Long Monotone Paths in Line Arrangements

József Balogh* Oded Regev† Clifford Smyth ‡ William Steiger §
Mario Szegedy ¶

May 27, 2003

Abstract

We show how to construct an arrangement of n lines having a monotone path of length $\Omega(n^{2-(d/\sqrt{\log n})})$, where $d > 0$ is some constant, and thus nearly settle the long standing question on monotone path length in line arrangements.

1 Introduction

Let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of n given lines in $\mathbb{R}^2$. A path in the arrangement $A(L)$ is a simple polygonal chain joining a set of distinct vertices in $V = \{\ell_i \cap \ell_j, i < j\}$ by segments which are on lines in L. The length of a path is one plus the number of vertices in V at which the path turns. A path is *monotone in direction* (a, b) if its sequence of vertices is monotone when projected orthogonally along the line with equation $ay - bx = 0$. An interesting open question asks for the value of λ_n, the maximal monotone path length that can occur in an arrangement of n lines[1]. Clearly $\lambda_n \leq \binom{n}{2}$.

A sequence of results by Sharir (see [2]), Matoušek [3], and Radoičić and Tóth [4] established that $\lambda_n = \Omega(n^{3/2})$, $\lambda_n = \Omega(n^{5/3})$, $\lambda_n = \Omega(n^{7/4})$, respectively. The last paper also showed $\lambda_n \leq 5n^2/12$. Here we give an explicit construction that proves

Theorem 1 *For any integers $n, m > 0$ such that $m \leq \frac{1}{2}\sqrt{\log n}$, there is an arrangement of at most $2n + 2(30^m)n$ lines in which there is a monotone path of length at least $2^{-m} \cdot n^{2-1/(m+1)}$.*

Notice that for $m = 3$ this gives the previously best bound $\lambda_n = \Omega(n^{7/4})$.

*The Ohio State University, Department of Mathematics, `jobal@math.ohio-state.edu`. Work done while in the Institute for Advanced Study.

†Institute for Advanced Study, Princeton, NJ, `odedr@ias.edu`. Research supported by NSF grant CCR-9987845.

‡Zeev Nehari Assistant Professor, Carnegie-Mellon University, `csmyth@andrew.cmu.edu`. Work done while in the Institute for Advanced Study and supported by NSF grant CCR-9987845.

§Rutgers University, `steiger@cs.rutgers.edu`.

¶Rutgers University, `szegedy@cs.rutgers.edu`. Research supported by NSF grant 0105692.

[1] Clearly, this is equivalent to the usual definition that considers paths monotone in the direction of the x-axis.

Corollary 1 *The maximal monotone path length satisfies*

$$\lambda_n = \Omega(n^{2-\frac{d}{\sqrt{\log n}}})$$

where $d > 0$ is some constant.

Proof: Let m be $\frac{1}{2}\sqrt{\log n}$. Then Theorem 1 gives a monotone path of length at least $n^{2-(3/(\sqrt{\log n}))}$ using at most $2n + 2(30^{\frac{1}{2}\sqrt{\log n}})n$ lines. A straightforward calculation gives the claimed bound on λ_n. ■

2 The Construction

2.1 The Basic Setup

Observe that k parallel horizontal lines and k parallel vertical lines give a path that is monotone in any direction (a, b) with $a, b > 0$, has length $n = 2k$, and uses n lines. We call this path a *"staircase"*.

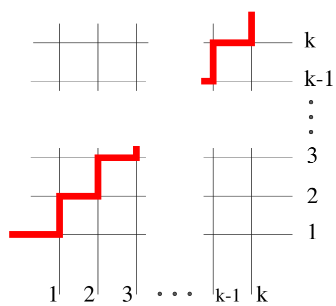

Figure 1. A "staircase" with $n = 2k$ lines, and having length n.

