An algebraic algorithm for non-commutative rank over any field.

In 1967, J. Edmonds introduced the problem of computing the rank over the rational function field of an $n \times n$ matrix $M_1x_1 + \ldots M_mx_m$ whose entries are homogeneous linear polynomials in commuting variables $x_1, x_2, \ldots, x_m$ with integer coefficients.

We consider the *non-commutative version* of Edmonds' problem over an arbitrary field. The "non-commutative" rank can be interpreted as the rank of the linear matrix $M_1x_1+\ldots+M_mx_m$ in the free skewfield generated by the non-commuting variables $x_1, \ldots, x_m$. This non-commutative rank is an upper bound for the rank of $M_1x_1 + \ldots + M_mx_m$ in the commuting variables $x_1, \ldots, x_m$.

We present a deterministic polynomial time algorithm which, given a collection $M_1, \ldots, M_m$ of n by n matrices over the field $\mathbb{F}$, computes the non-commutative rank r, and outputs d by d matrices $T_1, \ldots T_m$ such that the nd by nd block matrix $M_1 \otimes T_1 + \ldots + M_m \otimes T_m$ has rank rd.

When $r < n$ we also compute n by n invertible matrices L and R such that for some integer ℓ, the upper right $r - \ell$ by ℓ block of LM_jR is zero for all $j = 1, \ldots, m$ providing evidence to the fact that all these matrices compress a subspace of dimension l into a subspace of dimension $n - (r - l)$.

The key ingredient of the algorithm is an analogue of augmenting paths for matchings in bipartite graphs, combined with a *regularity* property of "blown up" matrix spaces. (The d-blowup of the matrix space generated by $M_1, \ldots, M_m$ is just the matrix space where the output sits: the space of nd by nd matrices of the form $M_1 \otimes T_1 + \ldots + M_m \otimes T_m$, where the T_j are arbitrary d by d matrices.)

It is known that this problem relates to the following ring of matrix semi-invariants denoted $R(n, m)$. For a field $\mathbb{F}$ it is the ring of semi-invariant polynomials for the action of $\mathrm{GL}(n, \mathbb{F}) \times \mathrm{GL}(n, \mathbb{F})$ on tuples of matrices :$(A, C) \in \mathrm{GL}(n, \mathbb{F}) \times \mathrm{GL}(n, \mathbb{F})$ sending $(M_1, \ldots, M_m)$ to $(AM_1C^{\mathrm{T}}, \ldots, AM_mC^{\mathrm{T}})$. Then $(M_1, M_2, \ldots, M_m)$ with non-commutative rank $r < n$, correspond to points where all non constant polynomials in $R(n, m)$ vanish.

This is joint work with Gábor Ivanyos and Youming Qiao.