Claw-free Graphs. II. Circular interval graphs

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Abstract

Construct a graph as follows. Take a circle, and a collection of intervals from it, no three of which have union the entire circle; take a finite set of points V from the circle; and make a graph with vertex set V in which two vertices are adjacent if they both belong to one of the intervals. Such graphs are "circular interval graphs", and they form an important subclass of the class of all claw-free graphs. In this paper we characterize them by excluded induced subgraphs. This is a step towards the main goal of this series, to find a structural characterization of all claw-free graphs.

1 Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) If $X \subseteq V(G)$, the subgraph $G|X$ *induced on* X is the subgraph with vertex set X and edge-set all edges of G with both ends in X. ($V(G)$ and $E(G)$ denote the vertex- and edge-sets of G respectively.) We say that $X \subseteq V(G)$ is a *claw in* G if $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. We say G is *claw-free* if no $X \subseteq V(G)$ is a claw in G. Our objective in this series of papers is to show that every claw-free graph can be built starting from some basic classes by means of some simple constructions. One important subclass of claw-free graphs are the "circular interval graphs", the topic of this paper.

Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be "intervals", that is, each is homeomorphic to the interval $[0,1]$. Assume that no three of $F_1, \ldots, F_k$ have union Σ. Now let $V \subseteq \Sigma$ be finite, and let G be the graph with vertex set V in which two vertices are adjacent if and only if some F_i contains both of them. Such a graph G is called a *circular interval graph.* (They are not to be confused with what are called "circular arc graphs"; one is a proper subclass of the other.) Our objective here is to characterize circular interval graphs by excluded induced subgraphs.

Circular interval graphs are claw-free, and these together with line graphs turn out to be the two "principal" basic classes of claw-free graphs. Our (lengthy) proof of that fact includes a characterization of circular interval graphs by excluded induced subgraphs, and it seemed sensible to spin that off into a separate paper, the present paper. Incidentally, one might ask why we insist that no three of the intervals $F_1, \ldots, F_k$ mentioned above can have union the circle. If we omit this condition, the graphs we produce are still claw-free; but we do not know how to characterize this larger class by excluded subgraphs, and for our application we do not need to do so.

Below we state our main result, but first, some terminology. A *hole* in G means an induced subgraph which is a cycle of length ≥ 4. A *centre* for a hole C is a vertex adjacent to every vertex of C. A hole is *dominating* if every vertex in $V(G) \setminus V(C)$ has a neighbour in C. Let us say a *trinity* is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, and six edges, the edges a_ib_i ($i = 1, 2, 3$) and the edges a_1a_2, a_2a_3, a_3a_1; and an *antitrinity* is a graph with six vertices whose complement graph is a trinity. A $(1,1,1)$-*prism* is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and nine edges, the edges a_ib_i ($i = 1, 2, 3$), and $a_1a_2, a_2a_3, a_3a_1, b_1b_2, b_2b_3, b_3b_1$. One form of our main result is the following:

1.1 *Let G be a graph. Then G is a circular interval graph if and only if no induced subgraph is a claw, trinity, antitrinity or* $(1,1,1)$*-prism, and every hole is dominating, and has no centre.*

However, for our application, we need to prove something stronger, connecting the size of the largest hole in G with the type of excluded subgraph that we use. This needs a number of further definitions, and we postpone the precise statement until 3.1.

Let G be a graph. If we direct its edges, we obtain a digraph D, called an *orientation* of G. If $u, v \in V(G)$, we write $uv \in E(D)$ to mean that there is an edge of G between u, v, and u is its tail and v its head in D. Let Σ be a circle, and assign an orientation to it called "clockwise". Let ϕ be a map from $V(G)$ into Σ; then for $\sigma \in \Sigma$, $\phi^{-1}(\sigma)$ denotes the set $\{v \in V(G) : \phi(v) = \sigma\}$. Let D be an orientation of G, and suppose the following conditions are satisfied:

- for each $\sigma \in \Sigma$, $\phi^{-1}(\sigma)$ is a clique of G
- if $\sigma_1, \sigma_2 \in \Sigma$ are distinct, there do not exist $u_i, v_i \in \phi^{-1}(\sigma_i)$ ($i = 1, 2$) such that $u_1v_2, u_2v_1 \in E(D)$

- if $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ are distinct and in clockwise order, let $X_i = \phi^{-1}(\sigma_i)$ $(i = 1, 2, 3)$, and suppose that there exist $u \in X_1$ and $v \in X_3$ such that $uv \in E(D)$. Then X_2 is complete to $X_1 \cup X_3$; the edges between X_1, X_2 all have tail in X_1; and the edges between X_2, X_3 all have tail in X_2
- for each $\sigma_1 \in \Sigma$, $|\phi^{-1}(\sigma_1)| > 1$ if and only if there exists $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$ such that $\phi^{-1}(\sigma_1)$ is neither complete nor anticomplete to $\phi^{-1}(\sigma_2)$. Moreover, for any such σ_2, every vertex in $\phi^{-1}(\sigma_1)$ has both a neighbour and a nonneighbour in $\phi^{-1}(\sigma_2)$.
- for each $\sigma_1 \in \Sigma$ there is at most one $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$ such that $\phi^{-1}(\sigma_1)$ is neither complete nor anticomplete to $\phi^{-1}(\sigma_2)$.

We call such a pair (ϕ, D) a *metacircular arrangement* of G; and if G admits such a pair (ϕ, D) we say that G is *metacircular*. Note that ϕ can be chosen to be an injection if and only if G is a circular interval graph; so metacircular graphs are just a slight variant of circular interval graphs. If (ϕ, D) is a metacircular arrangement of G, and C_0 is a hole in G, we say that C_0 is a *basis* for the arrangement if the points $\phi(v)$ $(v \in V(C_0))$ are all distinct.

An *n-hole* in G means a hole in G of length n. Let C be a n-hole, with vertices c_1-$\cdots$-c_n-c_1 in order; we call this a *numbering* of C. (We shall read these and similar subscripts modulo n, usually without saying so.) Let $v \in V(G) \setminus V(C)$, and let N be the set of neighbours of v in C. We say that (relative to C, and to the given numbering):

- v is a *hat* if $N = \{c_i, c_{i+1}\}$ for some i
- v is a *clone* if $N = \{c_{i-1}, c_i, c_{i+1}\}$ for some i
- v is a *star* if $n \geq 5$ and $N = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ for some i
- v is a *hub* if $n \geq 6$ and there exist i, j with $1 \leq i, j \leq n$ such that $i-1, i, i+1, j-1, j, j+1$ are all distinct (modulo n) and $N = \{c_i, c_{i+1}, c_j, c_{j+1}\}$.

For convenient reference, here are two lemmas that we frequently need. The proof of the first is clear.

1.2 *Let C be a hole in a claw-free graph G, and let $v \in V(G) \setminus V(C)$. Then either:*

- *v has no neighbours in $V(C)$, or*
- *v is a hat, clone, star or hub relative to C, or*
- *C has length ≤ 5 and v is a centre relative to C.*

If h_1, h_2 are adjacent hats, with no common neighbour in C, we call $\{h_1, h_2\}$ a *hat-diagonal* for C. The *length* of a path or cycle is the number of edges in it.

