

Quantum Communication Complexity of Symmetric Predicates

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Abstract

We completely (that is, up to a logarithmic factor) characterize the bounded-error quantum communication complexity of *every* predicate $f(x, y)$ depending only on $|x \cap y|$ ($x, y \subseteq [n]$). Namely, for a predicate D on $\{0, 1, \dots, n\}$ let $\ell_0(D) \stackrel{\text{def}}{=} \max \{ \ell \mid 1 \leq \ell \leq n/2 \wedge D(\ell) \neq D(\ell - 1) \}$ and $\ell_1(D) \stackrel{\text{def}}{=} \max \{ n - \ell \mid n/2 \leq \ell < n \wedge D(\ell) \neq D(\ell + 1) \}$. Then the bounded-error quantum communication complexity of $f_D(x, y) = D(|x \cap y|)$ is equal (again, up to a logarithmic factor) to $\sqrt{n\ell_0(D)} + \ell_1(D)$. In particular, the complexity of the set disjointness predicate is $\Omega(\sqrt{n})$. This result holds both in the model with prior entanglement and without it.

1. Introduction

The model of communication complexity, originally introduced by Yao [Yao79] has since evolved into a very intriguing and important branch of computational complexity that in particular links and unifies many different things. In this model, Alice holds an input $x \in X$, Bob holds $y \in Y$, and they exchange messages to evaluate a Boolean predicate $f : X \times Y \rightarrow \{0, 1\}$. The complexity is measured by the number of bits exchanged, and, like in many other areas of computational complexity, one distinguishes between deterministic and probabilistic modes.

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Just like the circuit complexity is quite often concerned with symmetric Boolean functions one class of problems that attracted a considerable interest in communication complexity is made by *symmetric predicates* which we define as those for which x, y are finite sets and $f_D(x, y) = D(|x \cap y|)$ for some predicate D on integers. Two most prominent members of this class are the disjointness predicate $DISJ_n$ ($D(s) \equiv (s = 0)$) and the inner product function IP_n ($D(s) \equiv s \pmod{2}$). The rank lower bound by Mehlhorn and Schmidt [MS82] immediately implies a tight $\Omega(n)$ lower bound on the deterministic communication complexity of both $DISJ_n$ and IP_n .

For the randomized algorithms, [Vaz87, CG88, BFS86] proved an $\Omega(n)$ lower bound on the complexity of the inner product IP_n , and [BFS86] also contained an $\Omega(\sqrt{n})$ lower bound for $DISJ_n$. The latter bound was improved to the optimal $\Omega(n)$ in [KS92], and their proof was further simplified in [Raz92].

The model of *quantum* communication complexity was also introduced by Yao [Yao93]. Suppose that Alice and Bob can employ the laws of quantum mechanics and are allowed to exchange qubits instead of classical bits. Can it help them to reduce the amount of communication?

Buhrman, Cleve and Wigderson [BCW98] observed that the rank lower bound for deterministic protocols extends to the quantum case (so, after all the answer for such protocols can be “NO”). In particular, both $DISJ_n$ and IP_n require $\Omega(n)$ qubits to be exchanged by quantum deterministic (= zero-error) protocols. The rank lower bound was extended in [BW01] to the stronger model with prior entanglement previously introduced in [CB97] (in that model, Alice and Bob share an unlimited number of entangled EPR-pairs before the communication even begins).

The question about the complexity of protocols that allow a small error is by far more interesting. As far as lower bounds are concerned, Kremer [Kre95], based upon some ideas from the seminal paper [Yao93], proved an $\Omega(n)$ lower bound for IP_n . This result was extended to the model with prior entanglement in [CDNT98]. Klauck [Kla01] looked at the threshold predicates ($D(s) \equiv (s \geq \ell)$) and exact- ℓ predicates ($D(s) \equiv (s = \ell)$) and proved an $\Omega(\ell / \log \ell)$ bound in both cases (without entanglement). The only *general* lower bound for $DISJ_n$ (that corresponds to $\ell = 0$) prior to this work was $\Omega(\log n)$ [AST⁺98, BW01]; we can also mention some partial results in this direction such as bounds for constant-round protocols [KNTZ01], protocols with exponentially small error [BW01] and some highly structured

protocols [HW02].

On the upper bounds frontier, the elegant paper [BCW98] established a strong connection between quantum search and quantum communication by showing how to convert *every* quantum search algorithm for *any* Boolean function g into a quantum communication algorithm for the associated predicate $f_g(x, y) = g(x \cap y)$ with only a logarithmic delay. Plugging into this procedure Grover’s search algorithm [Gro96] immediately gave an $O(\sqrt{n} \log n)$ upper bound on the bounded-error quantum communication complexity of disjointness (that was later slightly improved in [HW02] to $O(\sqrt{n} \exp(\log^* n))$). [BBC⁺98] proved that the quantum query complexity of every symmetric Boolean function g is equal, up to a constant factor, to its *approximate degree* $\widetilde{\deg}(g)$ (defined as the minimal degree of a real polynomial approximating g on $\{0, 1\}^n$ in the ℓ_∞ -norm within accuracy $1/3$). Combined with the BCW-reduction, this implies an $O(\widetilde{\deg}(g) \log n)$ upper bound on the bounded error quantum communication complexity of $f_g(x, y)$.

In this paper we prove that for *every* symmetric predicate $f_D(x, y)$ this communication algorithm is essentially optimal provided we take care of one “degenerate” case. More specifically, let

$$\ell_0(D) \stackrel{\text{def}}{=} \max \{ \ell \mid 1 \leq \ell \leq n/2 \wedge D(\ell) \neq D(\ell - 1) \} \quad (1)$$

and

$$\ell_1(D) \stackrel{\text{def}}{=} \max \{ n - \ell \mid n/2 \leq \ell < n \wedge D(\ell) \neq D(\ell + 1) \}. \quad (2)$$

If we let $g_D(x_1, \dots, x_n) = D(|x|)$, then the classical result by Paturi [Pat92] says that $\widetilde{\deg}(g_D) = \theta(\sqrt{n(\ell_0(D) + \ell_1(D))})$ which implies, via [BCW98], an upper bound of $O((\sqrt{n\ell_0(D)} + \sqrt{n\ell_1(D)}) \log n)$ on the quantum bounded-error communication complexity of f_D . This can be easily improved to $O((\sqrt{n\ell_0(D)} + \ell_1(D)) \log n)$ (large values of $|x \cap y|$ are taken care of by the trivial algorithm in which Alice sends to Bob her entire input). We prove the lower bound $\Omega(\sqrt{n\ell_0(D)} + \ell_1(D))$ matching this upper bound up to a logarithmic factor (Theorem 2.1). Our lower bound works also in the model with prior entanglement.

