Claw-free graphs I. Clique cutsets.

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Abstract

In this paper we give a full description of the class of claw-free graphs that admits a special kind of a clique cutset (called an internal clique cutset). This is a step toward the main goal of this series, to find a structural characterization of all claw-free graphs.

1 Introduction

All graphs in this paper are finite and simple. For a graph G and a set $X \subseteq V(G)$ we denote by $G|X$ the subgraph of G *induced* on X (that is the subgraph of G with vertex set X and edge set all edges of G with both ends in X.) We say that $X \subseteq V(G)$ is a *claw* if $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. G is *claw-free* if no subset of $V(G)$ is a claw. This is the first paper in a series of four,[1], [2], [3], the objective of which is to prove that every claw-free graph can be built starting from basic pieces, and combining them together in prescribed ways. In order to do that, we first prove a "decomposition theorem" for claw-free graphs, namely we prove that if G is claw-free, then either G belongs to one of a few basic classes, or it admits a certain kind of decomposition.

A *clique* in a graph G is a subset of $V(G)$ every two vertices of which are adjacent.A *clique cutset* in G is a clique C such that $G \setminus C$ is not connected. Two adjacent vertices of G are called *twins* if they have the same neighbors in G (apart from each other.) We say that a clique cutset C is *internal* if $V(G) \setminus C$ can be partitioned into two sets V_1, V_2 such that there are no edges between V_1 and V_2 and each V_i contains at least two vertices that are not twins. It turns out that claw-free graphs that admit an internal clique cutset are easier to handle than the general case (the basic classes and the types of decomposition needed are more restricted), and on the other hand, understanding them is useful in the remainder of the proof. The purpose of this paper is to prove the decomposition theorem for claw-free graphs admitting an internal clique cutset.

2 Definitions and the main theorem

We start with some definitions. A hole is an induced cycle of length at least 4. For a graph G and a vertex $v \in V(G)$ we denote by $N(x)$ the set of neighbors of x in G. For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$ we say that a is *complete* to B if a is adjacent to every vertex in B; and that a is *anticomplete* to B if a has no neighbor in B. For two disjoint subsets A and B of $V(G)$ we say that A is *complete*, respectively *anticomplete*, to B if every vertex in A is complete, respectively anticomplete, to B.

Let us now describe the basic classes we use.

- **The antitrinity** An antitrinity is a graph on six vertices

$$\{y_1, y_2, y_3, x_1, x_2, x_3\}$$

with the edge set

$$\{y_1y_2, y_2y_3, y_1y_3, x_1y_2, x_1y_3, x_2y_1, x_2y_3, x_3y_1, x_3y_2\}.$$

One of our basic classes in the decomposition theorem for all claw-free graphs is the class of line graphs. We therefore remark that an antitrinity is the linegraph of the graph with vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and edge set $\{a_1a_2, a_2a_3, a_1a_3, a_1b_1, a_2b_2, a_3b_3\}$.

- **Linear interval graphs.** Let I be an interval and let $F_1, \ldots, F_k$ be subintervals of I. Let V be a finite subset of I, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only $v_1, v_2 \in F_i$ for some i. Such a graph is called a *linear interval graph.* Linear interval graphs are a subclass of the class of *circular interval graphs* that appear in later papers in the series.

The complement of the antitrinity is a *trinity*. The trinity and the antitrinity will be of importance in [1].

Next, let us explain the decompositions that we use. The first is just that there are two vertices in G that are twins, or briefly, "G admits twins". For the second, let A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. Let (A, B) be a homogeneous pair, such that A, B are both cliques, and A is neither complete nor anticomplete to B. In these circumstances we call (A, B) a *W-join*. The pair (A, B) is *coherent* if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a clique. In some applications [4], coherent W-joins are easier to handle than general W-joins, and it turns out that throughout the proof in this paper, in every instance where we exhibit a W-join, it is coherent.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are nonempty and there are no edges between V_1 and V_2. We call the pair (V_1, V_2) a *0-join* in G. Thus G admits a 0-join if and only if it is not connected.

