal [3])**.** *For every natural number $n > 0$ and every $\rho \ge \aleph_0$, every $K(n, \rho^+)$-free graph has coloring number at most ρ.*

Erdős and Hajnal proved in the same paper, assuming the GCH, that for every infinite ν and $\rho \ge \nu^{++}$, every $K(\nu, \rho^+)$-free graph has coloring number at most ρ, so if one assumes the GCH then every graph with infinite $\chi_\ell(G) = \nu$ satisfies $\mathrm{Col}(G) \le \nu^{++}$. Without additional assumptions the following holds:

Theorem 2.3. *Suppose ν is infinite and $\rho \ge \beth_\omega(\nu)$. Every $K(\nu, \rho^+)$-free graph has coloring number at most ρ.*

Theorem 2.3 finishes the proof of Theorem 2.1. It will be proved by induction on $|V|$ after some preparation.

Definition 2.4. *Let $G = (V, E)$ be a graph.*

1. *Let $F : \mathcal{P}(V) \to \mathcal{P}(V)$ be defined by*
$$F(X) = \bigcap_{v \in X} G[v]$$
2. *For a cardinal κ let $F_\kappa = F \restriction [V]^\kappa$.*
3. *Observe that F and hence also F_κ, for each κ, is* anti-monotone*, that is*
$$X \subseteq Y \implies F(Y) \subseteq F(X).$$

(4) *A set $B \subseteq V$ is* κ-closed *for a cardinal κ if $F(X) \subseteq B$ for every $X \in [B]^\kappa$. If B is κ-closed it is κ'-closed for all $\kappa' \geq \kappa$ by anti-monotonicity of F.*

(5) *The* κ-closure *of $A \subseteq V$, denoted $\mathrm{cl}_\kappa A$, is the intersection of all κ-closed subsets of V which contain A and is itself κ-closed. If A is κ-closed it is κ'-closed for all $\kappa' \geq \kappa$.*

Fact 2.5. (1) *If $B \subseteq V$ is κ-closed then $|G[v] \cap B| < \kappa$ for all $v \in V \setminus B$.*

(2) *If $\langle B_i : i < \theta \rangle$ is a $\subseteq$-increasing sequence of κ-closed sets and $\mathrm{cf}\theta \neq \mathrm{cf}\kappa$, then $\bigcup_{i<\theta} B_i$ is κ-closed.*

(3) *If $\langle B_i : i < \theta \rangle$ is a $\subseteq$-increasing sequence of κ-closed sets and $\mathrm{cf}\theta = \kappa$ then $|G[v] \cap \bigcup_{i<\theta} B_i| \leq \kappa$ for all $v \in V \setminus \bigcup_{i<\theta} B_i$.*

Proof. The first clause is clear. For the second: replacing θ by its cofinality and the sequence by a cofinal subsequence we may assume θ is regular. Let $X \in [\bigcup_{i<\theta} B_i]^\kappa$ be given and assume that $|X \cap B_i| < \kappa$ for all i. Since $|X| = \kappa$ the case $\theta < \mathrm{cf}\kappa$ leads to a contradiction, hence $\theta \geq \mathrm{cf}\kappa$. If $\theta > \mathrm{cf}\kappa$ then $|X \cap B_i|$ stabilizes below κ, which again leads to a contradiction if $\theta < \kappa$. If $\theta > \kappa$ then $X \cap B_i$ stabilizes in $\subseteq$ and contradiction follows. We conclude, then, that $|X \cap B_i| = \kappa$ for some $i < \theta$. By anti-monotonicity $F(X) \subseteq F(X \cap B_i)$ and as B_i is κ-closed $F(X \cap B_i) \subseteq B_i$.