Given an integer $m > 0$ let $\alpha_k = 1/((k+1)(k+2))$, $0 \le k < m$, and $\alpha_m = 1/(m+1)$. Since $\alpha_0 + \ldots + \alpha_k = (k+1)/(k+2)$,

$$\alpha_0 + \cdots + \alpha_m = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{m(m+1)} + \frac{1}{m+1} = 1. \tag{1}$$

Let $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$. In the course of the proof we shall set an $\varepsilon > 0$ that will be suitably small. For now we treat ε as an infinitesimal quantity. We develop a notation to describe points in a hierarchical construction. For $\varepsilon > 0$, the vector-matrix product

$$(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^m) \begin{pmatrix} i_0 & i_1 & \ldots & i_m \\ j_0 & j_1 & \ldots & j_m \end{pmatrix}^T$$

is a point of the plane that we will denote by $\begin{bmatrix} i_0 & i_1 & \ldots & i_m \\ j_0 & j_1 & \ldots & j_m \end{bmatrix}$. The construction uses the set S of points for which $i_0, j_0, \ldots, i_m, j_m$ are integers with

$$\begin{aligned}
0 \le \quad i_0, j_0 \quad &\le \lfloor n^{\alpha_0} \rfloor - 1 \stackrel{def}{=} D_0 \\
0 \le \quad i_1, j_1 \quad &\le \lfloor n^{\alpha_1} \rfloor - 1 \stackrel{def}{=} D_1 \\
&\vdots \\
0 \le \quad i_m, j_m \quad &\le \lfloor n^{\alpha_m} \rfloor - 1 \stackrel{def}{=} D_m.
\end{aligned}$$

In view of (1), the number of points in S is at most $(n^{\alpha_0})^2(n^{\alpha_1})^2\cdots(n^{\alpha_m})^2 = n^2$.

For $k < m$ write B_k for the subset of S where $i_r = j_r = 0$, $r > k$. That is,

$$B_k = \left\{ P = \begin{bmatrix} i_0 & i_1 & \dots i_{k-1} & i_k & 0 & \dots & 0 \\ j_0 & j_1 & \dots j_{k-1} & j_k & 0 & \dots & 0 \end{bmatrix} \right\}. \tag{2}$$

There are at most $(n^{\alpha_0})^2\cdots(n^{\alpha_k})^2 = n^{2-2/(k+2)}$ such points.

Another way to think about B_k is as follows: let us call the square $[x, x+t)\times[y, y+t)\subseteq\mathbb{R}^2$ the "square of side t at (x,y)". The points of B_0 are the points of the integer lattice $\mathbb{Z}\times\mathbb{Z}\subseteq\mathbb{R}^2$ meeting the square of side $\lfloor\sqrt{n}\rfloor$ at $(0,0)$. To get the points of B_1, the next level of the hierarchy, replace each point $P\in B_0$ by the intersection of the square of side $\varepsilon\lfloor n^{\alpha_1}\rfloor$ at P with the points $P+\varepsilon(\mathbb{Z}\times\mathbb{Z})$. For $1\le k < m-1$ we construct B_{k+1} by replacing each point $P\in B_k$ by the intersection of the square of side $\varepsilon^{k+1}\lfloor n^{\alpha_{k+1}}\rfloor$ at P and the points $P+\varepsilon^{k+1}(\mathbb{Z}\times\mathbb{Z})$.

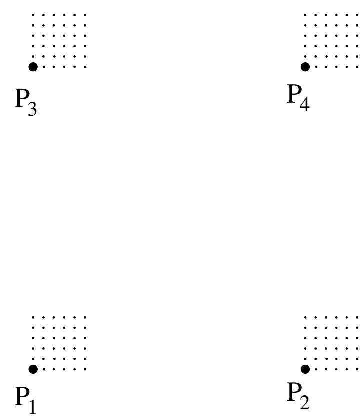

Figure 2. Some points in B_{k+1}

For example in Figure 2, P_1, P_2, P_3, P_4 are neighboring points in B_k, each the lower-left corner of a square of side $\varepsilon^{k+1}\lfloor n^{\alpha_{k+1}}\rfloor$ that contains $\lfloor n^{\alpha_{k+1}}\rfloor^2$ grid points. If in Figure 2 P_1 has coordinates