Next, suppose that V_1, V_2 partition $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- for $i = 1, 2$, A_i is a clique, and $A_i, V_i \setminus A_i$ are both nonempty
- A_1 is complete to A_2
- every edge between V_1 and V_2 is between A_1 and A_2.

In these circumstances, we say that (V_1, V_2) is a *1-join*.

We can now state our main result:

2.1 *Let G be a claw-free graph admitting a internal clique cutset. Then either G is s linear interval graph or G is the antitrinity, or G admits twins, or a 0-join, or a 1-join, or a coherent W-join.*

3 Internal clique cutsets

In this section we prove 2.1. We divide the proof into several lemmas.

3.1 *Let G be a claw-free graph and let C be a minimal clique cutset in G. Let V_1, V_2 be a partition of $V(G) \setminus C$ such that there are no edges between V_1 and V_2. Let $u, v \in C$. Then*

1. *for all $u \in C$ both $N(u) \cap V_1$ and $N(u) \cap V_2$ are cliques and*
2. *either $N(u) \cap V_1 \subseteq N(v) \cap V_1$ or $N(u) \cap V_2 \subseteq N(v) \cap V_2$.*

Proof.

1. Suppose for some vertex $u \in C$ there exist two non-adjacent vertices x, y in $N(u) \cap V_1$. By the minimality of C, there exists a vertex $x \in N(u) \cap V_2$. But now $\{u, x, y, z\}$ is a claw in G, a contradiction. This proves the first assertion of the theorem.

2. Assume for a contradiction that the second assertion of the theorem is false. Then there exist a vertex $v_1 \in (N(u) \setminus N(v)) \cap V_1$ and a vertex $v_2 \in (N(u) \setminus N(v)) \cap V_2$ Since C is a clique, u is adjacent to v. But then $\{u, v, v_1, v_2\}$ is a claw, a contradiction. This proves the second assertion of the theorem and completes the proof of 3.1. ▮

3.2 *Let G be a connected claw-free graph admitting an internal clique cutset C. If G contains a hole of length at least four, then either G admits twins or a 0-join or 1-join or a coherent W-join .*

Proof. We may assume no two vertices of G are twins. Let C be a clique cutset, and V_1, V_2 a partition of $V(G) \setminus C$, with no edges between V_1 and V_2 and such that each of V_1, V_2 contains at least two vertices. Since C is a clique, no hole of length at least four contains both a vertex of V_1 and a vertex of V_2. Choose a clique cutset C and a partition V_1, V_2 of $V(G) \setminus C$, such that

1. $G|(V_1 \cup C)$ contains a hole of length at least four,
2. $|V_2| \geq 2$, and
3. $|V_1 \cup C|$ minimum subject to (1) and (2).

We may assume that C is a minimal cutset in G. For a vertex $c \in C$ let $N_1(c)$ be the set of neighbors of c in V_1 and $N_2(c)$ the set of neighbors of c in V_2.

(1) *Let h_1-h_2-h_3-h_4-h_1 is a hole in G such that $h_1, h_2 \in V_1$ and $h_3, h_4 \in C$. Assume also that for every two vertices b_1, b_2 in C such that $N_2(b_1) = N_2(b_2) = N_2(h_3)$ the set $N_1(b_1) \cup N_1(b_2)$ is a clique. Then G admits a coherent W-join.*

A *strip* in G means a pair (A, B) of disjoint subsets of $V(G)$, such that

- A, B are nonempty cliques
- every vertex in A has a neighbor in B and vice versa.

Let (A, B) be a strip. A *step* (in this strip) means a hole a_1-a_2-b_2-b_1-a_1 where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. We say the strip is *step-connected* if for every partition (X, Y) of A or of B with $X, Y \neq \emptyset$, there is a step meeting both X, Y.