1.3 *Let G be claw-free, and let C be a hole in G. Let $v_1, v_2 \in V(G) \setminus V(C)$, and suppose that for $i = 1, 2$, the set of neighbours of v_i in C is the vertex set of a path P_i in C.*

1. *If P_1 is a subpath of P_2, and they have a common end, then u, v are adjacent.*
2. *If P_1 is a subpath of P_2 and they have no common end, then u, v are nonadjacent.*

3. *If neither of P_1, P_2 is a subpath of the other, and at least three vertices of P_1 do not belong to P_2, then u, v are nonadjacent.*

4. *If there is no hat-diagonal for C, and P_1, P_2 both have length 1 and are disjoint, then u, v are nonadjacent.*

5. *If P_1, P_2 both have length 1 and are different but with a common end, and C is a hole of maximum length in G, then u, v are nonadjacent.*

Proof. For the first assertion, let x be an end of both P_1, P_2, and let y be the neighbour of x in C that belongs to neither of P_1, P_2. Since $\{x, y, u, v\}$ is not a claw, u is adjacent to v. This proves the first assertion. For the second, let x, y be the ends of P_2; then $\{v_2, v_1, x, y\}$ is not a claw, and so v_1, v_2 are nonadjacent. For the third, suppose three vertices of P_1 do not belong to P_2; then some two of them, say x, y, are nonadjacent, and since $\{v_1, v_2, x, y\}$ is not a claw, it follows that v_1, v_2 are nonadjacent. For the fourth, this follows since if v_1, v_2 are adjacent, then $\{v_1, v_2\}$ is a hat-diagonal. Finally, if P_1, P_2 share one end c, then the subgraph induced on $(V(C) \setminus \{c\}) \cup \{v_1, v_2\}$ is not a longer hole than C, and so v_1, v_2 are nonadjacent. This proves 1.3. ■

2 Circle arrangements

Being metacircular implies being claw-free, but it is convenient to relax the definition of "metacircular" further, to a condition which is much easier to recognize, but which no longer implies being claw-free, as follows.

As before, let Σ be a circle, and assign an orientation to it called "clockwise". If ϕ is a map from $V(G)$ into Σ, we say a triple (u, v, w) of vertices of G is *clockwise* if $\phi(u), \phi(v), \phi(w)$ are distinct and in clockwise order in Σ. By a *circle arrangement* of G we mean a pair (ϕ, D), where ϕ is a map from $V(G)$ into Σ, and D is an orientation D of G, satisfying the following conditions:

- for all $\sigma \in \Sigma$, $\phi^{-1}(\sigma)$ is a clique of G
- there is no directed cycle in D of length 3
- for every clockwise triple (u, v, w), if $uw \in E(D)$ then $uv, vw \in E(D)$
- if $u, v, w \in V(G)$ and $uv, vw \in E(D)$, and $\phi(u) = \phi(w)$, then $\phi(u) = \phi(v)$.

We call the first condition the *first axiom* and so on. If C_0 is a hole in G, we say it is a *basis* for the circle arrangement if the points $\phi(v)$ $(v \in V(C_0))$ are all distinct. The next result implies that to show that a claw-free graph is metacircular, it suffices to exhibit a circle arrangement. (Incidentally, this result is closely related to a theorem of [1], but we were unable to formulate a reasonable common generalization of the two.)

2.1 *Every claw-free graph G that admits a circle arrangement is metacircular; and indeed, for every circle arrangement (ϕ_0, D_0) of G, there is a metacircular arrangement (ϕ, D) of G such that every basis for (ϕ_0, D_0) is a basis for (ϕ, D).*

Proof. If (ϕ, D) is a circle arrangement of G, we define $\Sigma(\phi)$ to be the set of all $\sigma \in \Sigma$ such that $\phi^{-1}(\sigma) \neq \emptyset$. Let (ϕ_0, D_0) be a circle arrangement of G; and choose a circle arrangement (ϕ, D) such that every basis for (ϕ_0, D_0) is a basis for (ϕ, D), and $|\Sigma(\phi)|$ is maximum with this property. (We call this property the "maximality" of (ϕ, D).) We shall prove that (ϕ, D) is in fact a metacircular arrangement. For a vertex u, let $A(u), B(u)$ be the sets of all $v \in V(G)$ such that $\phi(u) \neq \phi(v)$ and $uv \in E(G)$ (respectively, $vu \in E(G)$).

(1) *For all $s \in \Sigma(\phi)$, and all $X \subseteq \phi^{-1}(s)$, let $Y = \phi^{-1}(s) \setminus X$; if X, Y are both nonempty, then there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$.*

For suppose not. Choose two points r, t of $\Sigma \setminus \Sigma(\phi)$, immediately before and after s; more precisely, such that r, s, t are in clockwise order, and t, s', r are in clockwise order for all $s' \in \Sigma(\phi) \setminus \{s\}$. For $v \in V(G)$, define $\phi'(v)$ as follows: if $v \in X$, $\phi'(v) = r$; if $v \in Y$, $\phi'(v) = t$; and otherwise $\phi'(v) = \phi(v)$. Define a new orientation D' of G as follows: for each edge e of G with an end $x \in X$ and an end $y \in Y$, direct e so that x is its tail. For all other edges e let the tail of e in D' be the same as its tail in D. We claim that the pair (ϕ', D') is a circle arrangement of G. Let us check the axioms. The first axiom clearly holds. For the second, suppose that $uv, vw, wu \in E(D')$. Then one of them does not belong to $E(D)$, say wu, and so $w \in X$ and $u \in Y$. Since $uv \in E(D')$ it follows that $v \notin X$, and similarly $v \notin Y$; and so $v \notin \phi^{-1}(s)$. Consequently $uv, vw \in E(D)$; but then (ϕ, D) fails to satisfy the fourth axiom, a contradiction. For the third axiom, suppose that (u, v, w) is clockwise (with respect to ϕ'), and $uw \in E(D')$. If at most one of u, v, w belong to $\phi^{-1}(s)$, then (u, v, w) is also clockwise with respect to ϕ, and $uw \in E(D)$, and hence $uv, vw \in E(D)$ since (ϕ, D) satisfies the third axiom; but then $uv, vw \in E(D')$ as required. We may therefore assume that at least two of u, v, w belong to $\phi^{-1}(s)$. Since (u, v, w) is clockwise with respect to ϕ', at most one of u, v, w is in X and at most one in Y. There are therefore three possibilities: $u \in X$, $v \in Y$ and $w \notin \phi^{-1}(s)$; or $v \in X$, $w \in Y$ and $u \notin \phi^{-1}(s)$; or $w \in X, u \in Y$ and $v \notin \phi^{-1}(s)$. In the first case, $uv \in E(D')$; since the claim of (1) is false, $w \notin A(u) \setminus A(v)$, and consequently $vw \in E(D)$, and the axiom is satisfied. The second case is similar. The third case cannot occur since $uw \in E(D')$. Thus (ϕ', D') satisfies the third axiom. Finally, for the fourth axiom, suppose that $u, v, w \in V(G)$ and $uv, vw \in E(D')$, and $\phi'(u) = \phi'(w)$. Consequently $\phi(u) = \phi(w)$. We claim that $\phi(u) = \phi(v)$. For if $uv, vw \in E(D)$ then $\phi(u) = \phi(v)$, since (ϕ, D) satisfies the fourth axiom; and otherwise $X \cup Y$ contains v and at least one of u, w, and therefore contains both of u, w since $\phi(u) = \phi(w)$, and so again $\phi(u) = \phi(v)$. Thus in either case $\phi(u) = \phi(v) = \phi(w)$, and we may therefore assume that $u, v, w \in X \cup Y$. Since $\phi'(u) = \phi'(w)$, both u, w belong to X or both to Y; and since $uv, vw \in E(D')$, it follows that v belongs to the same set, and therefore $\phi'(u) = \phi'(v)$. This verifies the fourth axiom, and therefore proves that (ϕ', D') is a circle arrangement of G. But $|\Sigma(\phi)| < |\Sigma(\phi')|$, since X, Y are nonempty, and every basis for (ϕ, D) is a basis for (ϕ', D'), contrary to the maximality of (ϕ, D). This proves (1).