The last clause follows from the first. □

Definition 2.6. *A cardinal θ is* κ-stable *for a graph $G = (V, E)$ if for every set $A \in [V]^\theta$ the κ-closure of A has cardinality θ. If θ is κ-stable for G it is also κ'-stable for G for all $\kappa' \geq \kappa$ by Definition 2.4 (4).*

Definition 2.7 (Shelah [9])**.** *For cardinals κ, λ the* revised exponent $\lambda^{[k]}$ *is the least cardinality of a collection $\mathcal{A} \subseteq [\lambda]^\kappa$ with the property that for each $X \in [\lambda]^\kappa$ there exists $\mathcal{Y} \subseteq \mathcal{A}$ such that $|\mathcal{Y}| < \kappa$ and $X \subseteq \bigcup \mathcal{Y}$.*

Lemma 2.8. *Suppose that $\kappa = \mathrm{cf}\kappa$ and $2^\kappa \leq \lambda$. If $\lambda^{[\kappa]} = \lambda$ then λ is κ-stable for every $K(\kappa, \lambda^+)$-free graph G.*

Proof. Suppose that $A \in [V]^\lambda$ and fix $\mathcal{A} \subseteq [A]^\kappa$ of cardinality λ as guaranteed by $|A| = \lambda = \lambda^{[\kappa]}$. Given $X \in [A]^\kappa$ find $\mathcal{Y} \in [\mathcal{A}]^{<\kappa}$ such that $X \subseteq \bigcup \mathcal{Y}$. As κ is regular, there exists some $Z \in \mathcal{Y}$ such that $|X \cap Z| = \kappa$. Now $F(X) \subseteq F(X \cap Z)$ by anti-monotonicity of F.

This proves that for every $X \in [A]^\kappa$ there is some $Z \in \mathcal{A}$ and $W \in [Z]^\kappa$ such that $F(X) \subseteq F(W)$, that is:

$$\bigcup_{X \in [A]^\kappa} F(X) = \bigcup_{Z \in \mathcal{A}} \bigcup_{W \in [Z]^\kappa} F(W). \tag{4}$$

The total number of κ-subsets $X \in [A]^\kappa$ on the left hand side may be larger than λ, but $|\mathcal{A}| = \lambda^{[\kappa]} = \lambda$ and $|[Z]^\kappa| = 2^\kappa \leq \lambda$ for each $Z \in \mathcal{A}$, so the union on the right hand side is taken over $\lambda \times 2^\kappa = \lambda$ sets of the form $F(W)$.

As G is $K(\kappa,\lambda^+)$-free, for every $W\in[V]^\kappa$ it holds that $|F(W)|\le\lambda$ so we get that

$$\Big|\bigcup_{X\in[A]^\kappa} F(x)\Big|\le\lambda. \tag{5}$$

Given $A\in[V]^\lambda$ let $A_0=A$, $A_{i+1}=A_i\cup\bigcup_{X\in[A_i]^\kappa}F(X)$ and for limit i let $A_i=\bigcup_{j<i}A_j$. The set A_{κ^+} contains A and is κ-closed since for every $X\in[\bigcup_{i<\kappa^+}A_i]^\kappa$ there is some $i<\kappa^+$ for which $X\in[A_i]^\kappa$, so $F(X)\subseteq A_{i+1}$. Using (5) and $\kappa^+\le\lambda$ it follows by induction on $i<\kappa^+$ the set $|A_i|\le\lambda$ for all $i\le\kappa^+$, so we are done. □

Recall that by the König-Zermelo Lemma, for every ordinal α, $(\aleph_{\alpha+\omega})^\kappa>\aleph_{\alpha+\omega}$ for every cardinal $\aleph_\alpha$ and infinite κ, or, in other words, arbitrarily large cardinals $\aleph_\alpha$ are strictly smaller than their κ-th power. In contrast to this:

Theorem 2.9 (Shelah's Revised GCH in ZFC theorem [9], see also [10])**.** *For every infinite cardinal ν, for every $\lambda\ge\beth_\omega(\nu)$ there is some $\kappa(\lambda)<\beth_\omega(\nu)$ such that $\lambda^{[\kappa]}=\lambda$ for all κ such that $\kappa(\lambda)\le\kappa=\mathrm{cf}\kappa<\beth_\omega(\nu)$.*

Though the equation $\lambda^{[\kappa]}=\lambda$ does not hold for any fixed $\kappa<\beth_\omega(\nu)$ with *every* $\lambda\ge\beth_\omega(\nu)$, for every $\lambda\ge\beth_\omega(\nu)$ it holds with almost all regular $\kappa<\beth_\omega(\nu)$.