$$\begin{bmatrix} i_0 & \dots & i_{k-1} & I & 0 & \dots & 0 \\ j_0 & \dots & j_{k-1} & J & 0 & \dots & 0 \end{bmatrix} \in B_k,$$

then P_2 and P_4 have $i_k = I+1$, and P_3 and P_4 have $j_k = J+1$.

We now pick a direction in which we want our path to be monotone. Our choice is $\mathbf{w} = (\sqrt{2}, 1)$. Orthogonal to this is the direction $\mathbf{w}' = (\mathbf{-1}, \sqrt{\mathbf{2}})$. A vector is said to *point forward* if it has positive scalar product with $(\sqrt{2}, 1)$. In particular, $\mathbf{u}$ and $\mathbf{v}$ point forward. For $p, q > 0$ the vector $(-q, p)$ points forward iff $\frac{p}{q} > \sqrt{2}$, and $(q, -p)$ points forward iff $\frac{p}{q} < \sqrt{2}$. In the first case we say *p/q approximates $\sqrt{2}$ from above*; in the second, *p/q approximates $\sqrt{2}$ from below.*

Our monotone path is based on a set L of lines, one horizontal and one vertical, at each point of S. The points of S have at most n distinct x coordinates and at most n distinct y coordinates, so L **has at most $2n$ lines**. From each point

$$P = \begin{bmatrix} i_0 & \dots & i_{m-1} & 0 \\ j_0 & \dots & j_{m-1} & 0 \end{bmatrix} \in B_{m-1},$$

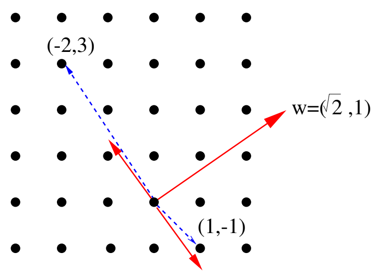

Figure 3. $\mathbf{w}$ is the chosen direction of monotonicity. $(-2,3)$ and $(1,-1)$ point forward, since $\frac{3}{2}$ approximates $\sqrt{2}$ from above and $1/1$ from below.

we follow the staircase to

$$Q = \begin{bmatrix} i_0 & \dots & i_{m-1} & D_m \\ j_0 & \dots & j_{m-1} & D_m \end{bmatrix} \in S.$$

This staircase is a monotone path because $\mathbf{u}$ and $\mathbf{v}$ both point forward. We use the following coarse lower bound on the number of staircases (which is good enough for our claim):

$$\lfloor n^{\alpha_0} \rfloor^2 \dots \lfloor n^{\alpha_{m-1}} \rfloor^2 \geq 2^{-m}(n^{\alpha_0})^2 \dots (n^{\alpha_{m-1}})^2 = 2^{-m} n^{2-2/(m+1)}$$

where the first inequality holds since $n^{\alpha_k} \geq n^{2/\log n} = 4$ for all $0 \leq k \leq m-1$. On each of these staircases the path makes $2\lfloor n^{1/(m+1)} \rfloor - 1 \geq n^{1/(m+1)}$ turns, so if we could move from staircase to staircase in a monotone fashion, the resulting path would have length at least $2^{-m} n^{2-1/(m+1)}$, as required.

2.2 Helping Lines

We now show how to connect the staircases using at most $2(30^m)n$ extra lines, and moving along each in a direction that points forward with respect to $\mathbf{w}$. Suppose we project the points of S orthogonally onto the line ℓ with equation $\sqrt{2}y - x = 0$. The ordering of the projections gives an ordering for the points of S, and we define $Q \in S$ to be the *successor* of $P \in S$ if it comes immediately after P in this ordering.