(2) *If $u, v \in V(G)$ and $\phi(u) = \phi(v)$ then either $A(u) \subseteq A(v)$ or $B(u) \subseteq B(v)$.*

Suppose not. Then there exist a vertex $w \in A(u) \setminus A(v)$ and a vertex $x \in B(u) \setminus B(v)$. Consequently, $uw, xu \in E(D)$, and $vw, xv \notin E(D)$. Since $\phi(u) = \phi(v)$, the first axiom implies that u, v are adjacent in G. Since $\{u, v, w, x\}$ is not a claw, we may assume from the symmetry that w is adjacent in G to one of v, x, and consequently one of $wv, vw, wx, xw \in E(D)$. The first, $wv \in E(D)$, is false by the fourth axiom. The second, $vw \in E(D)$, is false since $w \notin A(v)$. The third, $wx \in E(D)$, is false by the second axiom. Suppose the fourth holds, that is, $xw \in E(D)$. Now there are three subcases: (x, v, w) is clockwise; (v, x, w) is clockwise; and $\phi(w) = \phi(x)$. If (x, v, w) is clockwise, the third axiom implies that $xv, vw \in E(D)$, a contradiction; if (v, x, w) is clockwise, then (u, x, w) is clockwise, and the third axiom applied to uw implies that $ux \in E(D)$, which is false since $xu \in E(D)$; and if $\phi(x) = \phi(w)$, the fourth axiom is violated. This proves (2).

(3) *Let $s \in \Sigma(\phi)$, and let M be the set of all $v \in \phi^{-1}(s)$ such that $B(v)$ is maximal. Suppose that there exist $m_1, m_2 \in M$ such that $B(m_1), B(m_2)$ are different. Then $A(u) = A(v)$ for all $u, v \in \phi^{-1}(s)$.*

For if $x_1, x_2, y \in \phi^{-1}(s)$, not necessarily distinct, let us say that y is *between* x_1 and x_2 if $B(x_1) \not\subseteq B(y)$ and $B(y) \not\subseteq B(x_2)$. Let X be the minimal subset of $\phi^{-1}(s)$ such that $m_1 \in X$ and for all $x_1, x_2 \in X$ and $y \in \phi^{-1}(s)$, if y is between x_1 and x_2 then $y \in X$. Since $B(m_1), B(m_2)$ are different and both maximal, neither is a subset of the other, and so m_2 is between m_1 and m_1; and therefore $m_2 \in X$. For any other member $m_3 \in M$, $B(m_3)$ is different from one of $B(m_1), B(m_2)$, and so by the same argument, $m_3 \in X$. Consequently $M \subseteq X$.

Let $Y = \phi^{-1}(s) \setminus X$ and suppose that $Y \neq \emptyset$. Certainly X is nonempty, and so by (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. We claim that the second holds. For suppose that $B(y) \subseteq B(x)$. Since $B(y) \neq B(x)$ (since $x \in X$, $y \notin X$ and membership of X is determined by the value of the B function), it follows that $B(x) \not\subseteq B(y)$; and so by (2), $A(x) \subseteq A(y)$, contrary to the choice of x, y. This proves that $B(y) \not\subseteq B(x)$. Since $y \notin X$ and $M \subseteq X$, it follows that $y \notin M$, and so $B(m_1) \not\subseteq B(y)$. But then y is between m_1 and x, contradicting that $y \notin X$. We deduce that $Y = \emptyset$.

Let $A(m_1) = A_1$. We claim that $A(x) = A_1$ for every $x \in X$. To prove that, it is enough (because of the minimality of X) to show that if y is between x_1 and x_2 then $A(x_1) \subseteq A(y) \subseteq A(x_2)$. The first inclusion follows from (2) because $B(x_1) \not\subseteq B(y)$ and the second follows from (2) because $B(y) \not\subseteq B(x_2)$. Consequently, $A(x) = A_1$ for all $x \in \phi^{-1}(s)$. This proves (3).

(4) *Let $s \in \Sigma(\phi)$. Then either $A(u) = A(v)$ for all $u, v \in \phi^{-1}(s)$, or $B(u) = B(v)$ for all $u, v \in \phi^{-1}(s)$.*

For let M be the set of all $v \in \phi^{-1}(s)$ such that $B(v)$ is maximal. By (3) we may assume that $B(m_1) = B(m_2)$ for all $m_1, m_2 \in M$. Let $Y = \phi^{-1}(s) \setminus M$, and suppose that Y is nonempty. Certainly M is nonempty, and so by (1) there exist $x \in M$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. As in the proof of (3), the first statement implies the second, so we may assume that $B(y) \not\subseteq B(x)$. Choose $m \in M$ with $B(y) \subseteq B(m)$; then $m, x \in M$, and $B(m) \neq B(x)$, a contradiction. Thus $Y = \emptyset$. This proves (4).

Let $s, t \in \Sigma(\phi)$ be distinct. We write $s \Rightarrow t$ if all edges between $\phi^{-1}(s)$ and $\phi^{-1}(t)$ are directed in D from $\phi^{-1}(s)$ to $\phi^{-1}(t)$.

(5) *Let $s, t \in \Sigma(\phi)$ be different. Then either $s \Rightarrow t$ or $t \Rightarrow s$.*

For suppose there exist $u, u' \in \phi^{-1}(s)$ and $v, v' \in \phi^{-1}(t)$ such that $uv, v'u' \in E(D)$. By (4) and the symmetry, we may assume that $A(u) = A(u')$. Since $v \in A(u)$ it follows that $v \in A(u')$ and so

$u'v \in E(D)$. But $v'u' \in E(D)$, contrary to the fourth axiom. This proves (5).

(6) *Let $r, s, t \in \Sigma(\phi)$ be clockwise, and suppose that $r \Rightarrow t$. Then $r \Rightarrow s$ and $s \Rightarrow t$. Moreover, either $\phi^{-1}(r)$ is anticomplete to $\phi^{-1}(t)$, or $\phi^{-1}(s)$ is complete to both $\phi^{-1}(r)$ and $\phi^{-1}(t)$.*

For the first claim, we may assume from the symmetry that $r \not\Rightarrow s$. Choose $u \in \phi^{-1}(r)$ and $v \in \phi^{-1}(s)$, such that $vu \in E(D)$. Choose $w \in \phi^{-1}(t)$. Since (s, t, r) is clockwise, the third axiom implies that $wu \in E(D)$, contradicting that $r \Rightarrow t$. This proves the first claim.