For the proof of our result we use a multi-dimensional version of the ordinary discrepancy method (Section 5.2). That is, we measure the communication matrix against several probability distributions at the same time.

This allows us to reduce our problem to a classical problem in the discrete polynomial approximation that, quite fortunately, was solved in the above-mentioned paper [Pat92] (Section 5.3). Another specific feature of our approach is that we tend to apply spectral methods (as opposed to combinatorial ones) more systematically than it was done in the previous papers on the subject (this becomes especially critical for handling prior entanglement). In particular, we show a general lower bound on the quantum communication complexity of a function in terms of the approximate trace norm of its communication matrix (Section 5.1).

In the rest of the paper we formulate and prove our main result. Whenever possible, we try to present in reasonable generality those intermediate steps in our proof that might be of independent interest.

2. Quantum communication model and the main result

There are several equivalent definitions of the quantum communication model; in our description we follow [BW01] as this variant seems to be the most convenient to work with.

Let X, Y be finite sets and $f : X \times Y \rightarrow \{0, 1\}$ be a Boolean predicate. Let $\mathcal{H}_A, \mathcal{C}, \mathcal{H}_B$ be finite-dimensional Hilbert spaces representing Alice's part, the channel and Bob's part, respectively. Like in [BW01] we require that \mathcal{C} consists of a single qubit (that is, $\dim(\mathcal{C}) = 2$, and $|0\rangle, |1\rangle$ is its orthonormal basis).

The models with or without prior entanglement differ only in the unitary vector $\text{Input}(x, y) \in \mathcal{H}_A \otimes \mathcal{C} \otimes \mathcal{H}_B$ prepared at the beginning of the communication. We postpone its definition and describe first how the communication proceeds. A *c-qubit communication protocol* is completely determined by unitary operators U_1, U_2, \dots, U_c , where U_i acts on $\mathcal{H}_A \otimes \mathcal{C}$ if i is odd, and on $\mathcal{C} \otimes \mathcal{H}_B$ if it is even. The output (unitary) vector is then

$$\text{Output}(x, y) \stackrel{\text{def}}{=} \dots (U_3 \otimes I_B)(I_A \otimes U_2)(U_1 \otimes I_B)\text{Input}(x, y), \quad (3)$$

where I_A, I_B are identity operators on $\mathcal{H}_A, \mathcal{H}_B$, respectively. The *acceptance probability* of this protocol on x, y is the result of the measurement of $\text{Output}(x, y)$ with respect to \mathcal{C} , i.e., the squared ℓ_2 -norm of its orthogonal projection onto $\mathcal{H}_A \otimes |1\rangle \otimes \mathcal{H}_B$.

We are still left to describe $\text{Input}(x, y)$. In the model without prior entanglement, \mathcal{H}_A has the orthonormal basis $\{|a, x\rangle \mid a \in W_A, x \in X\}$, where W_A is a finite set with a distinguished element 0 (representing Alice's internal computations). Likewise, \mathcal{H}_B has the orthonormal basis $\{|y, b\rangle \mid b \in W_B, y \in Y\}$ and $\text{Input}(x, y) \stackrel{\text{def}}{=} |0, x\rangle|0\rangle|y, 0\rangle$.

In the model with prior entanglement, \mathcal{H}_A has the basis $\{|a, x, e\rangle \mid a \in W_A, x \in X, e \in E\}$ and \mathcal{H}_B has the basis $\{|e, y, b\rangle \mid b \in W_B, y \in Y, e \in E\}$, where E is a new finite set (corresponding to all possible pure states of entangled EPR-pairs). The beginning state in this case is

$$\text{Input}(x, y) \stackrel{\text{def}}{=} \frac{1}{|E|^{1/2}} \sum_{e \in E} |0, x, e\rangle|0\rangle|e, y, 0\rangle. \quad (4)$$

It is important that in this model we do not have any control of $|E|$ whatsoever, and it must not appear in our bounds.

A quantum protocol *computes* $f(x, y)$ *with error* ϵ if its acceptance probability on every (x, y) is at most ϵ whenever $f(x, y) = 0$ and at least $1 - \epsilon$ whenever $f(x, y) = 1$. Let $Q_\epsilon(f)$ [$Q_\epsilon^*(f)$] be the smallest c for which there exists a c -qubit communication protocol without [respectively, with] prior entanglement that computes f with error ϵ . Let $Q(f) \stackrel{\text{def}}{=} Q_{1/3}(f)$ and $Q^*(f) \stackrel{\text{def}}{=} Q_{1/3}^*(f)$.

Fix an integer n , and let $D : \{0, 1, \dots, n\} \rightarrow \{0, 1\}$ be any Boolean predicate. Let $f_{n,D}(x, y) \stackrel{\text{def}}{=} D(|x \cap y|)$, where $x, y \subseteq [n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$. Let $\ell_0(D)$ and $\ell_1(D)$ be given by (1), (2) (if no such ℓ exists, we naturally let $\ell_\epsilon(D) \stackrel{\text{def}}{=} 0$). The main result of this paper is the following

Theorem 2.1 *For every Boolean predicate* $D : \{0, 1, \dots, n\} \rightarrow \{0, 1\}$,

$$\Omega(\sqrt{n\ell_0(D)} + \ell_1(D)) \leq Q^*(f_{n,D}) \leq Q(f_{n,D}) \leq O((\sqrt{n\ell_0(D)} + \ell_1(D)) \log n).$$

Let $DISJ_n(x, y) \stackrel{\text{def}}{=} x \cap y = \emptyset$.

Corollary 2.2 $Q^*(DISJ_n) \geq \Omega(\sqrt{n})$.

Our lower bound proof essentially uses high symmetry of the predicate $f_{n,D}$ and, in particular, we need x, y to be of the same fixed cardinality

k . We formulate the corresponding intermediate result in this section since, although somewhat technical, it still might be of independent interest.

Let $k \leq n$ and $D : \{0, 1, \dots, k\} \rightarrow \{0, 1\}$. Let $X = Y \stackrel{\text{def}}{=} [n]^k$ be the set of all k -element subsets of $[n]$ and $f_{n,k,D} : X \times Y \rightarrow \{0, 1\}$; $f_{n,k,D}(x, y) \stackrel{\text{def}}{=} D(|x \cap y|)$ (thus, $f_{n,D} = f_{n,n,D}$).