Thus $(\{h_1, h_2\}, \{h_3.h_4\})$ is a step connected strip in G. By 3.1 $N_2(h_3) = N_2(h_4)$, and therefore we may choose a strip (A, B) such that

- (A, B) is step-connected, and $h_1, h_2 \in A$ and $h_3, h_4 \in B$,
- $A \subseteq V_1$,
- $B \subseteq C$,
- $N_2(b) = N_2(h_3)$ for all $b \in B$,
- $A \cup B$ is maximal.

We claim that (A, B) is a coherent W-join in G. We need to show that every vertex of $V(G) \setminus (A \cup B)$ with a neighbor in A is complete to A and same for B; and that the set of $A \cup B$-complete vertices in $V(G) \setminus (A \cup B)$ is a clique.

Assume first there is a vertex $v \in V(G) \setminus (A \cup B)$ with a neighbor and a non-neighbor in A. Let A_1 be the set of neighbors of v in A, and let $A_2 = A \setminus A_1$. So both A_1 and A_2 are non-empty. Since (A, B) is step-connected, there exists a step a_1-a_2-b_2-b_1-a_1 with $a_1 \in A_1$ and $a_2 \in A_2$. Since

$\{a_1, b_1, a_2, v\}$ is not a claw in G, v is adjacent to b_1. Suppose v is adjacent to b_2. Since a_1 belongs to $N_1(v) \setminus N_1(b_2)$ and a_2 belongs to $N_1(b_2) \setminus N_1(v)$, it follows from 3.1 that $N_2(v) = N_2(b_2) = N_2(h_3)$, and in particular $v \in C$. Now if we add v to B, then a_1-a_2-v-b_2-a_1 is a step of the strip $(A, B \cup \{v\})$, showing that this strip is step-connected, contrary to the maximality of $A \cup B$. This proves that v is non-adjacent to b_2. But now v belongs to V_1 (for it has a neighbor $a_1 \in V_1$ and a non-neighbor $b_2 \in C$); and a_2, v are two non-adjacent vertices in $N_1(b_1) \cup N_1(b_2)$, which is a contradiction since $b_1, b_2 \in B$ and $N_2(b_1) = N_2(b_2) = N_2(h_3)$. This proves that every vertex with a neighbor in A is complete to A.

Next assume that $v \in V(G) \setminus (A \cup B)$ with a neighbor and a non-neighbor in B. Let B_1 be the set of neighbors of v in B, and let $B_2 = B \setminus B_1$. So both B_1 and B_2 are non-empty. Since (A, B) is step-connected, there exists a step a_1-a_2-b_2-b_1-a_1 with $b_1 \in B_1$ and $b_2 \in B_2$. Then, since $\{b_1, a_1, b_2, v\}$ is not a claw in G, v is adjacent to a_1. v has a neighbor a_1 in V_1 and a non-neighbor b_2 in C, so v belongs to V_1. Suppose v is adjacent to a_2. If we add v to A, then v-a_1-b_2-b_2-v is a step of the strip $(A \cup \{v\}, B)$, showing that this strip is step-connected, contrary to the maximality of $A \cup B$. This proves that v is non-adjacent to a_2. But now a_2, v are two non-adjacent vertices in $N_1(b_1) \cup N_1(b_2)$, which is a contradiction since $b_1, b_2 \in B$ and $N_2(b_1) = N_2(b_2) = N_2(h_3)$. This proves that every vertex with a neighbor in B is complete to B.

Finally, let X be the set of $A \cup B$-complete vertices in $V(G) \setminus (A \cup B)$. Since $A \subseteq V_1$, it follows that $X \subseteq V_1 \cup C$. Suppose there exist two non-adjacent vertices a, b in X. By 3.1, the set of B-complete vertices in V_1 is a clique. Consequently, since C is a clique, we may assume exchanging a and b if necessary, that $a \in V_1$ and $b \in C$. Since $h_1 \in N_1(b) \setminus N_1(h_3)$ and $a \in N_1(h_3) \setminus N_1(b)$, it follows from 3.1 that $N_2(b) = N_2(h_3)$. If we add a to A and b to B, then a-h_1-b-h_3-a is a step of the strip $(A \cup \{a\}, B \cup \{b\})$, showing that this strip is step-connected, contrary to the maximality of $A \cup B$. So X is a clique and (A, B) is a coherent W-join in G. This proves (1).