For the second, we may assume that $\phi^{-1}(r)$ is not anticomplete to $\phi^{-1}(t)$, and (from the symmetry) not complete to $\phi^{-1}(s)$. Let X be the set of all $u \in \phi^{-1}(r)$ with a nonneighbour in $\phi^{-1}(s)$, and let $Y = \phi^{-1}(r) \setminus X$. By the third axiom, every vertex of $\phi^{-1}(r)$ with a neighbour in $\phi^{-1}(t)$ belongs to Y, and so X, Y are both nonempty. In particular, not all $v \in \phi^{-1}(r)$ have $A(v)$ the same, and so by (4), $B(x) = B(y)$ for all $x, y \in \phi^{-1}(r)$. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. The first holds, since we just saw that $B(x) = B(y)$. Choose $z \in A(x) \setminus A(y)$, and let $\phi(z) = t'$ say. Now $t' \neq s$ since y is complete to $\phi^{-1}(s)$ and $r \Rightarrow s$; if (r, s, t') is clockwise, then the third axiom implies that x is complete to $\phi^{-1}(s)$, contradicting that $x \in X$; and if (r, t', s) is clockwise, then since y has a neighbour in $\phi^{-1}(s)$ and $yz \notin E(D)$, the third axiom is violated. This proves (6).

Let $s, t \in \Sigma(\phi)$ be distinct. We say s, t is a *mixed pair* if $\phi^{-1}(s)$ is neither complete nor anticomplete to $\phi^{-1}(t)$.

(7) *Let s, t be a mixed pair. Then there exists $u \in \phi^{-1}(s)$ such that u has a neighbour and a nonneighbour in $\phi^{-1}(t)$.*

For suppose not. So every vertex in $\phi^{-1}(s)$ is either complete or anticomplete to $\phi^{-1}(t)$. Let X be the set of members of $\phi^{-1}(s)$ that are $\phi^{-1}(t)$-complete, and Y the set of those that are $\phi^{-1}(t)$-anticomplete. Since s, t is mixed, X, Y are both nonempty. By (5) and the symmetry, we may assume that $t \Rightarrow s$. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. But $B(x) \not\subseteq B(y)$, since $\phi^{-1}(t)$ is included in the first set and not the second; and so from (2), $A(x) \subseteq A(y)$. Consequently, $B(y) \not\subseteq B(x)$. Choose $v \in \phi^{-1}(t)$ and $w \in B(y) \setminus B(x)$. Since $B(y) \cap \phi^{-1}(t) = \emptyset$, $\phi(w) \neq t = \phi(v)$, and therefore one of $(v, w, y), (w, v, y)$ is clockwise. If (v, w, y) is clockwise then (v, w, x) is clockwise, and yet $vx \in E(D)$ and $wx \notin E(D)$, a contradiction to the third axiom. If (w, v, y) is clockwise then $wy \in E(D), vy \notin E(D)$ again contradicts the third axiom. This proves (7).

(8) *For every $s \in \Sigma(\phi)$ there is at most one $t \in \Sigma(\phi)$ such that s, t is a mixed pair.*

For let $s, t_1, t_2 \in \Sigma(\phi)$ be distinct, and suppose that the pairs s, t_1 and s, t_2 are both mixed. By (4) and the symmetry, we may assume that $B(x) = B(y)$ for all $x, y \in \phi^{-1}(s)$. By (7) for $i = 1, 2$ there exists $v_i \in \phi^{-1}(t_i)$ with a neighbour u_i and a nonneighbour u_i' in $\phi^{-1}(s)$. Since $v_i \notin B(u_i') = B(u_i)$ it follows that $v_i u_i \notin E(D)$, and so $u_i v_i \in E(D)$. By (6), $s \Rightarrow t_i$ for $i = 1, 2$. From the symmetry between t_1, t_2, we may assume that (s, t_1, t_2) is clockwise; but this contradicts (6). This proves (8).

(9) *If s, t is a mixed pair, then every $v \in \phi^{-1}(s)$ has both a neighbour and a nonneighbour in*

$\phi^{-1}(t)$.

For we may assume that $s \Rightarrow t$. Let X be the set of all vertices in $\phi^{-1}(s)$ with no neighbour in $\phi^{-1}(t)$, and suppose that X is nonempty. Let $Y = \phi^{-1}(s) \setminus X$. Since s, t is a mixed pair, $Y \neq \emptyset$. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. But $A(x) \subseteq A(y)$ by two applications of axiom 3, and so $A(y) \not\subseteq A(x)$, since $A(y) \cap \phi^{-1}(t) \neq \emptyset$; and so by (2), $B(y) \subseteq B(x)$, a contradiction. This proves that every vertex in $\phi^{-1}(s)$ has a neighbour in $\phi^{-1}(t)$. Now let X be the set of all vertices in $\phi^{-1}(s)$ that are not complete to $\phi^{-1}(t)$, and let $Y = \phi^{-1}(s) \setminus X$. Since s, t is a mixed pair, $X \neq \emptyset$. Suppose that Y is nonempty. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. By two applications of axiom 3, $A(x) \subseteq A(y)$, and equality does not hold; and so by (4), $B(y) = B(x)$, a contradiction. Thus $Y = \emptyset$. This proves (9).

(10) *If $s \in \Sigma$ and $|\phi^{-1}(s)| > 1$, then there exists $t \in \Sigma$ with $t \neq s$ such that s, t is a mixed pair.*

For suppose not; and partition $\phi^{-1}(s)$ into two nonempty subsets X, Y. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A(y)$, or $B(y) \not\subseteq B(x)$. But $A(x) = A(y)$ and $B(x) = B(y)$ since there is no t satisfying the claim, a contradiction. This proves (10).

By (5), (6), (8), (9), and (10), it follows that (ϕ, D) is a metacircular arrangement of G. This proves 2.1. ■

3 The main theorem

We begin with some more definitions. Let C be a hole with vertices $c_1\text{-}\cdots\text{-}c_n\text{-}c_1$ in order.

- We recall that if h_1, h_2 are adjacent hats, with no common neighbour in C, $\{h_1, h_2\}$ is a *hat-diagonal* for C
- If $n \geq 5$ and h, s are a hat and a star relative to C, and for some i h is adjacent to c_i, c_{i+1} and s is adjacent to $c_{i-1}, c_i, c_{i+1}, c_{i+2}$, we call $\{h, s\}$ a *coronet* for C.
- If $n \geq 5$ and s_1, s_2 are nonadjacent stars, adjacent respectively to $c_i, c_{i+1}, c_{i+2}, c_{i+3}$ and to $c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}$ for some i, we call $\{s_1, s_2\}$ a *crown* for C.
- If $n = 5$ or 6 and s_1, s_2 are adjacent stars, adjacent respectively to $c_i, c_{i+1}, c_{i+2}, c_{i+3}$ and to $c_{i+3}, c_{i+4}, c_{i+5}, c_{i+6}$ for some i, we call $\{s_1, s_2\}$ a *star-diagonal* for C.
- If $n = 6$ and s_1, s_2, s_3 are three pairwise adjacent stars, adjacent respectively to $c_i, c_{i+1}, c_{i+2}, c_{i+3}$, to $c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}$ and to $c_{i-2}, c_{i-1}, c_i, c_{i+1}$ for some i, we call $\{s_1, s_2, s_3\}$ a *star-triangle* for C.