Theorem 2.3 *Let $k \leq n/4$, $\ell \leq k/4$ and $D : \{0, 1, \dots, k\} \rightarrow \{0, 1\}$ be any predicate such that $D(\ell) \neq D(\ell - 1)$. Then $Q^*(f_{n,k,D}) \geq \Omega(\sqrt{k\ell})$.*

Remark 1 Nayak and Shi have observed (personal communication) that our lower bound extends to a more general model in which the entanglement need not necessarily be given in the form of shared EPR-pairs. More specifically, in this model (considered e.g. in [NS02]) the input vector $\text{Input}(x, y)$ has the form

$$\text{Input}(x, y) \stackrel{\text{def}}{=} \sum_{e \in E} \lambda_e |0, x, e\rangle |0\rangle |e, y, 0\rangle, \quad (5)$$

where $\{\lambda_e \mid e \in E\}$ is an *arbitrary* unitary vector (the case (4) of EPR-pairs corresponds to $\lambda_e = \frac{1}{|E|^{1/2}}$). With their kind permission, we include in Section 5.1 the adjustments to our basic proof needed for this generalization (Remark 4).

3. Preliminaries

In this section we compile together some definitions and previously known results needed for our proof.

3.1. Quantum search vs. quantum communication

For a precise definition of a *quantum decision tree* see e.g. [BW00]. Given a Boolean function $g(x_1, \dots, x_n)$ we will denote by $Q_{\text{DT}}(g)$ the minimal number of queries needed to compute g by a quantum decision tree with error at most $1/3$ at any input $x \in \{0, 1\}^n$.

Let us denote by $f_g : \mathcal{P}([n]) \times \mathcal{P}([n]) \rightarrow \{0, 1\}$ the predicate $f_g(x, y) \stackrel{\text{def}}{=} g(x \cap y)$, where $x \cap y$ is identified with its characteristic function. The following is probably the deepest general fact known about quantum communication:

Proposition 3.1 ([BCW98]) *For any Boolean function $g(x_1, \dots, x_n)$, $Q(f_g) \leq O(Q_{\text{DT}}(g) \log n)$.*

3.2. Matrix norms

All material in this section is classical and can be found e.g. in the excellent textbook [Bha97].

After we give up Dirac's notation (in Section 5.2), all vectors will be represented as columns. For a complex vector ξ [complex matrix A], let $x^* \stackrel{\text{def}}{=} (\bar{x})^\top$ [$A^* \stackrel{\text{def}}{=} (\bar{A})^\top$, respectively] be its conjugate transpose. Let $\|\xi\| \stackrel{\text{def}}{=} (\xi^* \xi)^{1/2}$ denote the ℓ_2 -norm of ξ .

For a complex matrix A , we will denote by $\|A\|$ its *operator norm* defined as $\|A\| \stackrel{\text{def}}{=} \max\{\|A\xi\| : \|\xi\| \leq 1\}$. Alternatively, $\|A\| = \max\{|\eta^\top A\xi| : \|\eta\|, \|\xi\| \leq 1\}$.

For two complex matrices A, B of the same size $m \times n$ we denote by $\langle A, B \rangle$ their entrywise scalar product, that is, $\langle A, B \rangle \stackrel{\text{def}}{=} \text{Tr}(A^* B) = \sum_{i=1}^m \sum_{j=1}^n \bar{a}_{ij} b_{ij}$. $\|A\|_{\text{F}}$ denotes the *Frobenius norm* corresponding to this scalar product, that is, $\|A\|_{\text{F}} \stackrel{\text{def}}{=} \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}$. We will also need the following (somewhat more exotic) *trace norm* $\|A\|_{\text{tr}}$ defined as

$$\|A\|_{\text{tr}} \stackrel{\text{def}}{=} \max_B \{ |\langle A, B \rangle| : \|B\| \leq 1 \},$$

where B runs over all (complex) matrices of the same size as A .

The following proposition summarizes some properties of these norms.

Proposition 3.2 1. *Let $\|\cdot\|$ be any one of the three norms $\|\cdot\|$, $\|\cdot\|_{\text{F}}$ or $\|\cdot\|_{\text{tr}}$, and A be a complex $m \times n$ matrix. Then:*

- (a) $\|A^*\| = \|A^\perp\| = \|A\|$;
- (b) if B is a submatrix of A then $\|B\| \leq \|A\|$;
- (c) $\|\cdot\|$ is invariant under left and right unitary transformations, that is, for every $m \times m$ unitary matrix U and every $n \times n$ unitary matrix V , $\|UAV\| = \|A\|$.

2. *Let now B be another complex $n \times k$ matrix, and AB stand for the ordinary matrix multiplication. Then:*

- (a) $\|AB\| \leq \|A\| \cdot \|B\|$;

(b) $\|AB\|_{\text{tr}} \leq \|A\|_{\text{F}} \cdot \|B\|_{\text{F}}$. (Hoelder inequality, see e.g. [Bha97, Corollary IV.2.6])

3. $\|A\| \leq \|A\|_{\text{F}} \leq (\min\{m, n\})^{1/2} \cdot \|A\|$.

4. For a square $n \times n$ matrix A , $\|A\|_{\text{tr}} \geq \sum_{i=1}^n |a_{ii}|$.

Remark 2 If $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_p(A)$, $p = \min\{m, n\}$ are the singular values of A then $\|A\| = \sigma_1(A)$, $\|A\|_{\text{F}} = (\sum_{t=1}^p \sigma_t^2(A))^{1/2}$ and $\|A\|_{\text{tr}} = \sum_{t=1}^p \sigma_t(A)$ (which, along with Proposition 3.2.1c) and the SVD-theorem almost immediately implies all non-trivial parts of that proposition). We, however, will not need this singular value characterization in our proof.

Remark 3 The same proposition 3.2.1c) implies that we can unambiguously talk of the operator, Frobenius or trace norm of an *operator* from one (finite-dimensional) Hilbert space to another.

Two more matrix norms we will be using are the ℓ_1 -norm and ℓ_∞ -norm defined entrywise:

$$\begin{aligned} \ell_1(A) &\stackrel{\text{def}}{=} \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |a_{ij}|; \\ \ell_\infty(A) &\stackrel{\text{def}}{=} \max_{1 \leq i \leq m} \max_{1 \leq j \leq n} |a_{ij}|. \end{aligned}$$

Of course, these norms are *not* invariant under unitary transformations. However, they are at least somewhat related to unitary invariant norms via the following (obvious) observation:

$$|\langle A, B \rangle| \leq \ell_1(A) \cdot \ell_\infty(B).$$

3.3. Decomposition of quantum communication protocols

Proposition 3.3 ([Yao93, Kre95]) *Let P be a c -qubit communication protocol, and let U_p be the unitary operator in the right-hand side of (3). Then there exist linear operators A_u on \mathcal{H}_A and B_u on \mathcal{H}_B ($u \in \{0, 1\}^c$) such that for every vector $a \in \mathcal{H}_A$ and every vector $b \in \mathcal{H}_B$,*

$$U_p(|a\rangle|0\rangle|b\rangle) = \sum_{u \in \{0, 1\}^c} |A_u(a)\rangle|u_c\rangle|B_u(b)\rangle.$$

Moreover, $\|A\|, \|B\| \leq 1$ for every $u \in \{0, 1\}^c$.