(2) *If there exist two vertices c_1 and c_2 in C with $N_2(c_1) \neq N_2(c_2)$ and such that $N_2(c_1)$ and $N_2(c_2)$ are both minimal, then G admits a 1-join or a coherent W-join*

We may assume that G admits no coherent W-join. For $u, v, w \in C$ (not necessarily distinct) we say that v is *between* u and w if $N_2(u) \not\subseteq N_2(v)$ and $N_2(v) \not\subseteq N_2(w)$.

Let X be a subset of C containing c_1, such that if v is between u and w and $u, w \in X$ then $v \in X$, and minimal with these properties. Now c_2 is between c_1 and c_1, so c_2 is in X. If $y \in C$ is a vertex with $N_2(y)$ minimal, then either $N_2(y) \neq N_2(c_1)$ or $N_2(y) \neq N_2(c_2)$. In the first case y is in X because it is between c_1 and c_1, and in the second case y is in X because it is between c_2 and c_2. So X contains every vertex y of C with $N_2(y)$ minimal.

Let $N_1(c_1) = N$. It follows from 3.1 that N is a clique. We claim that $N_1(y) = N$ for every $y \in X$. To prove that, it is enough to show that if y is between x and z then $N_1(x) \subseteq N_1(y)$ and $N_1(y) \subseteq N_1(c_z)$. The first inclusion follows from 3.1 because $N_2(x) \not\subseteq N_2(y)$ and the second follows from 3.1 because $N_2(y) \not\subseteq N_2(z)$.

Next we claim that $N_1(c) \subseteq N$ for every $c \in C \setminus X$. Since X contains all vertices $y \in C$ with $N_2(y)$ minimal, there exists $x \in X$ such that $N_2(x)$ is a proper subset of $N_2(c)$. But now, by 3.1, $N_1(c) \subseteq N_1(x) = N$ and the claim follows.

Let H be a hole of length at least four contained in $V_1 \cup C$. We now show that $V(H) \subseteq V_1$. For suppose $V(H) \cap C \neq \emptyset$. Since C is a clique, $|V(H) \cap C| \leq 2$ and if $|V(H) \cap C| = 2$, the two vertices of H in C are adjacent. If H is a hole of length at least five, or H is a hole of length four and $|V(H) \cap C| = 1$, then V_1 contains two non-adjacent vertices both with neighbors in C. On the other hand, we have shown that for all $c \in C$ we have $N_1(c) \subseteq N$, and N is a clique, a contradiction. This proves that H is a hole of length four, with two adjacent vertices in V_1 and two adjacent vertices in C. Now since $N_1(c) \subseteq N_1(x) = N$ for all $c \in C$ and N is a clique, it follows from (1) that G admits a coherent W-join, a contradiction. This proves that V_1 contains a hole of length at least four.

Now N is another clique cutset in G, and $V_1' = V_1 \setminus N$, and $V_2' = V_2 \cup C$ is a partition of $V(G) \setminus N$ with no edges between V_1' and V_2' , and such that$|V_2'| \geq 2$ and $G|(V_1' \cup N) = G|V_1$ contains a hole of length at least four. But $|V_1' \cup N| = |V_1| < |V_1 \cup C|$, contrary to the choice of C, V_1, V_2. This proves (2).

Let x be a vertex in C with $N_2(x)$ minimal. Let $X = \{x' \in C \ : \ N_2(x') = N_2(x)\}$. It follows from (1) that $N_2(x) \subseteq N_2(c)$ for all $c \in C$. Let $N = \bigcap_{x' \in X} N_1(x')$. By 3.1 $N_1(c) \subseteq N$ for every vertex $c \in C \setminus X$ and N is a clique.