Claim 2.10. *For every infinite cardinal ν, for every $\lambda\ge\beth_\omega(\nu)$ there exists $\kappa(\lambda)<\beth_\omega(\nu)$ such that λ is $\kappa(\lambda)$-stable for G for every $K(\nu,\beth_\omega(\nu)^+)$-free graph G.*

Proof. Let $\lambda\ge\beth_\omega(\nu)$ be given. By Shelah's revised GCH theorem, $\lambda^{[\kappa]}=\lambda$ for all sufficiently large regular $\kappa<\beth_\omega(\nu)$. Fix, then, $\kappa(\lambda)=\mathrm{cf}\kappa(\lambda)<\beth_\omega(\nu)$ for which $\lambda^{[\kappa(\lambda)]}=\lambda$. If G is any $K(\nu,\beth_\omega(\nu)^+)$-free graph it is also $K(\kappa(\lambda),\lambda^+)$-free. It now follows from Lemma 2.8 that λ is $\kappa(\lambda)$-stable for G. □

Proof of Theorem 2.3. Let now $\nu\ge\aleph_0$ be arbitrary and we prove 2.3 by induction on $\lambda=|V|$. For $\lambda\le\rho$ there is nothing to prove as $\mathrm{Col}(G)\le|G|$ for every graph G.

Let $G=(V,E)$ be a given $K(\nu,\rho^+)$-free graph with $|V|=\lambda>\rho$ and assume that $\mathrm{Col}(G')\le\beth_\omega(\nu)$ holds for all $K(\nu,\rho^+)$-free $G'=(V',E')$ with $|V'|<\lambda$.

For every cardinal θ satisfying $\rho\le\theta<\lambda$ fix, using Claim 2.10, some cardinal $\kappa(\theta)<\beth_\omega(\nu)$ such that θ is $\kappa(\theta)$-stable for G.

We need to define a well ordering $\prec$ on V that witnesses $\mathrm{Col}(G)\le\beth_\omega(\nu)$. This ordering will be defined as a well-ordered sum of well-orderings.

Case 1: $\mathrm{cf}\lambda=\aleph_0$. Fix a strictly increasing sequence of cardinals $\langle\theta_n:n<\omega\rangle$ with $\theta_0\ge\rho$ and with $\sup\{\theta_n:n<\omega\}=\lambda$. Present V as an increasing sequence of sets $\bigcup_n B_n$ such that $|B_n|=\theta_n$ and B_n is $\kappa(\theta_n)$-closed for each

n. (E.g, enumerate $V = \{v_\alpha : \alpha < \lambda\}$ and let $B_0 = \mathrm{cl}_{\kappa(\theta_0)}\{v_\alpha : \alpha < \theta_0\}$ and $B_{n+1} = \mathrm{cl}_{\kappa(\theta_{n+1})}\big(\{v_\alpha : \alpha < \theta_{n+1}\} \cup B_n\big)$.)

By the induction hypothesis and $\theta_n < \lambda$ we can fix enumerations $\{v^n_\alpha : \alpha < \alpha(n)\}$ of B_0, for $n = 0$, and of $B_n \setminus B_{n-1}$ for each $n > 0$ for suitable ordinals $\alpha(n)$ such that for every n and $\alpha < \alpha(n)$ it holds that

$$|G[v^n_\alpha] \cap \{v^n_\beta : \beta < \alpha\}| < \beth_\omega(\nu). \tag{6}$$

Put $C_0 = B_0$ and $C_{n+1} = B_{n+1} \setminus B_n$ for all n. Now $\{C_n : n \in \mathbb{N}\}$ is a partition of V and $|C_n| = \theta_n$. Define a well ordering $\prec$ on $V = \bigcup_n C_n$ by $v^m_\beta \prec v^n_\alpha$ iff $m < n$ or $m = n$ and $\beta < \alpha$. Let v^n_α be an arbitrary vertex in V and we need to show $|G_\prec[v^n_\alpha]| < \beth_\omega(\nu)$. The cardinality of $G_\prec[v^n_\alpha] \cap C_n$ is $< \beth_\omega(\nu)$ by (6) and $u \not\prec v^n_\alpha$ whenever $u \notin B_n$, so we need only count the vertices v^m_β for $m < n$. As, for $n > 0$, $\{v^m_\beta : m < n, \beta < \alpha(m)\} = B_{n-1}$ is $k(\theta_{n-1})$-closed and $v^n_\alpha \notin B_{n-1}$, it follows from Fact 2.5 (1) that $|G[v^n_\alpha] \cap B_{n-1}| < \kappa(\theta_{n-1})$. Thus $|G_\prec[v^n_\alpha]| < \beth_\omega(\nu)$.