Proof. It is only the last observation (about the operator norms) that is (apparently) new. This, however, immediately follows from Proposition 3.2.2a) and the fact that every operator A_u, B_u is composed from unitary operators and orthogonal projections onto the subspaces $\mathcal{H}_A \otimes |\epsilon\rangle \otimes \mathcal{H}_B$, $\epsilon \in \{0, 1\}$. ■

3.4. Symmetric functions and predicates

For a Boolean predicate $D : \{0, 1, \dots, n\} \rightarrow \{0, 1\}$, denote by $\widetilde{\deg}(D)$ the *approximate degree* of this predicate defined as the minimal degree of a univariate real polynomial $f(x)$ such that $|f(s) - D(s)| \leq 1/3$ for every $s \in \{0, 1, \dots, n\}$. Let $g_D(x_1, \dots, x_n)$ be the symmetric Boolean function defined as $g_D(x) \stackrel{\text{def}}{=} D(\sum_{i=1}^n x_i)$ (note that $f_{n,D}(x, y) = f_{g_D}(x, y)$). [NS94] observed that $\widetilde{\deg}(D) = \widetilde{\deg}(g_D)$, where $\widetilde{\deg}(g)$ is the minimal degree of a multi-variate polynomial approximating g on $\{0, 1\}^n$ within error $1/3$ in the ℓ_∞ -norm.

Proposition 3.4 ([Pat92]) $\widetilde{\deg}(D) = \theta(\sqrt{n(\ell_0(D) + \ell_1(D))})$.

It was proved in [BBC⁺98] that $\Omega(\widetilde{\deg}(g))$ is a general lower bound on $Q_{\text{DT}}(g)$. In the opposite direction, they show that for symmetric functions this bound is tight:

Proposition 3.5 ([BBC⁺98]) $Q_{\text{DT}}(g_D) \leq O(\widetilde{\deg}(D))$.

Assume now that $X = Y \stackrel{\text{def}}{=} [n]^k$. For $0 \leq s \leq k$, denote by $J_{n,k,s}$ the $0-1$ $\binom{n}{k} \times \binom{n}{k}$ matrix whose rows and columns are indexed by $[n]^k$ and $(J_{n,k,s})_{xy} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } |x \cap y| = s \\ 0 & \text{otherwise} \end{cases}$. The spectrum of these matrices is described by the so-called *Hahn polynomials* (see e.g. [Del78]). The latter, being classical objects, were re-discovered many times in different contexts; the expression that is the most convenient for our purposes was proposed by Knuth [Knu91]; remarkably, it is based on a direct computation of the eigenvalues.

Proposition 3.6 ([Knu91]) *Let $k \leq n/2$. The matrices $J_{n,k,s}$ ($0 \leq s \leq k$) share the same eigenspaces E_0, E_1, \dots, E_k . The eigenvalue of $J_{n,k,s}$ corresponding to the eigenspace E_t is given by*

$$\sum_{i=\max\{0, s+t-k\}}^{\min\{s,t\}} (-1)^{t-i} \binom{t}{i} \binom{k-i}{s-i} \binom{n-k-t+i}{k-s-t+i}.$$

4. Upper bound

In this section we show that the upper bound $Q(f_{n,D}) \leq O((\sqrt{n\ell_0(D)} + \ell_1(D)) \log n)$ in Theorem 2.1 is almost immediately implied by the previously known results from Section 3.

Let $D : \{0, 1, \dots, n\} \rightarrow \{0, 1\}$ be any predicate. D is constant on the interval $[\ell_0(D), n - \ell_1(D)]$. Negating it if necessary, we can assume that D takes on value 0 in this interval. Then $D = D_0 \vee D_1$, where $D_0^{-1}(1) \subseteq [0, \ell_0(D) - 1]$ and $D_1^{-1}(1) \subseteq [n - \ell_1(D) + 1, n]$. Also, $f_D = f_{D_0} \vee f_{D_1}$, and Alice and Bob compute f_{D_0} and f_{D_1} separately.

For computing f_{D_0} , they apply the BCW-reduction (Proposition 3.1) and Propositions 3.5, 3.4:

$$Q(f_{D_0}) \leq O(Q_{\text{DT}}(g_{D_0}) \log n) \leq O(\widetilde{\text{deg}}(D_0) \log n) \leq O(\sqrt{\ell_0(D)} \log n).$$

For computing f_{D_1} , Alice and Bob use the following trivial (classical) protocol. Alice first checks whether her input x has $\leq n - \ell_1(D)$ ones or not. In the first case $f_{D_1}(x, y) = 0$ and she declares the result. Otherwise she sends her entire input to Bob. This will take at most $\log_2 \left(\sum_{k=n-\ell_1(D)+1}^n \binom{n}{k} \right)$ bits which is $O(\ell_1(D) \log n)$ since $\ell_1(D) \leq n/2$. Then Bob computes $f_{D_1}(x, y)$.

5. Lower bounds

In this section we prove the lower bound in Theorem 2.1 and Theorem 2.3. First we show that the latter implies the first, and this is done by a straightforward reduction.

Definition 5.1 For a Boolean predicate D on $\{0, 1, \dots, n\}$ and $0 \leq r \leq n$, let $D - r : \{0, 1, \dots, n - r\} \rightarrow \{0, 1\}$ be given by $(D - r)(s) \stackrel{\text{def}}{=} D(r + s)$. Let also $D|_k$ be the restriction of D onto $\{0, 1, \dots, k\}$, $k \leq n$.

Lemma 5.2 For every predicate D on $\{0, 1, \dots, n\}$ and every integers k, r satisfying $0 \leq r \leq n$, $k \leq n - r$, we have $Q^*(f_{n,D}) \geq Q^*(f_{n-r,k,(D-r)|_k})$.

