Recent progress on the Kakeya problem

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1 The finite field Kakeya problem

Let $\mathbb{F}_q$ denote the finite field of cardinality q. A set $K \subseteq \mathbb{F}_q^n$ is said to be a *Kakeya set* if it "contains a line in every direction". In other words, for every "direction" $\mathbf{b} \in \mathbb{F}_q^n$ there should exist an "offset" $\mathbf{a} \in \mathbb{F}_q^n$ such that the "line" through $\mathbf{a}$ in direction $\mathbf{b}$, i.e., the set $\{\mathbf{a} + t\mathbf{b} | t \in \mathbb{F}_q\}$, is contained in K. A question of interest in combinatorics/algebra/geometry, posed originally by Wolff [Wol99], is: "What is the size of the smallest Kakeya set, for a given choice of q and n?". The original motivation for this question comes from the Euclidean Kakeya problem in which one is interested in bounding the Housdorff dimension of an Euclidean Kakeya set (a set that contains a unit line segment in each direction). However, the finite field version is motivated also from the perspective of theoretical computer science using its connection to the explicit construction of pseudorandom objects.

The trivial upper bound on the size of a Kakeya set is q^n and this can be improved to roughly $\frac{1}{2^{n-1}}q^n$ [SS08]. An almost trivial lower bound is $q^{n/2}$ (every Kakeya set "contains" at least q^n lines, but there are at most $|K|^2$ lines that intersect K at least twice). Till recently even the exponent of q was not known precisely. This changed with the result of [Dvi08] who showed that for every n, $|K| \geq c_n q^n$, for some constant c_n depending only on n.

Subsequently the work [SS08] explored the growth of the constant c_n as a function of n. The result of [Dvi08] shows that $c_n \geq 1/n!$, and [SS08] improve this bound to show that $c_n \geq 1/(2.6)^n$. An even better bound was proved in [DKSS09] (extending the proof technique of earlier works) showing that $|K| \geq (q/2)^n$ for every q and n. This bound is tight to within a $2 + o(1)$ multiplicative factor as long as $q = \omega(2^n)$ and in particular when $n = O(1)$ and $q \to \infty$.

A generalization of the Kakeya problem dealing with curves in algebraic varieties appears in [EOT09].

1.1 Applications

The new technique used in [Dvi08] to prove the finite field Kakeya problem had applications in several directions.

1. In [DW08, DKSS09] the technique was used to give explicit constructions of 'extractors' which are functions that transform arbitrary random sources into uniform ones. It is easy to

show that such functions exist, but it has proven to be a difficult task to come up with explicit construction. Explicit constructions of extractors have many applications in complexity, cryptography, algorithms and other fields.

2. In [Gut08] the proof technique was used (in combination with other tools) to prove a special case of the Euclidean Kakeya conjecture (the Bennett-Carbery-Tao multilinear Kakeya conjecture). This was the first application of the finite field solution to a similar problem over the reals.

3. In [GK08, EKS09, KSS09] the ideas from [Dvi08] were developed further to prove the 20 year old 'Joints conjecture'. Here one deals with a configuration of n lines in d-dimensional real space. A 'joint' is a point in which at least d lines, that do not belong to the same hyperplane, intersect. The conjecture (which is now a theorem) states that the number of joints is at most $O(n^{d/(d-1)})$ which is tight up to constant factors.

References

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