WORKSHOP ON EMERGING TOPICS: TORSION CLASSES AND MODULARITY

Description of the topic. Between October 31 and November 4 2016, the Institute hosted a workshop on emerging topics focused on the Langlands program for number fields. At the heart of the Langlands program heart lies *reciprocity*, a mysterious relationship between automorphic forms and Galois representations. The Galois representations of interest are essentially those arising from the cohomology of varieties defined over number fields. A famous instance of reciprocity is the modularity of elliptic curves defined over $\mathbb{Q}$. The specific goal of the IAS workshop was to investigate further instances of reciprocity, such as the modularity of elliptic curves defined over *imaginary quadratic fields*, as well as applications of any newly discovered reciprocity laws.

The general strategy for establishing new reciprocity laws is to prove *modularity lifting theorems* using a technique known as the *Taylor-Wiles method*. The Taylor-Wiles method uses congruences in a systematic fashion to match Galois representations to automorphic forms in families. Traditionally, this depended on a certain numerical criterion, which forced one to work over totally real extensions of $\mathbb{Q}$ and excluded the imaginary quadratic case. Understanding the imaginary quadratic case has been an open question for many years, and seemed to require significantly new ideas.

Timeliness of the workshop. In 2011, Calegari and Geraghty proposed an extension of the Taylor-Wiles method which was expected to work in vastly more general situations, including over imaginary quadratic fields, provided that certain preliminary steps were in place. The preliminary steps needed by Calegari and Geraghty involve a precise understanding of the *torsion classes* which occur in the cohomology of certain locally symmetric spaces, such as arithmetic hyperbolic 3-manifolds. These torsion classes should be thought of as mod p (or mod p^n) versions of automorphic forms.

In 2012, Scholze proved, using his theory of perfectoid spaces, that all such torsion classes are themselves associated to Galois representations. This was a major breakthrough and provided the first ingredient needed to make the Calegari-Geraghty method go through. Nevertheless, one needed to have more precise control over these Galois representations to apply Scholze's theorem. In late 2015, Caraiani and Scholze found a strategy for obtaining this stronger control. By 2016, it became clear to experts that one could finally hope to carry out the Calegari-Geraghty method, prove modularity lifting theorems in new settings, and understand something new about elliptic curves over imaginary quadratic fields.

Participants. The workshop was co-organized by Ana Caraiani (University of Bonn) and Richard Taylor (IAS). The participants were Patrick Allen (UIUC), Frank Calegari (University of Chicago), Matthew Emerton (University of Chicago), Toby Gee (Imperial College), David Helm (Imperial College), Bao Le Hung (University of Chicago), James Newton (King's College), Peter Scholze (University of Bonn), and Jack Thorne (Cambridge University).

Organization. The meeting was organized as a close-knit working group. The first two days were devoted to talks scheduled ahead of time, aimed at bringing all the participants up to speed with the different aspects of the problem: by Caraiani on the on-going work of Caraiani-Scholze, by Scholze on getting started with local-global compatibility, by Calegari on the extension of the Taylor-Wiles method, and by Newton on the work of Khare-Thorne (which, in certain settings, supplied another crucial ingredient needed for the Calegari-Geraghty method). These talks were relatively informal, often developed into extended discussions, and thus led to new insights. The third day was devoted to a discussion in the whole group with a series of participants going to the blackboard for varying periods of time. This seems to have led to key progress. On the fourth day the working group split into 3 subgroups each of which worked on sub-problems which had been identified. Twice during the day the whole working group reconvened to share progress and align goals. The fifth day was largely devoted to taking stock of what had been achieved and to planning the writing of a joint paper.

In order to encourage the sharing of more speculative ideas, outside participation in the lectures was not encouraged. However the weeklong workshop also included two colloquium-style talks, by Caraiani and Calegari, aimed at explaining the emerging topics described above to a general mathematical audience.

Outcome. During the workshop, we obtained one major breakthrough, namely a more general version of Taylor's *Ihara avoidance* argument. This argument had played a key part in the proof (by Clozel, Harris, Shepherd-Barron, Taylor and others) of the Sato-Tate conjecture for elliptic curves defined over $\mathbb{Q}$, as it greatly amplifies the available modularity lifting results. By making Ihara avoidance compatible with the Calegari-Geraghty method and the work of Khare-Thorne, we were able to prove the following:

Theorem 1. *Let F be a CM field (for example, an imaginary quadratic field).*

(1) *Let E/F be a non-CM elliptic curve. Then E is* potentially *modular (i.e. becomes modular over an extension of F) and satisfies the Sato-Tate conjecture.*
(2) *Let π be a regular algebraic cuspidal automorphic representation of GL_2 over F of weight 0. Then π satisfies the generalized Ramanujan-Petersson conjecture at all but finitely many places.*

These results are being written up in a joint paper by Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, and Thorne. This paper is expected to become publically available in 2017.