Proof. Alice and Bob use the optimal protocol for $f_{n,D} : \mathcal{P}([n]) \times \mathcal{P}([n]) \rightarrow \{0, 1\}$ to compute $f_{n-r,k,(D-r)|_k} : [n-r]^k \times [n-r]^k \rightarrow \{0, 1\}$. For this they simply map their inputs $x, y \in [n-r]^k$ to the inputs $\phi(x), \phi(y) \in \mathcal{P}([n])$ using the mapping $\phi(x) \stackrel{\text{def}}{=} x \cup \{n-r+1, \dots, n\}$, and feed $\phi(x), \phi(y)$ into the protocol for $f_{n,D}$. ■

Proof of lower bound in Theorem 2.1 from Theorem 2.3. We need to establish two separate bounds, $Q^*(f_{n,D}) \geq \Omega(\sqrt{n\ell_0(D)})$ and $Q^*(f_{n,D}) \geq \Omega(\ell_1(D))$, and both are proved via a reduction from Lemma 5.2 (with different values r, k of course). In choosing r, k we must satisfy the two conditions

$$k \leq (n-r)/4, \quad (\ell-r) \leq k/4 \tag{6}$$

(arising from the statement of Theorem 2.3), where $\ell \stackrel{\text{def}}{=} \ell_0(D)$ for the first bound and $\ell \stackrel{\text{def}}{=} n - \ell_1(D)$ for the second. As long as they are satisfied, Theorem 2.3 gives $Q^*(f_{n,D}) \geq \Omega(\sqrt{k(\ell-r)})$.

If $\ell \leq n/16$ (and, in particular, $\ell = \ell_0(D)$), we simply let $r \stackrel{\text{def}}{=} 0$ and $k \stackrel{\text{def}}{=} n/4$. Then the bound of Theorem 2.3 becomes $\Omega(\sqrt{n\ell})$; i.e., exactly what what we are proving.

If $\ell \geq n/16$, we satisfy the conditions (6) with equality for which we set $r \stackrel{\text{def}}{=} \frac{16\ell-n}{15}$ and $k \stackrel{\text{def}}{=} \frac{4}{15}(n-\ell)$. Then $\ell-r \geq \Omega(n-\ell)$, and Theorem 2.3 still gives us the required bound $Q^*(f_{n,D}) \geq \Omega(n-\ell)$. ■

In the rest of the paper we prove Theorem 2.3. The proof splits into three fairly independent blocks.

5.1. Approximate trace norm lower bound

Definition 5.3 For a real matrix M , let $\|M\|_{\text{tr}}^\epsilon \stackrel{\text{def}}{=} \min\{\|P\|_{\text{tr}} : \ell_\infty(M-P) \leq \epsilon\}$ be its ϵ -approximate trace norm (P runs over all real matrices of the same size as M).

Definition 5.4 For a predicate $f : X \times Y \rightarrow \{0, 1\}$, M_f denotes its communication 0-1 matrix $(M_f)_{xy} \stackrel{\text{def}}{=} f(x, y)$.

Theorem 5.5 For any predicate $f : X \times Y \rightarrow \{0, 1\}$, where $|X| = |Y| = N$, and any $\epsilon > 0$, $Q_\epsilon^*(f) \geq \Omega(\log(\|M_f\|_{\text{tr}}^\epsilon/N))$.

Proof. Fix a c -qubit communication protocol with prior entanglement computing f with probability ϵ . Let p_{xy} be the acceptance probabilities of this protocol on the input (x, y) ; arrange them into an $(N \times N)$ matrix P . Then, clearly, $\ell_\infty(M_f - P) \leq \epsilon$, and we only have to prove that $\|P\|_{\text{tr}} \leq N \cdot \exp(O(c))$.

Apply the decomposition from Proposition 3.3 to the input string (4). We get:

$$\text{Output}(x, y) = \frac{1}{|E|^{1/2}} \sum_{e \in E} \sum_{u \in \{0,1\}^c} A_u |0, x, e\rangle |u_c\rangle B_u |e, y, 0\rangle$$

and then

$$\begin{aligned} p_{xy} &= \frac{1}{|E|} \left\| \sum_{e \in E} \sum_{u \in \Pi} A_u |0, x, e\rangle B_u |e, y, 0\rangle \right\|^2 \\ &= \frac{1}{|E|} \cdot \sum_{e, f \in E} \sum_{u, v \in \Pi} (\langle f, x, 0 | A_v | A_u | 0, x, e \rangle \cdot \langle f, y, 0 | B_v | B_u | 0, y, e \rangle), \end{aligned}$$

where $\Pi \stackrel{\text{def}}{=} \{u \in \{0, 1\}^c \mid u_c = 1\}$.

Let us now define $N \times (|E|^2 \times |\Pi|^2)$ -matrices A, B by letting $a_{x, (efuv)} \stackrel{\text{def}}{=} \langle f, x, 0 | A_v | A_u | 0, x, e \rangle$ and $b_{y, (efuv)} \stackrel{\text{def}}{=} \langle f, y, 0 | B_v | B_u | 0, y, e \rangle$. Then $P = \frac{1}{|E|} AB^\perp$, and Proposition 3.2.2b) implies

$$\|P\|_{\text{tr}} \leq \frac{1}{|E|} \cdot \|A\|_{\text{F}} \cdot \|B\|_{\text{F}}. \quad (7)$$

For estimating $\|A\|_{\text{F}}, \|B\|_{\text{F}}$, we divide these matrices into $N \cdot |\Pi|^2$ blocks, and interpret every block as an $(|E| \times |E|)$ matrix. Namely, for any fixed $x \in X$ and $u, v \in \Pi$, let A^{xuv} be the square $(|E| \times |E|)$ matrix given by $a_{ef}^{xuv} \stackrel{\text{def}}{=} a_{x, (efuv)} = \langle f, x, 0 | A_v | A_u | 0, x, e \rangle$. Then

$$\|A\|_{\text{F}}^2 \leq N \cdot |\Pi|^2 \cdot \max_{x, u, v} \|A^{xuv}\|_{\text{F}}^2. \quad (8)$$

For bounding $\|A^{xuv}\|_{\text{F}}$ we first use Proposition 3.2.3:

$$\|A^{xuv}\|_{\text{F}} \leq |E|^{1/2} \cdot \|A^{xuv}\|. \quad (9)$$

Finally we claim that

$$\|A^{xuv}\| \leq 1. \quad (10)$$

Indeed, let η, ξ be any vectors of length $|E|$ with $\|\eta\|, \|\xi\| \leq 1$. Then we have

$$\eta^\top A^{xuv} \xi = \langle \sum_{f \in E} \eta_f f, x, 0 | A_v | A_u | 0, x, \sum_{e \in E} \xi_e e \rangle$$

and, since $\|A_u\|, \|A_v\| \leq 1$,

$$\begin{aligned} \|\eta^\top A^{xuv} \xi\| &\leq \|A_u | 0, x, \sum_{e \in E} \xi_e e\| \cdot \|A_v | 0, x, \sum_{f \in E} \eta_f f\| \\ &\leq \| | 0, x, \sum_{e \in E} \xi_e e \rangle\| \cdot \| | 0, x, \sum_{f \in E} \eta_f f \rangle\| = \|\xi\| \cdot \|\eta\| \leq 1. \end{aligned}$$