The points in B_0 project to distinct points on ℓ and are ordered by these projections. When each point in B_0 is replaced by a square of side $\varepsilon \lfloor n^{\alpha_1} \rfloor$, each square projects to an interval, and if ε is suitably small, these intervals will be disjoint. This gives an ordering for the points in B_1 based first on the ordering for B_0, and then on the ordering for points with the same i_0, j_0. Inductively, the points in B_k are ordered, and when we replace each by a square of side $\varepsilon^{k+1} \lfloor n^{\alpha_{k+1}} \rfloor$, each square projects to an interval; if ε is suitably small, these intervals will be disjoint. This gives an ordering for the points in B_{k+1}, first based on the ordering of points in B_k, and then on the ordering of points with the same values of i_r, j_r, $r \leq k$. These observations imply that the set of staircases can be connected in a monotone manner. Specifically,

Lemma 1 *Let*

$$P = \begin{bmatrix} i_0 & \dots & i_{k-1} & i_k & D_{k+1} & \dots & D_m \\ j_0 & \dots & j_{k-1} & j_k & D_{k+1} & \dots & D_m \end{bmatrix}$$

be a point in S with either $i_k \neq D_k$ or $j_k \neq D_k$, and $k < m$. The successor of P is a point

$$Q = \left[\begin{array}{cccccc} i_0 & \dots i_{k-1} & i'_k & 0 & \dots & 0 \\ j_0 & \dots j_{k-1} & j'_k & 0 & \dots & 0 \end{array} \right]$$

with either $i'_k \neq i_k$, $j'_k \neq j_k$, or both.

The point P is the *top of a staircase at level k.*

We now discuss the issues concerning the choice of lines used to move from a point to its successor. We call these lines *helping lines.* Fix $k < m$.

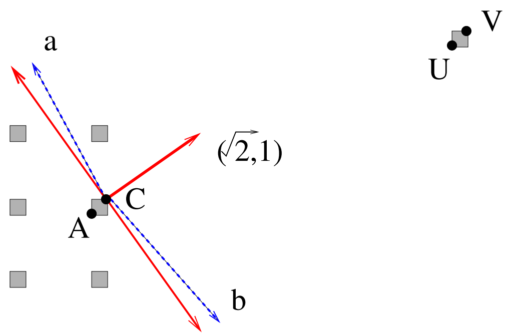

Figure 4. Successors at level k.

In Figure 4, the shaded squares are situated at points in B_k, $k < m$, that have the same i_r, j_r, $r < k$. A and C have the same values of i_k, j_k and C is the top of the last staircase whose points have these values. U and V also have $i_k = j_k = D_k$, and V is the top of the last staircase having these values. We want to move to the successor of

$$C = \left[\begin{array}{cccccc} i_0 & \dots & i_k & D_{k+1} & \dots & D_m \\ j_0 & \dots & j_k & D_{k+1} & \dots & D_m \end{array} \right] \in S.$$

From C we will follow the line in direction **a** or **b**, both pointing forward as in Figure 4. There are two issues:

- To be able to move to the successor of C, it is necessary that no point in B_k that follows C and has the same i_r, j_r, $r \leq k-1$, can be below **a** or **b**. Also,
- We will need two such helping lines at the top of every staircase at level k. Since there are as many as $n^{2-2/(k+2)}$ ($\gg 2(30^m)n$) such staircases, we will have to *re-use each helping line, many times.* The idea is the following: when we move to the successor of V, the top of the last staircase in B_k having the same values of i_r, j_r, $r \leq k-1$, we must do so on a line that points *less forward than* **a** *or* **b**.

These issues are handled as follows.