Now we prove the main result of this paper, the following. (Please note that for some values of n, certain hypotheses of the theorem are vacuous. For instance, if $n > 5$ then no n-hole has a centre; star-triangles are only possible if $n = 6$; and so on.)

3.1 *Let G be claw-free, with a hole, and let n be the maximum length of holes in G. Suppose that every hole of length n is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Then G is metacircular. Indeed, for any n-hole C, there is a metacircular arrangement (ϕ, D) of G with basis C.*

Proof. Let C be a hole in G with length n, with a numbering c_2-c_4-c_6-$\cdots$-c_{2n}-c_2 say. Since C is dominating and has no hub or centre, it follows that every vertex of G not in $V(C)$ is a hat, clone or star relative to C. For every $v \in V(G)$, we define $\pi(v)$ as follows. Let N be the set of neighbours of v in $V(C)$, together with v if $v \in V(C)$. Since either $v \in V(C)$ or v is a hat, clone or star, there is a unique choice of $i \in \{1, \ldots, 2n\}$ such that N is one of the three sets $\{c_{i-1}, c_{i+1}\}$, $\{c_{i-2}, c_i, c_{i+2}\}$, and $\{c_{i-3}, c_{i-1}, c_{i+1}, c_{i+3}\}$. We define $\pi(v) = i$.

Let Σ be a circle, with circumference $2n$, and let $\sigma_1, \ldots, \sigma_{2n}$ be equally spaced points of it, in order. For each vertex v of G, we define $\phi(v) = \sigma_{\pi(v)}$. If $u, v \in V(G)$, we define $d(u,v)$ to be the length of the shorter arc of Σ joining $\phi(u), \phi(v)$ (or 0, if $\phi(u) = \phi(v)$); thus $d(u,v) \leq n$. (For example, $d(c_2, c_4) = 2$.)

For $i = 1, \ldots, 2n$ with i even, let C_i be the set of all clones v with $\pi(v) = i$, and for i odd, let H_i and S_i be respectively the set of all hats and stars v with $\pi(v) = i$. (If $n = 4$ then the sets S_i are all empty, by definition.) We read all these subscripts modulo $2n$; so for instance H_{2n+1} means H_1 and so on.

(1) *If $u, v \in V(G)$ are adjacent and $u \in V(C)$ then $d(u,v) \leq 3$.*

The proof is clear.

(2) *If $u, v \in V(G)$ are adjacent then $d(u,v) \leq 4$.*

For since $d(u,v) \leq n$, we may assume that $n \geq 5$; and by (1) we may assume that $u, v \notin V(C)$. If either of u, v is a hat, the claim follows from 1.3, since there is no hat-diagonal for C; so we assume that u, v both have at least three neighbours in C. Suppose first that u is a star, say $u \in S_i$ where i is odd. If v is adjacent to c_{i+1}, then

$$v \in S_{i-2} \cup C_{i-1} \cup S_i \cup C_{i+1} \cup S_{i+2} \cup C_{i+3} \cup S_{i+4},$$

and consequently $d(u,v) \leq 4$ as required. We assume then that v is nonadjacent to c_{i+1}, and similarly to c_{i-1}. Since $\{u, v, c_{i-3}, c_{i+1}\}$ is not a claw, it follows that v is adjacent to c_{i-3}, and similarly to c_{i+3}. Since v is not a hub, it follows that $n \leq 6$. If $n = 6$, then $v \in S_{i+6}$, and $\{u, v\}$ is a star-diagonal for C, a contradiction; while if $n = 5$, then $u \in C_{i+5}$, and u is a centre for the 5-hole v-c_{i-3}-c_{i-1}-c_{i+1}-c_{i+3}-v, a contradiction. We may therefore assume that u is not a star, and similarly v is not a star. They are therefore both clones; but then $d(u,v) \leq 4$ by 1.3. This proves (2).

(3) *If $u, v \in V(G)$ and $d(u,v) \leq 1$ then u, v are adjacent.*

For if one of $u, v \in V(C)$, this is clear, so we assume both are in $V(G) \setminus V(C)$. If $d(u,v) = 0$, then either u, v are both hats, or both clones, or both stars, or one is a star and one is a hat. In the first three cases, u, v are adjacent by 1.3, and in the last case, $n \geq 5$ and $\{u, v\}$ is a coronet for C, a contradiction. Thus if $d(u,v) = 0$ then the claim holds. We assume then that $u \in C_i$ and $v \in H_{i+1} \cup S_{i+1}$. But then u,v are adjacent by 1.3. This proves (3).

If $u,v,w \in V(G)$, let us say that v is *between* u,w if $d(u,w) < n$, and $d(u,v), d(v,w) > 0$, and $d(u,v) + d(v,w) = d(u,w)$ (and so $\phi(v)$ lies on the shorter circle arc between $\phi(u), \phi(w)$).

(4) *If $u,v,w \in V(G)$ and v is between u,w, and $d(u,w) \geq 4$, and u,w are adjacent, then v is adjacent to both u,w.*

For then $u,w \notin V(C)$ by (1), and $d(u,w) = 4$ by (2). Since $d(u,w) < n$ it follows that $n \geq 5$. It suffices from the symmetry to prove that u,v are adjacent, and we therefore assume they are nonadjacent, for a contradiction. Let $\pi(u) = i$. We may assume that $\pi(w) = i+4$, and $\pi(v) \in \{i+1, i+2, i+3\}$. By (3), $\pi(v) \neq i+1$, and so $\pi(v) = i+2$ or $i+3$. There are two cases, depending whether i is odd or even.

Suppose first that i is even. If $\pi(v) = i+2$, then v,w are nonadjacent, since $\{w,u,v,c_{i+6}\}$ is not a claw; but then $\{u,w\}$ is a hat-diagonal for the n-hole induced on $(V(C)\setminus\{c_{i+2}\}) \cup \{v\}$, a contradiction. If $\pi(v) = i+3$, then by (3), v,w are adjacent; since $\{w,u,v,c_{i+6}\}$ is not a claw, v is adjacent to c_{i+6} and hence to c_i; and then $\{u,v\}$ is a crown for the n-hole induced on $(V(C) \setminus \{c_{i+4}\}) \cup \{w\}$, a contradiction.

Now suppose that i is odd. Since u,w are adjacent and $n \geq 5$, it follows from 1.3 that u,w are both stars, and so $u \in S_i, w \in S_{i+4}$. If $n = 5$, then $\{u,w\}$ is a star-diagonal for C, a contradiction, and so $n \geq 6$. If $\pi(v) = i+2$, then by 1.3, $v \notin H_{i+2}$, and so $v \in S_{i+2}$; and then $\{u,v\}$ is a crown for C, a contradiction. If $\pi(v) = i+3$, then v,w are adjacent by (3), but then $\{w,u,v,c_{i+7}\}$ is a claw, a contradiction. This proves (4).