(10) is proved. Along with (9) and (8) this implies $\|A\|_F \leq N^{1/2} \cdot |\Pi| \cdot |E|^{1/2}$, and the same bound holds for $\|B\|_F$. Substituting them into (7), we get $\|P\|_{\text{tr}} \leq N \cdot |\Pi|^2 \leq N \cdot \exp(O(c))$ which completes the proof of Theorem 5.5. ■

Remark 4 (Nayak, Shi) Theorem 5.5 (and, hence, all lower bounds following from it) extends to the case of more general entanglement in which the input vector is given by (5). In order to see this, first note the following generalization of the right-hand side in Proposition 3.2.3:

$$\|LA\|_F \leq \|L\|_F \cdot \|A\| \quad (11)$$

(the original statement corresponds to $L = I_{\min\{m,n\}}$). If $\hat{a}_{x,(efuv)} \stackrel{\text{def}}{=} \lambda_e a_{x,(efuv)}$ and $\hat{b}_{y,(efuv)} \stackrel{\text{def}}{=} \lambda_f b_{y,(efuv)}$ then $\hat{P} = \hat{A} \hat{B}^\top$, where \hat{P} is the matrix of acceptance probabilities relative to the input vector (5), and $\|\hat{P}\|_{\text{tr}} \leq \|\hat{A}\|_F \cdot \|\hat{B}\|_F$. As before, $\|\hat{A}\|_F^2 \leq N \cdot |\Pi|^2 \cdot \max_{x,u,v} \|\hat{A}^{xuv}\|_F^2$. We, however, know that $\hat{A}^{xuv} = LA^{xuv}$, where L is the diagonal matrix with elements $\{\lambda_e \mid e \in E\}$. Since λ is unitary, $\|L\|_F = 1$ and (11) implies $\|\hat{A}^{xuv}\|_F \leq \|A^{xuv}\| \leq 1$. The remaining calculations are the same as in the basic proof.

5.2. Multi-dimensional discrepancy bound

This section is central to our argument, so we begin with a brief overview of the ordinary discrepancy bound.

Suppose that we want to get a lower bound on the approximate trace norm (or any other approximate norm) of a matrix M . That is, we need to rule out the existence of a decomposition $M = P + \Delta$, where $\|P\|_{\text{tr}}$ is small and $\ell_\infty(\Delta)$ is small. The ordinary discrepancy method [Yao93, Kre95] proceeds as follows. Assume that M is a ± 1 -matrix, take any probability distribution μ on its entries and form the Hadamard product $M \circ \mu$ ($(M \circ \mu)_{ij} \stackrel{\text{def}}{=} M_{ij}\mu_{ij}$). Then $\langle M, M \circ \mu \rangle = 1$ and $|\langle \Delta, M \circ \mu \rangle| \leq \ell_1(M \circ \mu) \cdot \ell_\infty(\Delta) = \ell_\infty(\Delta)$. Therefore, if $|\langle P, M \circ \mu \rangle|$ is small for every matrix P with small trace norm (in other words, $M \circ \mu$ has a *low discrepancy* with such matrices), we are done.

The next logical step was taken by Klauck in [Kla01, Theorem 4] who observed that the “test matrix” need not be of the particular form $M \circ \mu$. As long as μ is *any* matrix with $\ell_1(\mu) = 1$ and of low discrepancy, we are still in a good shape for *all* matrices M for which $|\langle M, \mu \rangle|$ is at least *somewhat* large.

It is well known, however, that even in this form the discrepancy method does not work for (say) the disjointness predicate. In this paper we take it one step further and instead of considering the linear functional $X \mapsto \langle X, \mu \rangle$ for a *single* “test matrix” μ , we consider the multi-dimensional “trace operator” $X \mapsto (\langle X, \mu_1 \rangle, \dots, \langle X, \mu_r \rangle)$ for a *family* of matrices μ_1, \dots, μ_r with $\ell_1(\mu_s) \leq 1$. In order to be able to apply spectral methods, we will assume that μ_1, \dots, μ_r are real symmetric commuting matrices (although it would be sufficient to assume that they allow singular value decompositions $U\mu_1V, \dots, U\mu_rV$ with common unitary matrices U, V).

Definition 5.6 An *r-dimensional discrepancy test* consists of real symmetric matrices μ_1, \dots, μ_r with $\ell_1(\mu_s) \leq 1$ ($1 \leq s \leq r$) that have the same size $N \times N$ and commute with each other, along with an orthogonal decomposition

$$\mathbb{R}^N = E_1 \oplus E_2 \oplus \dots \oplus E_k \tag{12}$$

of \mathbb{R}^N into their shared eigenspaces E_1, E_2, \dots, E_k .

Note that the commutativity alone implies the existence of at least one decomposition (12). For our application we, however, need $k \ll N$ (that is, eigenvalues substantially repeat themselves), and for this reason we prefer to fix the decomposition explicitly in the definition.

Given a discrepancy test $(\mu_1, \dots, \mu_r, E_1, \dots, E_k)$, denote by λ_{st} the eigenvalue of μ_s corresponding to E_t . Let the *trace of E_t* be the r -dimensional vec-

tor λ^t naturally defined as $(\lambda^t)_s \stackrel{\text{def}}{=} \lambda_{st}$, and let $\text{Trace}(\bar{\mu}, \bar{E}) \stackrel{\text{def}}{=} \{\lambda^t \in \mathbb{R}^r \mid 1 \leq t \leq k\}$ be the set of all these vectors.

Definition 5.7 Given a set of vectors $T \subseteq \mathbb{R}^r$ and $C > 0$, let $\text{Conv}_C(T) \stackrel{\text{def}}{=} \{\sum_{\lambda \in T} a_\lambda \lambda : \sum_{\lambda \in T} |a_\lambda| \leq C\}$ be the convex hull of the segments $\{[-C, C]\lambda \mid \lambda \in T\}$. Given another vector $\xi \in \mathbb{R}^r$ and $\epsilon > 0$, let $\phi^\epsilon(\xi, T) \stackrel{\text{def}}{=} \min \{C \mid \rho_\infty(\xi, \text{Conv}_C(T)) \leq \epsilon\}$, where ρ_∞ is the distance in the ℓ_∞ -norm.