Let H be a hole of length at least four contained in $V_1 \cup C$. We claim that $V(H) \subseteq V_1 \cap X$. For suppose $V(H) \cap (C \setminus X) \neq \emptyset$. Since C is a clique, $|V(H) \cap C| \leq 2$ and if $|V(H) \cap C| = 2$, the two vertices of H in C are adjacent. First assume that $V(H) \cap (C \setminus X) = V(H) \cap C$. If H is a hole of length at least five or H is a hole of length four and $|V(H) \cap C| = 1$ then V_1 contains two non-adjacent vertices both with neighbors in $C \cap V(H)$. But $N_1(c) \subseteq N$ for all $c \in C \setminus X$ and N is a clique, so H is a hole of length four, say h_1-h_2-h_3-h_4-h_1, and $V(H) \cap C = V(H) \cap (C \setminus X) = \{h_3, h_4\}$. We claim that for $b_1, b_2 \in C$ with $N_2(b_1) = N_2(b_2) = N_2(h_3)$ the set $N_1(b_1) \cup N_1(b_2)$ is a clique. Indeed, since h_3 belongs to $C \setminus X$, so do both b_1, b_2 and $N_1(b_1) \cup N_1(b_2)$ is a subset of N which is a clique. But now G admits a coherent W-join by (1), a contradiction. This proves that $V(H) \cap (C \setminus X) \neq V(H) \cap C$. So $V(H) \cap C = \{u, v\}$ such that $u \in X$ and $v \in C \setminus X$. But now since H is a hole, there exists a vertex $w \in V_1$ adjacent to v and not to u, contrary to the fact that $N_1(v) \subseteq N \subseteq N_1(u)$. This proves that $V(H) \subseteq V_1 \cap X$.

So $N \cup X$ is a clique in G, $V_1' = V_1 \setminus N$ and $V_2' = V_2 \cup C \setminus X$ is a partition of $V(G) \setminus (N \cup X)$ with no edges between V_1' and V_2', $|V_2'| \geq 2$. By the choice of C, V_1, V_2 it follows that $|V_1' \cup X \cup N| = |V_1 \cup C|$, so $C = X$. So $X \cup N_2(x)$ is a clique, and the only edges between $V_1 \cup C$ and V_2 are between C and $N_2(x)$. Since no two vertices of G are twins and $|V_2| \geq 2$, it follows that $V_2 \setminus N_2(x)$ is non-empty, and so $(V_1 \cup C, V_2)$ is a 1-join in G. This proves 3.2. ▮

3.3 *Let G be a claw-free graph with no hole. Then either*

- *G is a linear interval graph, or*
- *G is an antitrinity, or*
- *G admits twins, or*
- *G admits a 0-join, or*
- *G admits a 1-join and contains an induced subgraph isomorphic to the trinity or the antitrinity.*

Proof. Since a disconnected graph admits a 0-join, we may assume that G is connected and $|V(G)| > 1$. We may also assume that no two vertices of G are twins. Since G has no hole, it is a chordal graph, and therefore (by a well-known folklore theorem) there is a tree T and a family

$(T_v : v \in V(G))$ of subtrees of T such that for all distinct $u, v \in V(G)$, $V(T_u \cap T_v) \neq \emptyset$ if and only if u, v are adjacent in G. Choose such a tree T with $|V(T)|$ minimum.

(1) *For every edge st of T, there exist $u, v \in V(G)$ such that T_u contains s and not t, and T_v contains t and not s.*

For let e be the edge st. Let T' be the tree obtained from T by contracting the edge st. For each $v \in V(G)$, let $T'_v = T_v$ if $s, t \notin V(T_v)$, and otherwise let S be the subtree of T with edge-set $E(T_v) \cup \{e\}$, and let T'_v be the subtree of T' obtained from S by contracting e. For all distinct $u, v \in V(G)$, if u, v are adjacent then $V(T_u \cap T_v) \neq \emptyset$ and so $V(T'_u \cap T'_v) \neq \emptyset$; and hence from the minimality of $|V(T)|$, the converse is false. In other words, there exist nonadjacent $u, v \in V(G)$ such that $V(T_u \cap T_v) = \emptyset$ and $V(T'_u \cap T'_v) \neq \emptyset$. This proves (1).

