Monotone separations for constant degree

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Abstract

We prove a separation between monotone and general arithmetic formulas for polynomials of constant degree. We give an example of a polynomial C in n variables and degree k which is computable by an arithmetic formula of size $O(k^2n^2)$, but every monotone formula computing C requires size $(n/k^c)^{\Omega(\log k)}$, with $c \in (0,1)$. This also gives a separation between monotone and homogeneous formulas, for polynomials of constant degree.

1 Introduction

Facing the unyielding challenge of proving lower bounds on arithmetic circuit or formula size, researchers have focused on several restricted models of computation. The first and most notable of such restrictions is the case of monotone computation. For example, lower bounds on monotone circuit size were proved in [2], and on monotone formula size in [3]. An exponential separation between monotone and general arithmetic circuits was given in [4]; this implies an exponential separation between monotone and general formulas as well.

An interesting class of polynomials is that of polynomials of constant degree. Proving nontrivial lower bounds for constant degree polynomials is, apparently, a much harder task. Nevertheless, Shamir and Snir proved a lower bound of $n^{\Omega(\log k)}$ on the monotone formula size of a polynomial of degree k – multiplication of k $n \times n$ matrices. It is not known whether this polynomial can be computed by a small arithmetic formula, and hence this result does not imply a separation. Also note that Valiant's construction in [4] involves a

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high degree polynomial and does not imply a separation for constant degree. The purpose of this note is to fill this gap, and to give a separation between monotone and general arithmetic formulas for constant degree polynomials.

2 Counting polynomials

We are interested in *arithmetic formulas* with fan-in at most two over the field of real numbers (see, e.g., [1] for a formal definition). We define formula *size* as the number of leaves in the formula. A *monotone formula* is a formula with only non-negative constants. A *monotone polynomial* is a polynomial with only non-negative coefficients.

Let $n, k, \ell \in \mathbb{N}$ and let $S \subseteq [n]$, where $[n] = \{0, 1, \ldots, n\}$. Denote by $\mathcal{I}(S, k, \ell)$ the set of k-tuples $\langle i_1, i_2, \ldots, i_k \rangle \in S^k$ such that $i_1 + i_2 + \cdots + i_k = \ell$. Let $C_{n,k,\ell}$ be a polynomial in variables $x_0, \ldots, x_n$ defined as

$$C_{n,k,\ell} = \sum_{I \in \mathcal{I}([n],k,\ell)} x_I, \tag{2.1}$$

where x_I denotes the monomial $\prod_{i \in I} x_i$. We call $C_{n,k,\ell}$ a *counting* polynomial. It is a homogeneous polynomial of degree k in $n+1$ variables.

The following theorem implies the separation between monotone and general formulas for constant degree.

Theorem 1. *Let $C = C_{n,k,n}$. Then*

(i). every monotone formula for C has size at least $(n/k^c)^{\Omega(\log k)}$, where $0 < c < 1$ is a universal constant, and

(ii). there exists a formula of size $O(k^2 n^2)$ for C; this formula is homogeneous.

2.1 Lower bound

We use some terminology from [1]. Let f be a homogeneous polynomial of degree k. We say that f is *balanced* if there exist p homogeneous polynomials $f_1, \ldots, f_p$ such that $f = f_1 f_2 \cdots f_p$ with

(i). $(1/3)^i k < \deg f_i \leq (2/3)^i k$, $i = 1, \ldots, p-1$, and

(ii). $\deg(f_p) = 1$.

The following lemma shows that a small monotone formula can be written as a short sum of balanced polynomials. It is a straightforward adaptation of the lemma from [1] to the case of monotone formulas.

Lemma 2. *Let Φ be a monotone formula of size s computing a homogeneous polynomial f of degree $k > 0$. Then there exist balanced monotone polynomials $f_1, \ldots, f_{s'}$ of degree k such that $s' \leq s$ and $f = f_1 + \cdots + f_{s'}$.*

The following proposition and Lemma 2 imply part (i) of Theorem 1.

Proposition 3. *Let $n, k \in \mathbb{N}$ and let $C = C_{n,k,n}$. If $C = f_1 + \cdots + f_s$ with $f_1, \ldots, f_s$ balanced monotone polynomials of degree k, then $s \geq (n/k^c)^{\Omega(\log k)}$, where $0 < c < 1$ is a universal constant.*

Before proving the proposition we recall the following estimate from [1].

Lemma 4. *Let $n \geq 2k$ and $k_1, \ldots, k_p$ be non-zero natural numbers such that $k_1 + \cdots + k_p = k$. Then for every natural numbers $n_1, \ldots, n_p$ such that $n_1 + \cdots + n_p = n$,*

$$\binom{n_1}{k_1} \cdots \binom{n_p}{k_p} \leq 3k^{1/2}(k_1 \cdots k_p)^{-1/2}\binom{n}{k}.$$

Proof of Proposition 3. Since C is homogeneous of degree k and $f_1, \ldots, f_s$ are monotone, $f_1, \ldots, f_s$ are homogeneous polynomials of degree k. Fix $t = 1, \ldots, s$ and denote $f = f_t$. Since f is a product polynomial, we can write $f = g_1 g_2 \cdots g_p$.