Theorem 5.8 Let M be a real squared matrix, and $(\mu_1, \dots, \mu_r, E_1, \dots, E_k)$ be an arbitrary r -dimensional test of the same size. Let $\xi_M \in \mathbb{R}^r$ be defined as $(\xi_M)_s \stackrel{\text{def}}{=} \langle M, \mu_s \rangle$. Then

$$\|M\|_{\text{tr}}^\epsilon \geq \phi^\epsilon(\xi_M, \text{Trace}(\bar{\mu}, \bar{E})).$$

Proof. Let $\|M\|_{\text{tr}}^\epsilon = C$ and $M = P + \Delta$, where $\|P\|_{\text{tr}} = C$ and $\ell_\infty(\Delta) \leq \epsilon$. Then $\xi_M = \xi_P + \xi_\Delta$ and, moreover, $|(\xi_\Delta)_s| = |\langle \Delta, \mu_s \rangle| \leq \ell_1(\mu_s) \cdot \ell_\infty(\Delta) \leq \epsilon$ for every $s \in [r]$ which implies $\ell_\infty(\xi_\Delta) \leq \epsilon$. Thus, we only need to prove that $\xi_P \in \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E}))$.

Let U be the orthogonal matrix corresponding to the decomposition (12), so that all $(U^\top \mu_s U)$ are diagonal. Consider the matrix $(U^\top P U)$, for every $t \in [k]$ let $(U^\top P U)_t$ be its principal submatrix corresponding to the eigenspace E_t , and let $a_t \stackrel{\text{def}}{=} \text{Tr}((U^\top P U)_t)$. Then $\xi_P \in \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E}))$ is implied by the following two facts:

$$\xi_P = \sum_{t=1}^k a_t \lambda^t$$

and

$$\sum_{t=1}^k |a_t| \leq C.$$

Both are proved by easy matrix manipulations (with heavy use of Proposition 3.2):

$$(\xi_P)_s = \langle P, \mu_s \rangle = \langle (U^\top P U), (U^\top \mu_s U) \rangle = \sum_{t=1}^k \text{Tr}((U^\top P U)_t) \cdot \lambda_{st} = \sum_{t=1}^k a_t \lambda_{st}$$

and

$$\sum_{t=1}^k |a_t| \leq \sum_{i=1}^N |(U^\top P U)_{ii}| \leq \|(U^\top P U)\|_{\text{tr}} = \|P\|_{\text{tr}} = C.$$

■

5.3. Putting things together

Now we are ready to finish the proof of Theorem 2.3. Fix integers n and $k \leq n/4$. Set $N \stackrel{\text{def}}{=} \binom{n}{k}$. Let $D : \{0, 1, \dots, k\} \longrightarrow \{0, 1\}$ be any predicate such that $D(\ell) \neq D(\ell - 1)$ for some $\ell \leq k/4$. Applying Theorem 5.5 (and observing that the error probability can be always reduced from $1/3$ to $1/4$ with an increase in complexity by at most a constant multiplicative factor), we get

$$Q^*(f_{n,k,D}) \geq \Omega(\log(\|M_{f_{n,k,D}}\|_{\text{tr}}^{1/4}/N)). \quad (13)$$

Let now $\mu_s \stackrel{\text{def}}{=} N^{-1} \binom{k}{s}^{-1} \binom{n-k}{k-s}^{-1} J_{n,k,s}$, and let E_0, \dots, E_k be the shared eigenspaces of these matrices from Proposition 3.6. Note that $\ell_1(\mu_s) = 1$ and $\langle M_{f_{n,k,D}}, \mu_s \rangle = D(s)$. Applying Theorem 5.8 with the $(k/2 + 1)$ -dimensional test $(\mu_0, \mu_1, \dots, \mu_{k/2}, E_0, E_1, \dots, E_k)$, we get

$$\|M_{f_{n,k,D}}\|_{\text{tr}}^{1/4} \geq \phi^{1/4}(D|_{k/2}, \text{Trace}(\bar{\mu}, \bar{E})). \quad (14)$$

Claim 5.9 *Let λ_{st} be the eigenvalue of the matrix μ_s corresponding to the eigenspace E_t . Then:*

1. $\lambda_{st} = F_t(s)$, where F_t is a polynomial of degree t (known, up to a normalizing factor, as Hahn polynomial);
2. whenever $k \leq n/4$ and $s \leq k/2$, $|\lambda_{st}| \leq N^{-1} \cdot \exp(-\Omega(t))$.

Proof. By Proposition 3.6,

$$\begin{aligned} \lambda_{st} &= N^{-1} \binom{k}{s}^{-1} \binom{n-k}{k-s}^{-1} \cdot \sum_{i=\max\{0, s+t-k\}}^{\min\{s,t\}} (-1)^{t-i} \binom{t}{i} \binom{k-i}{s-i} \binom{n-k-t+i}{k-s-t+i} \\ &= N^{-1} \sum_{i=\max\{0, s+t-k\}}^{\min\{s,t\}} (-1)^{t-i} \binom{t}{i} \frac{\binom{k-i}{s-i} \binom{n-k-t+i}{k-s-t+i}}{\binom{k}{s} \binom{n-k}{k-s}} \\ &= N^{-1} \sum_{i=0}^t \left((-1)^{t-i} \binom{t}{i} \frac{s(s-1)\dots(s-i+1)}{k(k-1)\dots(k-i+1)} \times \right. \\ &\quad \left. \times \frac{(k-s)(k-s-1)\dots(k-s-t+i+1)}{(n-k)(n-k-1)\dots(n-k-t+i+1)} \right). \end{aligned}$$

Part 1 is already obvious from this expression. Part 2 is also easy:

$$\begin{aligned}
|\lambda_{st}| &\leq N^{-1} \sum_{i=0}^t \left(\binom{t}{i} \frac{s(s-1)\dots(s-i+1)}{k(k-1)\dots(k-i+1)} \times \right. \\
&\quad \left. \times \frac{(k-s)(k-s-1)\dots(k-s-t+i+1)}{(n-k)(n-k-1)\dots(n-k-t+i+1)} \right) \\
&\leq N^{-1} \sum_{i=0}^t \binom{t}{i} \left(\frac{s}{k} \right)^i \cdot \left(\frac{k-s}{n-k} \right)^{t-i} = N^{-1} \cdot \left(\frac{s}{k} + \frac{k-s}{n-k} \right)^t \leq N^{-1} \cdot \left(\frac{1}{2} + \frac{1}{3} \right)^t.
\end{aligned}$$

■

This claim implies that for every $t_0 \leq k$, $\{\lambda^t \mid t \leq t_0\} \subseteq P(t_0)$, where $P(t_0)$ is the set of all real-valued functions on $\{0, 1, \dots, k/2\}$ representable by (real) polynomials of degree $\leq t_0$. Whereas $\ell_\infty(\lambda^t) \leq N^{-1} \exp(-\Omega(t_0))$ for $t \geq t_0$. Hence,

$$\forall \xi \in \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E})), \rho_\infty(\xi, P(t_0)) \leq N^{-1} \cdot C \cdot \exp(-\Omega(t_0)). \quad (15)$$