(2) *If T is a path then the theorem holds.*

For let T be a path, with vertices $t_1, \ldots, t_m$ in order. For each $v \in V(G)$, T_v is a nonempty subpath of T, say from $t_{a(v)}$ to $t_{b(v)}$, where $1 \leq a(v) \leq b(v) \leq m$. For $u, v \in V(G)$, write $u \leq v$ if $a(u) \leq a(v)$ and $b(u) \leq b(v)$. We claim that for all $u, v \in V(G)$, either $u \leq v$ or $v \leq u$. For suppose not; then we may assume that $a(u) < a(v) \leq b(v) < b(u)$. By (1), there exist $x, y \in V(G)$ such that T_x contains $t_{a(u)}$ and not $t_{a(u)+1}$, and T_y contains $t_{b(u)}$ and not $t_{b(u)-1}$.But then $\{u, v, x, y\}$ is a claw in G, a contradiction. This proves our claim. Hence we may number the vertices of G as $v_1, \ldots, v_n$ such that $v_i \leq v_j$ for $1 \leq i < j \leq n$. Now suppose that $1 \leq i < k \leq n$ and v_i, v_k are adjacent. It follows that

$$a(v_i) \leq a(v_k) \leq b(v_i) \leq b(v_k);$$

choose h with $a(v_k) \leq h \leq b(v_i)$. We claim that $t_h \in V(T_{v_j})$ for all j with $i \leq j \leq k$. For $v_i \leq v_j$ implies that $b(v_i) \leq b(v_j)$, and $v_j \leq v_k$ implies that $a(v_j) \leq a(v_k)$. Since $a(v_k) \leq h \leq b(v_i)$, this proves that $t_h \in V(T_{v_j})$. Hence $\{v_i, \ldots, v_k\}$ is a clique, and so G is a linear interval graph. This proves (2).

In view of (2), we may assume that there is a vertex t_0 of T with d neighbors $t_1, \ldots, t_d$ in T, where $d \geq 3$. Let Y be the set of all $v \in V(G)$ such that $t_0 \in V(T_v)$; thus, Y is a clique of G. For $1 \leq i \leq d$, let X_i be the set of all $v \in V(G)$ such that $V(T_v)$ contains t_i and not t_0, and let Y_i be the set of all $v \in Y$ with $t_0, t_i \in V(T_v)$. By (1), $X_i \neq \emptyset$, for $1 \leq i \leq d$; choose $x_i \in X_i$. Since $t_i \in V(T_{x_i})$, it follows that every member of Y_i is adjacent to x_i.

(3) *For $1 \leq i \leq d$, $\emptyset \neq Y_i \neq Y$.*

For suppose that $Y_1 = \emptyset$ say. Let S_1, S_2 be the two components of $T \setminus e$, where e is the edge t_0t_1. For $j = 1, 2$, let A_j be the set of all $v \in V(G)$ such that $V(T_v) \subseteq V(S_j)$; then since $Y_1 = \emptyset$, $A_1 \cup A_2 = V(G)$. Since each T_v has at least one vertex, $A_1 \cap A_2 = \emptyset$. Moreover, by (1) $A_1, A_2 \neq \emptyset$; and if $v_1 \in A_1$ and $v_2 \in A_2$ then $V(T_{v_1} \cap T_{v_2}) = \emptyset$, and so v_1, v_2 are nonadjacent. It follows that G is not connected, a contradiction. This proves that $Y_i \neq \emptyset$, for $1 \leq i \leq d$. Moreover, by (1) there exists $v \in V(G)$ such that T_v contains t_0 and not t_i, and therefore $v \in Y \setminus Y_i$; and consequently $Y_i \neq Y$ for $1 \leq i \leq d$. This proves (3).