(5) *If $u,v,w \in V(G)$ and v is between u,w, and u,w are adjacent, then v is adjacent to both u,w.*

For by (4) we may assume that $d(u,w) \leq 3$, and by (3) the claim is true if $d(u,w) \leq 2$, so we may assume that $d(u,w) = 3$. Thus exactly one of $\pi(u), \pi(w)$ is even; and so from the symmetry we may assume that $\pi(u)$ is even, $\pi(u) = i$ say, and $\pi(w) = i+3$, and $\pi(v)$ is one of $i+1, i+2$. Thus $u \in C_i \cup \{c_i\}$ and $w \in H_{i+3} \cup S_{i+3}$, and $v \in H_{i+1} \cup S_{i+1} \cup C_{i+2} \cup \{c_{i+2}\}$. Suppose first that $v \in H_{i+1} \cup S_{i+1}$. Then by (3), u,v are adjacent, and so we may assume that v,w are nonadjacent. Now w is not adjacent to c_{i-2}, since $w \in H_{i+3} \cup S_{i+3}$ (and if $n = 4$ then $w \in H_{i+3}$, by definition of a star). Since $\{u,v,w,c_{i-2}\}$ is not a claw, v is adjacent to c_{i-2}, and therefore $v \in S_{i+1}$. Since v,w are nonadjacent, 1.3 implies that w is not a hat, and so $w \in S_{i+3}$; and then $\{v,w\}$ is a crown for C, a contradiction. Suppose now that $v \in C_{i+2} \cup \{c_{i+2}\}$. Since u,w are both adjacent to c_{i+2}, we may assume that $v \in C_{i+2}$. By (1), v,w are adjacent, and so we may assume that u,v are nonadjacent. If $w \in S_{i+3}$ then $n \geq 5$ and $\{w,u,v,c_{i+6}\}$ is a claw, a contradiction; and if $w \in H_{i+3}$ then $\{u,w\}$ is a hat-diagonal for the n-hole induced on $(V(C) \setminus \{c_{i+2}\}) \cup \{v\}$, a contradiction. This proves (5).

(6) *If $u,v \in V(G)$ are adjacent, then $d(u,v) < n$.*

By (2) we assume that $d(u,v) = n = 4$. From (1), $u,v \notin V(C)$, and since $n = 4$, there are no stars relative to C. Thus u,v are both hats or both clones. If they are both hats, then $\{u,v\}$ is a hat-diagonal for C, a contradiction. If they are both clones, say $u \in C_2, v \in C_6$, then v is a centre for the 4-hole u-c_4-c_6-c_8-u, a contradiction. This proves (6).

Fix an orientation of Σ, called clockwise. For each edge uv of G, if $\phi(u) \neq \phi(v)$ then by (6), one of the arcs of Σ between $\phi(u), \phi(v)$ has length $< n$ (and therefore the other has length $> n$). We call this shorter arc the *route* between $\phi(u), \phi(v)$. Direct uv from u to v if moving clockwise from u to v traverses the route between u, v; and otherwise direct uv from v to u. For each $\sigma \in \Sigma$, direct the edges of G with both ends in $\phi^{-1}(\sigma)$ in such a way that there is no directed cycle with vertex set in $\phi^{-1}(\sigma)$. Let D be the orientation of G just defined.

(7) *There is no directed cycle in D of length ≤ 3.*

For certainly there is none of length < 3. Suppose that a_1-a_2-a_3-a_1 is a directed cycle of length 3. Not all of $\phi(a_1), \phi(a_2), \phi(a_3)$ are equal, and so we may assume that $\phi(a_2), \phi(a_3) \neq \phi(a_1)$. Thus moving clockwise along Σ from $\phi(a_1)$ to $\phi(a_2)$, and then moving clockwise from $\phi(a_2)$ to $\phi(a_3)$, traverses the route between $\phi(a_1)$, $\phi(a_2)$ and then traverses the route between $\phi(a_2)$, $\phi(a_3)$. Since both these arcs have length $< n$ by (6), it follows that $\phi(a_2), \phi(a_3)$ are different, and $\phi(a_1), \phi(a_2), \phi(a_3)$ are in clockwise order, and the three routes joining pairs of them have union Σ. Since a_1, a_2 are adjacent, it follows that $d(a_1, a_2) \leq 4$ by (2); and similarly $d(a_2, a_3), d(a_3, a_1) \leq 4$. Since the three routes have union the whole circle of length $2n$, it follows $2n \leq 12$, and so $n \leq 6$.

Suppose first that $n = 6$. Then equality holds throughout. Hence we may assume that either $\pi(a_1) = 1$, $\pi(a_2) = 5$ and $\pi(a_3) = 9$, or $\pi(a_1) = 2$, $\pi(a_2) = 6$ and $\pi(a_3) = 10$. In the first case, 1.3 implies that none of a_1, a_2, a_3 is a hat, and so $\{a_1, a_2, a_3\}$ is a star-triangle for C, a contradiction. In the second case, a_3-c_{12}-c_2-c_4-c_6-c_8-a_3 is a 6-hole, and $\{a_1, a_2\}$ is a star-diagonal for it, a contradiction. Thus $n < 6$.

Suppose next that $n = 5$. Then the sum of $d(a_1, a_2)$, $d(a_2, a_3)$ and $d(a_3, a_1)$ is at least 10, and so we may assume that $d(a_1, a_2) = 4$. Suppose that a_1 is a star, say $a_1 \in S_1$; then we may assume that $a_2 \in H_5 \cup S_5$. By 1.3, $a_2 \notin H_5$, and if $a_2 \in S_5$ then $\{a_1, a_2\}$ is a star-diagonal for C, a contradiction. Thus a_1 is not a star, and similarly a_2 is not a star. Not both a_1, a_2 are hats, since $\{a_1, a_2\}$ is not a hat-diagonal for C. Thus we may assume that $a_1 \in C_2 \cup \{c_2\}$, and $a_2 \in C_6 \cup \{c_6\}$, and therefore

$$c_3 \in C_8 \cup \{c_8\} \cup H_9 \cup S_9 \cup C_{10} \cup \{c_{10}\}.$$

Since a_1, a_2 are adjacent, $a_1 \neq c_2$ and $a_2 \neq c_6$. Now $a_3 \neq c_8$ since c_8 is not adjacent to a_1, and similarly $a_3 \neq c_{10}$. If a_3 is nonadjacent to c_2, then a_1 is a centre for the 5-hole a_2-a_3-c_{10}-c_2-c_4-a_2, a contradiction. Thus a_3 is adjacent to c_2 and similarly to c_6, and so $a_3 \in S_9$. But then $\{a_2, a_3\}$ is a star-diagonal for the 5-hole a_1-c_4-c_6-c_8-c_{10}-a_1, a contradiction. Thus $n \neq 5$.

Consequently $n = 4$. By (6), we may assume that $a_1 \in C_2 \cup \{c_2\}$, $a_2 \in H_5$, and $a_3 \in H_7 \cup C_8 \cup \{c_8\}$. Then a_2 is a centre for the 4-hole a_1-c_4-c_6-a_3-a_1, a contradiction. This proves (7).