Set now $t_0 \stackrel{\text{def}}{=} \widetilde{\text{deg}}(D|_{k/2}) - 1$ and $C \stackrel{\text{def}}{=} \phi^{1/4}(D|_{k/2}, \text{Trace}(\bar{\mu}, \bar{E}))$. Note that since $\ell \leq k/4$,

$$t_0 \geq \Omega(\sqrt{k\ell}) \quad (16)$$

by Proposition 3.4. Also, by definition of the approximate degree, $\rho_\infty(D|_{k/2}, P(t_0)) > 1/3$. On the other hand, by (15),

$$\begin{aligned}
\rho_\infty(D|_{k/2}, P(t_0)) &\leq N^{-1} \cdot C \cdot \exp(-\Omega(t_0)) + \rho_\infty(D|_{k/2}, \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E}))) \\
&\leq N^{-1} \cdot C \cdot \exp(-\Omega(t_0)) + 1/4.
\end{aligned}$$

Combining these two bounds with (16), we get

$$\phi^{1/4}(D|_{k/2}, \text{Trace}(\bar{\mu}, \bar{E})) = C \geq N \cdot \exp(\Omega(\sqrt{k\ell})). \quad (17)$$

Theorem 2.3 now follows from (13), (14) and (17).

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References

- [AST⁺98] A. Ambainis, L. Schulman, A. Ta-Shma, U. Vazirani, and A. Wigderson. The quantum communication complexity of sampling. In *Proceedings of the 39th IEEE Symposium on Foundations of Computer Science*, pages 342–351, 1998.
- [BBC⁺98] R. Beals, H. Buhrman, R. Cleve, M. Mosca, and R. de Wolf. Quantum lower bounds by polynomials. In *Proceedings of the 39th IEEE Symposium on Foundations of Computer Science*, pages 352–361, 1998. Preliminary version available at [quant-ph/9802049](#).
- [BCW98] H. Buhrman, R. Cleve, and A. Wigderson. Quantum vs. classical communication and computation. In *Proceedings of the 30th ACM Symposium on the Theory of Computing*, pages 63–86, 1998. Preliminary version available at [quant-ph/9802040](#).
- [BFS86] L. Babai, P. Frankl, and J. Simon. Complexity classes in communication complexity theory. In *Proceedings of the 27th IEEE FOCS*, pages 337–347, 1986.
- [Bha97] R. Bhatia. *Matrix Analysis. Graduate texts in mathematics, 169*. Springer-Verlag, 1997.
- [BW00] H. Buhrman and R. de Wolf. Complexity measures and decision tree complexity: a survey. Manuscript to appear in *Theoretical Computer Science*, 2000.
- [BW01] H. Buhrman and R. de Wolf. Communication complexity lower bounds by polynomials. In *Proceedings of the 16th IEEE Conference on Computational Complexity*, pages 120–130, 2001. Preliminary version available at [cs.CC/9910010](#).
- [CB97] R. Cleve and H. Buhrman. Substituting quantum entanglement for communication. *Physical Reviews A*, 56:1201–1204, 1997. Preliminary version available at [quant-ph/9704026](#).
- [CDNT98] R. Cleve, W. van Dam, M. Nielsen, and A. Tapp. Quantum entanglement and the communication complexity of the inner

- product function. In *Proceedings of the 1st NASA QCQC Conference, Lecture Notes in Computer Science*, 1509, pages 61–74, New York/Berlin, 1998. Springer-Verlag. Preliminary version available at quant-ph/9708019.
- [CG88] B. Chor and O. Goldreich. Unbiased bits from sources of weak randomness and probabilistic communication complexity. *SIAM Journal on Computing*, 17(2):230–261, 1988.
- [Del78] Ph. Delsarte. Hahn polynomials, discrete harmonics and t -designs. *SIAM Journal on Applied Mathematics*, 34(1):157–166, 1978.
- [Gro96] L. K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the 28th ACM Symposium on the Theory of Computing*, pages 212–219, 1996. Preliminary version available at quant-ph/9605043.
- [HW02] P. Høyer and R. de Wolf. Improved quantum communication complexity bounds for disjointness and equality. In *Proceedings of the 19th Annual Symposium on Theoretical Aspects of Computer Science*, pages 299–310, 2002. Preliminary version available at quant-ph/0109068.
- [Kla01] H. Klauck. Lower bounds for quantum communication complexity. In *Proceedings of the 42nd IEEE Symposium on Foundations of Computer Science*, pages 288–297, 2001. Preliminary version available at quant-ph/0106160.
- [KNTZ01] H. Klauck, A. Nayak, A. Ta-Shma, and D. Zuckerman. Interaction in quantum communication and the complexity of set disjointness. In *Proceedings of the 33rd ACM Symposium on the Theory of Computing*, pages 124–133, 2001.
- [Knu91] D. Knuth. Combinatorial matrices. Manuscript available at <http://www-cs-faculty.stanford.edu/~knuth/preprints.html#unpub>, 1991.
- [Kre95] I. Kremer. Quantum communication. Master’s thesis, Hebrew University, Jerusalem, 1995.

- [KS92] B. Kalyanasundaram and G. Schnitger. The probabilistic communication complexity of set intersection. *SIAM Journal on Discrete Mathematics*, 5(4):545–557, 1992.
- [MS82] K. Mehlhorn and E. M. Schmidt. Las Vegas is better than determinism in VLSI and distributive computing. In *Proceedings of the 14th ACM STOC*, pages 330–337, 1982.
- [NS94] N. Nisan and M. Szegedy. On the degree of Boolean functions as real polynomials. *Computational complexity*, 4:301–313, 1994.
- [NS02] A. Nayak and J. Salzman. On communication over an entanglement-assisted quantum channel. In *Proceedings of the 34th ACM Symposium on the Theory of Computing*, 2002.
- [Pat92] R. Paturi. On the degree of polynomials that approximate symmetric Boolean functions. In *Proceedings of the 24th ACM Symposium on the Theory of Computing*, pages 468–474, 1992.
- [Raz92] A. Razborov. On the distributional complexity of disjointness. *Theoretical Computer Science*, 106:385–390, 1992.
- [Vaz87] U. Vazirani. Towards a strong communication complexity theory of generating quasi-random sequences from two communicating slightly-random sources. *Combinatorica*, 7(4):375–392, 1987.
- [Yao79] A. Yao. Some complexity questions related to distributive computing. In *Proceedings of the 11th ACM STOC*, pages 209–213, 1979.
- [Yao93] A. Yao. Quantum circuit complexity. In *Proceedings of the 34th IEEE Symposium on Foundations of Computer Science*, pages 352–361, 1993.