(4) *For $1 \leq i < j \leq d$, either $Y_i \cap Y_j = \emptyset$ or $Y_i \cup Y_j = Y$; and for $1 \leq i < j < k \leq d$, $Y_i \cap Y_j \cap Y_k = \emptyset$.*

For if $y \in Y_i \cap Y_j$ and $y' \in Y \setminus (Y_i \cup Y_j)$, then $\{y, y', x_i, x_j\}$ is a claw, a contradiction; and if $y \in Y_i \cap Y_j \cap Y_k$, then $\{y, x_i, x_j, x_k\}$ is a claw, a contradiction. This proves (4).

(5) *If $Y_1, \ldots, Y_d$ are all pairwise disjoint, then G admits a 1-join and contains a trinity as an induced subgraph.*

We observe that since T is a tree the graph $G \setminus Y$ is disconnected, each of $x_1, \ldots, x_d$ belongs to a different component of $G \setminus Y$, say $C_1, \ldots, C_d$ respectively, and $V(C_i)$ is anticomplete to $Y \setminus Y_i$. Thus if all $Y_1, \ldots, Y_d$ are pairwise disjoint, $(V(C_1) \cup Y_1, V(G) \setminus (V(C_1) \cup Y_1))$ is a 1-join in G. Also, choosing a vertex y_i form Y_i for $1 \leq i \leq 3$, the subgraph $G|\{y_1, y_2, y_3, x_1, x_2, x_3\}$ is a trinity. This proves (5).

(6) *If $Y_1 \cap Y_2 \neq \emptyset$ then G contains an antitrinity as an induced subgraph.*

Choose $y_3 \in Y_1 \cap Y_2$. By (4), $Y_1 \cup Y_2 = Y$. Suppose that $Y_1 \cap Y_3 = \emptyset$. Then $Y_3 \subseteq Y_2$, and since $Y_3 \neq \emptyset$ by (3), it follows that $Y_2 \cap Y_3 \neq \emptyset$. By (4), $Y_2 \cup Y_3 = Y$, and since $Y_3 \subseteq Y_2$, we deduce that $Y_2 = Y$, contrary to (3). This proves that there exists $y_2 \in Y_1 \cap Y_3$, and similarly there exists $y_1 \in Y_2 \cap Y_3$. Thus y_1 is adjacent to x_2, x_3; and not to x_1, since $y_1 \notin Y_1$ by (4). Similarly y_i is not adjacent to x_i for $i = 2, 3$ and hence $G|\{y_1, y_2, y_3, x_1, x_2, x_3\}$ is an antitrinity. This proves (6)

(7) *If $V(T) \neq \{t_0, t_1, \ldots, t_d\}$ then G admits a 1-join*

For suppose t_1 has a neighbor s_1 in T, different from t_0. Then by (1)for some vertex v of G, $s_1 \in T_v$ and $t_1 \notin T_v$. Then the graph $G \setminus X_1 \cup Y_1$ is disconnected, for there is no path in $T \setminus t_1$ from s_1 to t_2, and so v and x_2 are in different components of $G \setminus X_1$. Let C_1 be the component containing v, and let C_2 be the union of all other components. Then $V(C_1)$ is anticomplete to Y_1, and $V(C_2)$ is anticomplete to X_1, and so $(V(C_1) \cup X_1, V(C_2) \cup Y_1)$ is a 1-join in G. This proves (7)

By (5) we may assume that $Y_1 \cap Y_2 \neq \emptyset$ and by (6) and (7) that $V(T) = \{t_0, t_1, \ldots, t_d\}$. Assume $d \geq 4$. Then, from the symmetry, there exists a vertex $z \in Y_4 \cap Y_1$. By (4) $z \notin Y_2 \cup Y_3$, so z is non-adjacent to x_2 or x_3. But then $\{y_1, z, x_2, x_3\}$ is a claw in G, a contradiction. This proves that $d = 3$. Since no two vertices of G are twins, for $1 \leq i \leq 3$ we have $X_i = \{x_i\}$ and $Y_i = \{y_i\}$, and hence G is an antitrinity. This completes the proof of 3.3. ∎

Proof of 2.1. Now 2.1 follows from 3.2 if G contains a hole of length at least four, and from 3.3 otherwise. ∎

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