We claim that (ϕ, D) is a circle arrangement of G. The first axiom holds by (3), the second by (7), the third by (5), and the fourth by (6). Hence (ϕ, D) is a circle arrangement of G. Moreover, C is a basis for (ϕ, D), and the result follows from 2.1. This proves 3.1. ■

4 Circular interval graphs

In this section we derive 1.1 from 3.1. We say that G is a *linear interval graph* if the vertices of G can be numbered $v_1, \dots, v_n$ such that for all i, j with $1 \le i < j \le n$, if v_i is adjacent to v_j then $\{v_i, v_{i+1}, \dots, v_j\}$ is a clique. (Linear interval graphs are a subclass of circular interval graphs, as may easily be checked.) We start with the following lemma.

4.1 *Let G be a graph with no hole, and with no induced subgraph which is a claw, trinity or antitrinity. Then G is a linear interval graph.*

Proof. If every component of G is a linear interval graph, then so is G; and we may therefore assume that G is connected and $|V(G)| > 1$. Since G has no hole, it is a chordal graph, and therefore (by a well-known folklore theorem) there is a tree T and a family $(T_v : v \in V(G))$ of subtrees of T such that for all distinct $u, v \in V(G)$, $V(T_u \cap T_v) \neq \emptyset$ if and only if u, v are adjacent in G. Choose such a tree T with $|V(T)|$ minimum.

(1) *For every edge st of T, there exist $u, v \in V(G)$ such that T_u contains s and not t, and T_v contains t and not s.*

For let e be the edge st. Let T' be the tree obtained from T by contracting the edge st. For each $v \in V(G)$, let $T'_v = T_v$ if $s, t \notin V(T_v)$, and otherwise let S be the subtree of T with edge-set $E(T_v) \cup \{e\}$, and let T'_v be the subtree of T' obtained from S by contracting e. For all distinct $u, v \in V(G)$, if u, v are adjacent then $V(T_u \cap T_v) \neq \emptyset$ and so $V(T'_u \cap T'_v) \neq \emptyset$; and hence from the minimality of $|V(T)|$, the converse is false. In other words, there exist nonadjacent $u, v \in V(G)$ such that $V(T_u \cap T_v) = \emptyset$ and $V(T'_u \cap T'_v) \neq \emptyset$. This proves (1).

(2) *If T is a path then the theorem holds.*

For let T be a path, with vertices $t_1, \dots, t_m$ in order. For each $v \in V(G)$, T_v is a nonempty subpath of T, say from $t_{a(v)}$ to $t_{b(v)}$, where $1 \le a(v) \le b(v) \le m$. For $u, v \in V(G)$, write $u \le v$ if $a(u) \le a(v)$ and $b(u) \le b(v)$. We claim that for all $u, v \in V(G)$, either $u \le v$ or $v \le u$. For suppose not; then we may assume that $a(u) < a(v) \le b(v) < b(u)$. By (1), there exist $x, y \in V(G)$ such that T_x contains $t_{a(u)}$ and not $t_{a(u)+1}$, and T_y contains $t_{b(u)}$ and not $t_{b(u)-1}$. But then $\{u, v, x, y\}$ is a claw in G, a contradiction. This proves our claim. Hence we may number the vertices of G as $v_1, \dots, v_n$ such that $v_i \le v_j$ for $1 \le i < j \le n$. Now suppose that $1 \le i < k \le n$ and v_i, v_k are adjacent. It follows that

$$a(v_i) \le a(v_k) \le b(v_i) \le b(v_k);$$

choose h with $a(v_k) \le h \le b(v_i)$. We claim that $t_h \in V(T_{v_j})$ for all j with $i \le j \le k$. For $v_i \le v_j$ implies that $b(v_i) \le b(v_j)$, and $v_j \le v_k$ implies that $a(v_j) \le a(v_k)$. Since $a(v_k) \le h \le b(v_i)$, this proves that $t_h \in V(T_{v_j})$. Hence $\{v_i, \dots, v_k\}$ is a clique, and so G is a linear interval graph. This proves (2).

In view of (2), we may assume that there is a vertex t_0 of T with d neighbours $t_1, \dots, t_d$ in T, where $d \ge 3$. Let Y be the set of all $v \in V(G)$ such that $t_0 \in V(T_v)$; thus, Y is a clique of G. For $1 \le i \le d$, let X_i be the set of all $v \in V(G)$ such that $V(T_v)$ contains t_i and not t_0, and let Y_i be the set of all $v \in Y$ with $t_0, t_i \in V(T_v)$. By (1), $X_i \neq \emptyset$, for $1 \leq i \leq d$; choose $x_i \in X_i$. Since $t_i \in V(T_{x_i})$, it follows that every member of Y_i is adjacent to x_i.

(3) *For $1 \leq i \leq d$, $\emptyset \neq Y_i \neq Y$.*

For suppose that $Y_1 = \emptyset$ say. Let S_1, S_2 be the two components of $T \setminus e$, where e is the edge t_0t_1. For $j = 1, 2$, let A_j be the set of all $v \in V(G)$ such that $V(T_v) \subseteq V(S_j)$; then since $Y_1 = \emptyset$, $A_1 \cup A_2 = V(G)$. Since each T_v has at least one vertex, $A_1 \cap A_2 = \emptyset$. Moreover, by (1) $A_1, A_2 \neq \emptyset$; and if $v_1 \in A_1$ and $v_2 \in A_2$ then $V(T_{v_1} \cap T_{v_2}) = \emptyset$, and so v_1, v_2 are nonadjacent. It follows that G is not connected, a contradiction. This proves that $Y_i \neq \emptyset$, for $1 \leq i \leq d$. Moreover, by (1) there exists $v \in V(G)$ such that T_v contains t_0 and not t_i, and therefore $v \in Y \setminus Y_i$; and consequently $Y_i \neq Y$ for $1 \leq i \leq d$. This proves (3).

(4) *For $1 \leq i < j \leq d$, either $Y_i \cap Y_j = \emptyset$ or $Y_i \cup Y_j = Y$; and for $1 \leq i < j < k \leq d$, $Y_i \cap Y_j \cap Y_k = \emptyset$.*

For if $y \in Y_i \cap Y_j$ and $y' \in Y \setminus (Y_i \cup Y_j)$, then $\{y, y', x_i, x_j\}$ is a claw, a contradiction; and if $y \in Y_i \cap Y_j \cap Y_k$, then $\{y, x_i, x_j, x_k\}$ is a claw, a contradiction. This proves (4).