Definition 1 *A best upper approximator of $\sqrt{2}$ is a rational number $\frac{p}{q} > \sqrt{2}$ such that no other rational $\frac{p'}{q'}$ with $q' \leq q$ approximates $\sqrt{2}$ better from either above or below. A best lower approximator of $\sqrt{2}$ is a rational $\frac{r}{s} < \sqrt{2}$ such that no other rational $\frac{r'}{s'}$ with $s' \leq s$ approximates $\sqrt{2}$ better from either above or below.*

It is easy to see that

Lemma 2 *For every $t \geq 1$ there is a best upper approximator $\frac{p}{q}$ and a best lower approximator $\frac{p'}{q'}$ of $\sqrt{2}$ such that $t < q, q' \leq 10t$.*

Proof: The convergents of the simple continued fraction for $\sqrt{2}$ define a sequence $r_0/s_0, r_1/s_1, \ldots$ of rational approximations that are alternately best lower approximations, then best upper approximations. Specifically, for $j \geq 0$,

$$\frac{r_{2j}}{s_{2j}} < \frac{r_{2j+2}}{s_{2j+2}} < \sqrt{2} < \frac{r_{2j+3}}{s_{2j+3}} < \frac{r_{2j+1}}{s_{2j+1}}.$$

Since $s_{j+1} \leq 3s_j$ there are two in every interval $(t, 10t]$, $t \geq 1$. ■

For $0 \leq k < m$ let $\frac{p_k}{q_k}$ be a best upper approximator of $\sqrt{2}$ such that $n^{\alpha_k} < q_k \leq 10n^{\alpha_k}$ and let $\frac{p'_k}{q'_k}$ be a best lower approximator of $\sqrt{2}$ such that $n^{\alpha_k} < q'_k \leq 10n^{\alpha_k}$. We use them to define the helping lines that monotonely connect staircases.

For $0 \leq k < m$ we define the set $T_k \subseteq S$ as follows:

$$T_k = \left\{ P = \begin{bmatrix} i_0 & \ldots & i_k & D_{k+1} & \ldots & D_m \\ j_0 & \ldots & j_k & D_{k+1} & \ldots & D_m \end{bmatrix} \in S : (i_k, j_k) \neq (D_k, D_k) \right\} \tag{3}$$

These points are the **tops of staircases at level** k of the hierarchy (in Figure 4, $C \in T_k$ and $V \in T_{k-1}$).

For $0 \leq k \leq m-1$ and for every point $P \in T_k$ we define two lines that are incident with P, one in direction $(-q_k, p_k)$ (an upper helping line) and one in direction $(q'_k, -p'_k)$ (a lower helping line). Formally, L_k^{up} denotes the set of lines of slope $\frac{-p_k}{q_k}$ through the points of T_k and L_k^{down}, the lines of slope $\frac{-p'_k}{q'_k}$ through these points. The monotone path will follow one of these lines.

Lemma 3 *From each point in $P \in T_k$ there is a monotone path to its successor, Q, that either follows the line in L_k^{up} through P and then a horizontal line to Q, or the line in L_k^{down} through P, and then a horizontal (see Figure 5).*

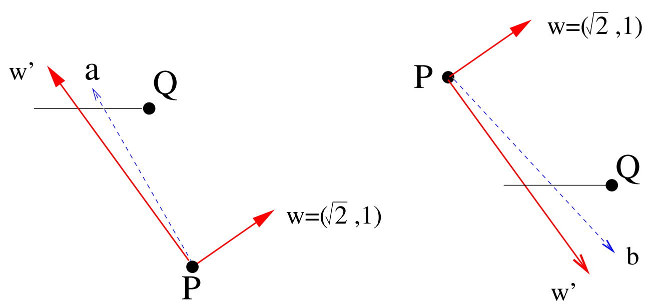

Figure 5. Helping lines precede successors.

Proof: The choice of p_k/q_k and p'_k/q'_k as best approximators with $q_k, q'_k > n^{\alpha_k}$ guarantee that if ε is small enough, the successor of P is on a line from P of slope less than $-p_k/q_k$ in the upper case, or greater than $-p'_k/q'_k$ in the lower case. ■

2.3 Counting

To complete the proof of the theorem we count the number of distinct helping lines used in the construction.