Case 2: $\mathrm{cf}\lambda > \aleph_0$. Fix a $\leq$-increasing sequence $\langle \theta_i : i < \mathrm{cf}\lambda\rangle$ of cardinals satisfying $\beth_\omega(\nu) \leq \theta_i < \lambda$ for each i such that $(\forall\theta < \lambda)(\exists i < \mathrm{cf}\lambda)(\theta \leq \theta_i)$. In the case $\lambda = \theta^+$ for some θ we fix $\theta_i = \theta$ for all $i < \lambda$.

By increasing $\kappa(\theta_i) < \beth_\omega(\nu)$ we may assume that $\kappa(\theta_i) = (\beth_{n(i)}(\nu))^+$ for some $n(i)$ for each $i < \theta$. As $\mathrm{cf}\lambda > \aleph_0$ is regular, by restricting to a sub-sequence we get that $\kappa(\theta_i) = \kappa$ for all $i < \mathrm{cf}\lambda$ for some fixed regular $\kappa < \beth_\omega(\nu)$.

Using the κ-stability of θ_i for G, define inductively on $i < \mathrm{cf}\lambda$ a $\subsetneq$-increasing sequence of subsets $B_i \subseteq V$ satisfying:

- $|B_i| = \theta_i < \lambda$ and B_i is κ-closed.
- $V = \bigcup_{i<\theta} B_i$

Denote $C_i = B_i \setminus \bigcup_{j<i} B_j$ for $i \in I$ and let $I = \{i < \mathrm{cf}\lambda : B_i \setminus \bigcup_{j<i} B_j \neq \emptyset\}$. The collection $\{C_i : i \in I\}$ is a partition of V with $|C_i| < \mu$ for each $i \in I$.

By the induction hypothesis fix, for $i \in I$, an enumeration $C_i = \{v^i_\alpha : \alpha < \alpha(i)\}$ for some suitable ordinal $\alpha(i)$ so that $|G[v^i_\alpha] \cap \{v^i_\beta : \beta < \alpha\}| < \beth_\omega(\nu)$ for each $\alpha < \alpha(i)$. Define a well ordering $\prec$ on V by $v^j_\beta \prec v^i_\alpha$ iff $j < i$ or $j = i$ and $\beta < \alpha$. To verify that $\prec$ witnesses $\mathrm{Col}(G) \leq \beth_\omega(\nu)$ we only check that an arbitrary vertex v^i_j has fewer than $\beth_\omega(\nu)$ neighbors v^j_β with $j < i$. If $i = i' + 1$ is a successor ordinal, the union $\bigcup_{j<i} B_j = B_{i'}$ is κ-closed. Also if $\mathrm{cf} i \neq \kappa$ this union is κ-closed by Fact 2.5 (2). In these two cases, then, v^i_α has strictly fewer than κ neighbors v^j_β with $j < i$ by Fact 2.5 (1). In case $\mathrm{cf} i = \kappa$ the vertex v^i_α has $\leq \kappa$ many neighbors in $\bigcup_{j<i} B_j$ by Fact 2.5 (3). □

Corollary 2.11 (Testability of $\mathrm{Col}(G) \leq \mu$ in bounded subgraphs)**.** *Suppose ν is infinite and $\mu = \beth_\omega(\nu)$. For every graph G, $\mathrm{Col}(G) \leq \mu$ if one of the following conditions holds:*

(1) *$\mathrm{Col}(G') \leq \nu$ for every subgraph $G' \leq G$ of cardinality ν^+.*

(2) $\chi_\ell(G') \le \nu$ *for every subgraph* $G' \le G$ *of cardinality* 2^ν.
(3) $\mathrm{Col}(G') < \mu$ *for every subgraph* $G' \le G$ *of cardinality* μ^+

Proof. The complete bipartite graph $K(\nu, \nu^+)$ satisfies $\mathrm{Col}(K(\nu, \nu^+)) = \nu^+$. So if every subgraph of G of size ν^+ has coloring number μ, the graph G omits $K(\nu, \nu^+)$, and hence $\mathrm{Col}(G) \le \mu$. This proves (1), and (2) is similar, by $\chi_\ell(K(\nu, 2^\nu)) = \nu^+$.