If there exist i, j, k with $1 \leq i < j < k \leq d$ such that Y_i, Y_j, Y_k are pairwise disjoint, then the subgraph induced on $\{y_i, y_j, y_k, x_i, x_j, x_k\}$ is a trinity (where $y_i \in Y_i$, $y_j \in Y_j$ and $y_k \in Y_k$), a contradiction. Since $d \geq 3$, we may therefore assume that $Y_1 \cap Y_2 \neq \emptyset$. Choose $y_3 \in Y_1 \cap Y_2$. By (4), $Y_1 \cup Y_2 = Y$. Suppose that $Y_1 \cap Y_3 = \emptyset$. Then $Y_3 \subseteq Y_2$, and since $Y_3 \neq \emptyset$ by (3), it follows that $Y_2 \cap Y_3 \neq \emptyset$. By (4), $Y_2 \cup Y_3 = Y$, and since $Y_3 \subseteq Y_2$, we deduce that $Y_2 = Y$, contrary to (3). This proves that there exists $y_2 \in Y_1 \cap Y_3$, and similarly there exists $y_1 \in Y_2 \cap Y_3$. Thus y_1 is adjacent to x_2, x_3; and not to x_1, since $y_1 \notin Y_1$ by (4). Similarly y_i is not adjacent to x_i for $i = 2, 3$, and so the subgraph induced on $\{y_i, y_j, y_k, x_i, x_j, x_k\}$ is an antitrinity, a contradiction. This proves 4.1. ■

Now we prove 1.1, which we restate:

4.2 *Let G be a graph. Then G is a circular interval graph if and only if no induced subgraph is a claw, trinity, antitrinity or $(1,1,1)$-prism, and every hole is dominating and has no centre.*

Proof. The "only if" statement is easy, and we omit it. For the "if" part, let G be a graph such that no induced subgraph is a claw, trinity, antitrinity or $(1,1,1)$-prism, and every hole in G is dominating and has no centre. By 4.1, we may assume that G has a hole. Let the longest hole in G have length n, and let C_0 be some such longest hole.

(1) *There is a metacircular arrangement (ϕ, D) of G with basis C_0.*

For by 3.1, it suffices to show that every hole of length n is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Let C be a hole of length n, with vertices c_1-$\cdots$-c_n-c_1 say. By hypothesis, C is dominating and has no centre. If it has a hub v, then there are two holes with vertex set in $V(C) \cup \{v\}$ different from C, and both are nondominating, a contradiction. Similarly C has no coronet.

Suppose that there is a crown for C; then $n \geq 5$ and there are nonadjacent stars s_1, s_2, adjacent say to $c_1, \ldots, c_4$ and to $c_2, \ldots, c_5$ respectively. If $n \geq 6$, the hole s_1-c_2-s_2-c_4-s_1 is not dominating, since c_6 has no neighbour in it. If $n = 5$, then the subgraph induced on $\{c_1, c_2, c_4, c_5, s_1, s_2\}$ is a $(1, 1, 1)$-prism, a contradiction. Thus there is no crown for C.

Suppose that there is a hat-diagonal for C; that is, there are adjacent hats h_1, h_2, where h_1 is adjacent to c_1, c_2 say, and h_2 is adjacent to c_i, c_{i+1}, for some i with $3 \leq i < n$. If $i \geq 4$ then the hole h_1-h_2-c_{i+1}-c_{i+2}-$\cdots$-c_1-h_1 is not dominating, since c_3 has no neighbour in it, a contradiction. Thus $i = 3$ and similarly $i + 1 = n$, and so $n = 4$. But then the subgraph induced on $V(C) \cup \{h_1, h_2\}$ is a $(1, 1, 1)$-prism, a contradiction. Thus there is no hat-diagonal for C.

Suppose that there is a star-diagonal for C; that is, $n = 5$ or 6, and there are adjacent stars s_1, s_2 adjacent respectively to c_{n-1}, c_n, c_1, c_2 and c_2, c_3, c_4, c_5 say. Then s_2 is a centre for the $(n-1)$-hole s_1-c_2-c_3-$\cdots$-c_{n-1}-s_1, a contradiction. Finally, suppose that there is a star-triangle for C; that is, $n = 6$ and there are three pairwise adjacent stars s_1, s_2, s_3 adjacent respectively to $c_1, \ldots, c_4$, to $c_3, \ldots, c_6$ and to c_5, c_6, c_1, c_2. Then s_2 is a centre for the 4-hole s_1-c_4-c_5-s_3-s_1, a contradiction. Thus the hypotheses of 3.1 are satisfied, and consequently this proves (1).

Let (ϕ, D) be as in (1). Suppose that $|\phi^{-1}(\sigma_1)| > 1$ for some $\sigma_1 \in \Sigma$. From the fourth axiom for a metacircular arrangement, there exists $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$, such that $\phi^{-1}(\sigma_1)$ is neither complete nor anticomplete to $\phi^{-1}(\sigma_2)$. Let $X_i = \phi^{-1}(\sigma_i)$ $(i = 1, 2)$. Consequently, from the third axiom for a metacircular arrangement, every vertex in X_1 has both a neighbour and a nonneighbour in X_2, and vice versa. Choose $u_1 \in X_1$ with as many neighbours in X_2 as possible. Choose $v_2 \in X_2$ nonadjacent to u_1. Let $v_1 \in X_1$ be a neighbour of v_2. Since v_1 does not have more neighbours in X_2 than u_1, and v_1 is adjacent to v_2 and u_1 is not, it follows that there is a neighbour $u_2 \in X_2$ of u_1 that is nonadjacent to v_1. Since X_1, X_2 are cliques by the first axiom, it follows that u_1-u_2-v_2-v_1-u_1 is a hole, C say. We may assume that in D, all edges between X_1 and X_2 have tail in X_1. By hypothesis, C has no centre, and so there does not exist $\sigma_3 \in \Sigma$ different from σ_1, σ_2 and such that $\sigma_1, \sigma_3, \sigma_2$ are in clockwise order. Since σ_2 is the only point $\sigma \in \Sigma$ such that $\phi^{-1}(\sigma_1)$ is neither complete nor anticomplete to $\phi^{-1}(\sigma)$, it follows that every $v \in V(G) \setminus (X_1 \cup X_2)$ is either complete or anticomplete to X_1, and similarly is either complete or anticomplete to X_2. But no such v is complete to both X_1, X_2, since C has no centre, and no v is anticomplete to both X_1, X_2 since C is dominating. For $i = 1, 2$, let A_i be the set of all $v \in V(G) \setminus (X_1 \cup X_2)$ that are complete to X_i; then A_1, A_2, X_1, X_2 are pairwise disjoint and have union $V(G)$, and A_1 is complete to X_1 and anticomplete to X_2, and A_2 is complete to X_2 and anticomplete to X_1. Since G contains no $(1, 1, 1)$-prism, it follows that A_1 is anticomplete to A_2. Since G is claw-free, and A_1 is complete to $\{u_1\}$ and anticomplete to $\{u_2\}$, it follows that A_1 is a clique, and similarly so is A_2. Hence for $i = 1, 2$, and for every vertex $v \in A_i$, all neighbours of v are pairwise adjacent, and therefore v belongs to no hole. We recall that C_0 is a basis for (ϕ, D). It follows that $V(C_0) \subseteq X_1 \cup X_2$, which is impossible since $\phi(v)$ $(v \in V(C_0))$ are all different. This proves that there is no such σ_1.

We have therefore proved that $|\phi^{-1}(\sigma_1)| \leq 1$ for all $\sigma_1 \in \Sigma$. Consequently G is a circular interval graph. This proves 4.2. ■

References

[1] Maria Chudnovsky and Paul Seymour, "A structure theorem for claw-free perfect graphs", manuscript, June 2003.