Lemma 4 *Let $|L_k^{up}|$ and $|L_k^{down}|$ denote the number of distinct lines in the respective sets, $k < m$. Then the total number of helping lines is*

$$\leq \sum_{k=0}^{m-1} (|L_k^{up}| + |L_k^{down}|) \leq 2(30^m)n. \tag{4}$$

Proof: We just treat $|L_k^{up}|$, the down case being completely analogous. Fix non-negative $I_r, J_r \leq D_r$, $r < k$, and consider the points in

$$A = \{P \in T_k : (i_r, j_r) = (I_r, J_r) \text{ for all } r < k\}.$$

There are at most $N = n^{2\alpha_k}$ such points, one for each possible pair $(i_k, j_k) \neq (D_k, D_k)$, and they require N distinct lines in L_k^{up}. Let R be the points in T_k which have the same values of i_r, j_r as do the points in A, for all $r < k-1$; i.e.,

$$R = \{P \in T_k : (i_r, j_r) = (I_r, J_r) \text{ for all } r < k-1\}.$$

The N lines just considered (for A) will also meet all points in R for which both $i_{k-1} = I_{k-1} - cq_k \geq 0$ and $j_{k-1} = J_{k-1} + cp_k \leq D_{k-1}$ for some integer c. For example, in Figure 6, the square B is located q_k squares to the left of A and p_k squares above it and therefore the N lines going through A are the same as the N lines going through B. Similarly, C is located $2q_k$ squares to the left and $2p_k$ squares above A and also shares the same N lines.

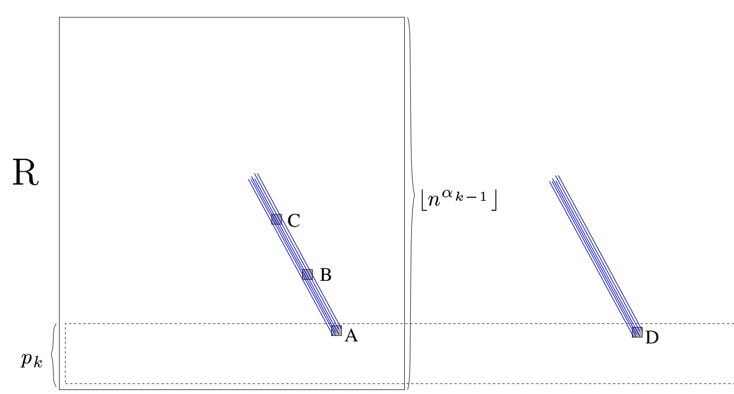

Figure 6. Lines in L_k^{up} for points with the same $i_r, j_r, r < k-1$

This indicates that the number of *distinct* lines in L_k^{up} needed for all points in R is less than the trivial bound of $n^{2\alpha_{k-1}} \cdot N$. Indeed, consider the lines of slope $-p_k/q_k$ at those points with $(i_r, j_r) = (I_r, J_r)$, $r < k-1$ and with $i_{k-1} = 0, \ldots, 2\lfloor n^{\alpha_{k-1}} \rfloor$ and $j_{k-1} = 0, \ldots, p_k - 1$ (in Fig. 6, these are the lines emanating from the squares inside the dashed rectangle, such as A and D). Because $p_k > q_k$ (i.e., the lines form an angle of more than $45°$ with the x-axis), *all* points in R will be covered. Each square uses at most N lines in L_k^{up} and we cover R with at most $2p_k n^{\alpha_{k-1}}$ squares. Hence, the number of distinct lines in L_k^{up} needed for all the points in $R \subseteq T_k$ is at most

$$2p_k \cdot n^{\alpha_{k-1}} N \leq (30n^{\alpha_k}) n^{\alpha_{k-1}} n^{2\alpha_k},$$

where we used the fact that $p_k \leq 1.5q_k$ and $q_k \leq 10n^{\alpha_k}$.