Claim 5. *There exist natural numbers $n_1, n_2, \ldots, n_p, k_1, k_2, \ldots, k_p$ such that $n_1 + \cdots + n_p = n$ and $k_1 + \cdots + k_p = k$ and for every $j = 1, \ldots p$, all the monomials that occur in g_j are of the form x_I with $I \in \mathcal{I}([n_j], k_j, n_j)$.*

Proof. Define k_j to be the degree of g_j. Since f is homogeneous of degree k, $k_1 + \cdots + k_p = k$ and each g_j is homogeneous. Hence if a monomial x_I occurs in g_j then $|I| = k_j$. Let us fix n_j as some natural number such that there exists a monomial x_I which occurs in g_j and $\sum_{i \in I} i = n_j$. Monotonicity implies that for every monomial x_L occurring in g_j, $\sum_{i \in L} i = n_j$. For assume otherwise, and let x_M be a monomial that occurs in $g_1 \cdots g_{j-1} g_{j+1}, \ldots g_p$. Then both the monomials $x_I x_M, x_L x_M$ occur in C, which is impossible since $\sum_{i \in I \cup M} i \neq \sum_{i \in L \cup M} i$. For a similar reason, $n_1 + \cdots + n_p = n$. Finally, since $\sum_{i \in L} i = n_j$ implies that, as a set, $L \subseteq [n_j]$, we have $L \in \mathcal{I}([n_j], k_j, n_j)$ for every x_L occurring in g_j. $\square$

Claim 5 shows that for every $j = 1, \ldots, p$, the number of monomials that occur in g_j is at most $|\mathcal{I}([n_j], k_j, n_j)|$. The size of $\mathcal{I}([n_j], k_j, n_j)$ is

$$\binom{n_j + k_j - 1}{k_j - 1}.$$

If $k_j = 1$, g_j contains exactly one monomial. Setting q to be the maximal j such that $k_j \geq 2$, Lemma 4 shows that the number of monomials in f is at most

$$\begin{aligned}\binom{n_1 + k_1 - 1}{k_1 - 1} \cdots \binom{n_q + k_q - 1}{k_q - 1} &\leq 3k^{1/2} \prod_{i=1,\ldots,q} (k_i - 1)^{-1/2} \binom{n + k - q}{k - q} \\ &= 3k^{1/2} \prod_{i=1,\ldots,q} (k_i - 1)^{-1/2} \prod_{i=1,\ldots,q-1} \frac{k - i}{n + k - i} \cdot \binom{n + k - 1}{k - 1}.\end{aligned}$$

For every $1 \leq i \leq \log k/(2 \log 3) - 1$, we have $k_i \geq 3k^{1/2}$, and so $k_i - 1 \geq k^{1/2}$. Hence

$$k^{1/2} \prod_{i=1,\ldots,q} (k_i - 1)^{-1/2} \leq k^{-c_1 \log k + 1}$$

with a constants $c_1 > 0$. Since $q \leq k$, we have

$$\prod_{i=1,\ldots,q-1} \frac{k - i}{n + k - i} \leq \left(\frac{k}{n}\right)^{q-1}$$

Since f is balanced, q is at least $c_2 \log k - 2$ with $c_2 > 0$ a universal constant. Hence the number of monomials in $f = f_t$ is at most

$$3k^{-c_1 \log k + 1} \left(\frac{k}{n}\right)^{c_2 \log k - 3} \binom{n + k - 1}{k - 1} \leq \left(\frac{k^c}{n}\right)^{-\Omega(\log k)} \binom{n + k - 1}{k - 1},$$

with an adequate constant $c \in (0, 1)$. Since this holds for every t and since the number of monomials in C is $\binom{n+k-1}{k-1}$, we have that s is at least $(n/k^c)^{\Omega(\log k)}$. □

2.2 Upper bound

We now construct polynomial size formulas for $C_{n,k,\ell}$.

Proof of part (ii) of Theorem 1. The proof follows by interpolation. Fix $n, k \in \mathbb{N}$. Let Z be the polynomial

$$Z(t) = (x_0 t^0 + x_1 t^1 + \cdots + x_n t^n)^k ,$$

where t is an auxiliary variable. Observe that

$$Z(t) = \sum_{0 \le \ell \le nk} t^\ell C_{n,k,\ell}.$$

Evaluating at $t = 0, \ldots, nk$,

$$\begin{bmatrix} Z(0) \\ Z(1) \\ \cdots \\ Z(nk) \end{bmatrix} = A \begin{bmatrix} C_{n,k,0} \\ C_{n,k,1} \\ \cdots \\ C_{n,k,nk} \end{bmatrix}$$

with

$$A = \begin{bmatrix} 1 & 0^1 & \cdots & 0^{nk} \\ 1^0 & 1^1 & \cdots & 1^{nk} \\ & & \cdots & \\ (nk)^0 & (nk)^1 & \cdots & (nk)^{nk} \end{bmatrix}.$$

Since the matrix A is invertible, we can express every $C_{n,k,\ell}$ as a linear combination of $Z(0), \ldots, Z(nk)$. For a particular number a, $Z(a)$ has a homogeneous formula of size roughly kn computing it, hence we can compute $C_{n,k,\ell}$ by a homogeneous formula of size roughly $k^2 n^2$. □

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