To prove (3), assume that $\mathrm{Col}(G) > \mu$, so by Claim 2.3 there are copies $G_n \le G$ of $K(\beth_n(\nu), \rho^+)$ for all n. Let $G' = \bigcup_n G_n$. The cardinality of G' is μ^+, but $\mathrm{Col}(G') \ge \mu$. □

2.1. **Tighter bounds from additional assumptions.** As for every $\lambda < (2^\nu)^{+}\omega$ it holds that $\lambda^\nu = \lambda$, the proof in the previous section gives that $\mathrm{Col}(G) \le 2^\nu$ for all $K(\nu, 2^\nu)$-free $G = (V, E)$ with $|V| \le (2^\nu))^{+\omega}$.

Getting tighter upper bounds on colorability for all $K(\nu, 2^\nu)$-free graphs is also possible if one assume some mild restrictions on cardinal arithmetic as in [4].

Definition 2.12. *For an infinite regular cardinal* κ *let* $(*)_\kappa$ *denote the statement: every cardinal* $\mu > 2^\kappa$ *with* $\mathrm{cf}\mu = \kappa$ *is closed under* κ*-powers, that is,* $\theta < \mu \Rightarrow \theta^\kappa < \mu$.

The statement "$(*)_\kappa$ for all regular κ" is weaker than the Singular Cardinal Hypothesis, which is weaker than the Generalized Continuum Hypothesis. By Gitik's theorem [5], for every regular κ the statement $(*)_\kappa$ is not provable in ZFC modulo the consistency of large cardinal axioms.

Lemma 2.13. *Assuming* $(*)_\kappa$ *for a regular cardinal* κ*, every cardinal* $\lambda \ge 2^\kappa$ *is* κ^+*-stable for every* $K(\kappa, \lambda^+)$*-free graph* G.

Proof. Let us first state the simple

Fact 2.14. *If* $\lambda^\kappa = \lambda$ *and* G *is* $K(\kappa, \lambda^+)$*-free,* λ *is* κ*-stable for* G.

Proceed now by induction on $\lambda \ge 2^\kappa$. The cardinal $\lambda = 2^\kappa$ itself is κ-stable for every $K(\kappa, \lambda^+)$-free graph G by Corollary 2.14, hence it is also κ^+-stable. If $\mathrm{cf}\lambda \ne \kappa$ then κ^+-stability of λ follows from the induction hypothesis by Fact 2.5 (2). The remaining case is $\mathrm{cf}\lambda = \kappa$. Using the assumption $(*)_\kappa$, fix an increasing sequence of cardinals $\langle \theta_i : i < \kappa \rangle$ with limit λ such that $\theta_i^\kappa = \theta_i$. Given $A \in [V]^\lambda$ present $A \subseteq \bigcup_{i<\kappa^+} B_i$ with each B_i a κ-closed set and now the union is κ-stable, hence κ^+-stable, by 2.5 (2). □

Theorem 2.15. *Assuming* $(*)\kappa$*, every graph* G *with* $\chi_\ell(G) = \kappa$ *satisfies* $\mathrm{Col}(G) \le \max\{2^\kappa, \kappa^{++}\}$.

Proof. By induction on $\lambda = |V| \ge 2^\kappa$, using κ^+-stability. Present $V = \bigcup_{i<\mathrm{cf}\lambda} B_i$, an increasing union of κ^+-closed sets B_i with $|B_i| < \lambda$ and let $C_i = B_i \setminus \bigcup_{j<i} B_j$, $I = \{i : C_i \ne \emptyset\}$. As each B_i is κ^+-closed, for every $i \in I$ and $v \notin \bigcup_{j<i} B_j$ it holds that $|G[v] \cap \bigcup_{j<i} B_j| \le \kappa^+$. □

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[1]The author is not certain whether it should be 'patience for the infinite' or 'infinite patience'.