Applying this argument again to points in T_k that have $(i_r, j_r) = (I_r, J_r)$ for $r < k-2$ we deduce that at most

$$(30n^{\alpha_k})^2 n^{\alpha_{k-2}} n^{\alpha_{k-1}} n^{2\alpha_k}$$

lines in L_k^{up} are needed, and continuing inductively, we see that T_k needs at most

$$(30n^{\alpha_k})^k n^{(\alpha_0+\ldots+\alpha_{k-1})} n^{2\alpha_k} = (30)^k n^{(\alpha_0+\ldots+\alpha_{k-1}+(k+2)\alpha_k)}$$

lines in L_k^{up}. Using the fact that $(k+2)\alpha_k = 1/(k+1)$ and $\alpha_0 + \ldots + \alpha_{k-1} = k/(k+1)$, we obtain

$$|L_k^{up}| \leq (30)^k n.$$

Applying this estimate for each k, we establish the bound in (4) and prove the lemma.

■

Proof of Theorem 1. We have constructed an arrangement of at most $2n + 2(30^m)n$ lines, at most n horizontal and at most n vertical lines used in the staircases, and the helping lines. Also, as mentioned above, the staircases alone comprise part of a monotone path of length at least $2^{-m} \cdot n^{2-1/(m+1)}$. ■

3 Remarks

1. One interesting open question concerns the quantity $\lambda_n(k)$, the length of the longest monotone path in an arrangement of n lines with at most k distinct slopes. Clearly, $\lambda_n(k)$ increases with k and is at most λ_n. The construction of Sharir used $k = 4$ different slopes, so $\lambda_n(4) \geq \Omega(n^{3/2})$. Matoušek's construction gives $\lambda_n(5) \geq \Omega(n^{5/3})$. For any constant m, our construction uses a set of $O(n)$ lines with $2m + 2$ distinct slopes. Hence, it implies $\lambda_n(2m+2) \geq \Omega(n^{2-1/(m+1)})$. Recently, Dumitrescu [1] showed that $\lambda_n(k) \leq O(n^{2-1/F_{k-1}})$ where F_k is the k'th Fibonacci number ($F_1 = F_2 = 1, F_3 = 2, F_4 = 3$, etc.). In particular, this provides tight upper bounds for $k = 4, 5$.

2. Matoušek [3] also studied arrangements of *pseudo-lines*; i.e., n continuous functions $f_1, \ldots, f_n$ with the same intersection rules as lines. Specifically, for each $i < j$ there is a point x_{ij} (a vertex) such that $(f_i(u) - f_j(u))(f_i(t) - f_j(t)) < 0$ whenever $(u - x_{ij})(t - x_{ij}) < 0$. General position would impose the condition that the vertices be distinct. A "path" moves along a function and may turn at a vertex. Matoušek constructed a pseudoline arrangement with an x-monotone path of length $\Omega(n^2/\log n)$. He also had conjectured that $\lambda_n = O(n^{5/3})$, i.e., that his lower bound for monotone path length in line arrangements was optimal. If this were true we would have a neat combinatorial separation of line and pseudoline

arrangements based on monotone path length. In view of the subsequent improvements in λ_n, the hope for such a strong separation has all but disappeared. However it still remains an important issue to know if there is an $o(n^2)$ upper bound for λ_n.

References

[1] A. Dumitrescu. *Monotone Paths in Line Arrangements with a Small Number of Directions.* Manuscript.

[2] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Berlin, 1987.

[3] J. Matoušek. Lower Bounds on the Length of Monotone Paths in Arrangements. *Discrete and Computational Geometry 6*, 129-134 (1991).

[4] R. Radoičić and G. Tóth. Monotone Paths in Line Arrangements. *Proc.* 17^{th} *ACM Symp. Comp. Geom.*, 312-314 (2001)