Calculational Aspects

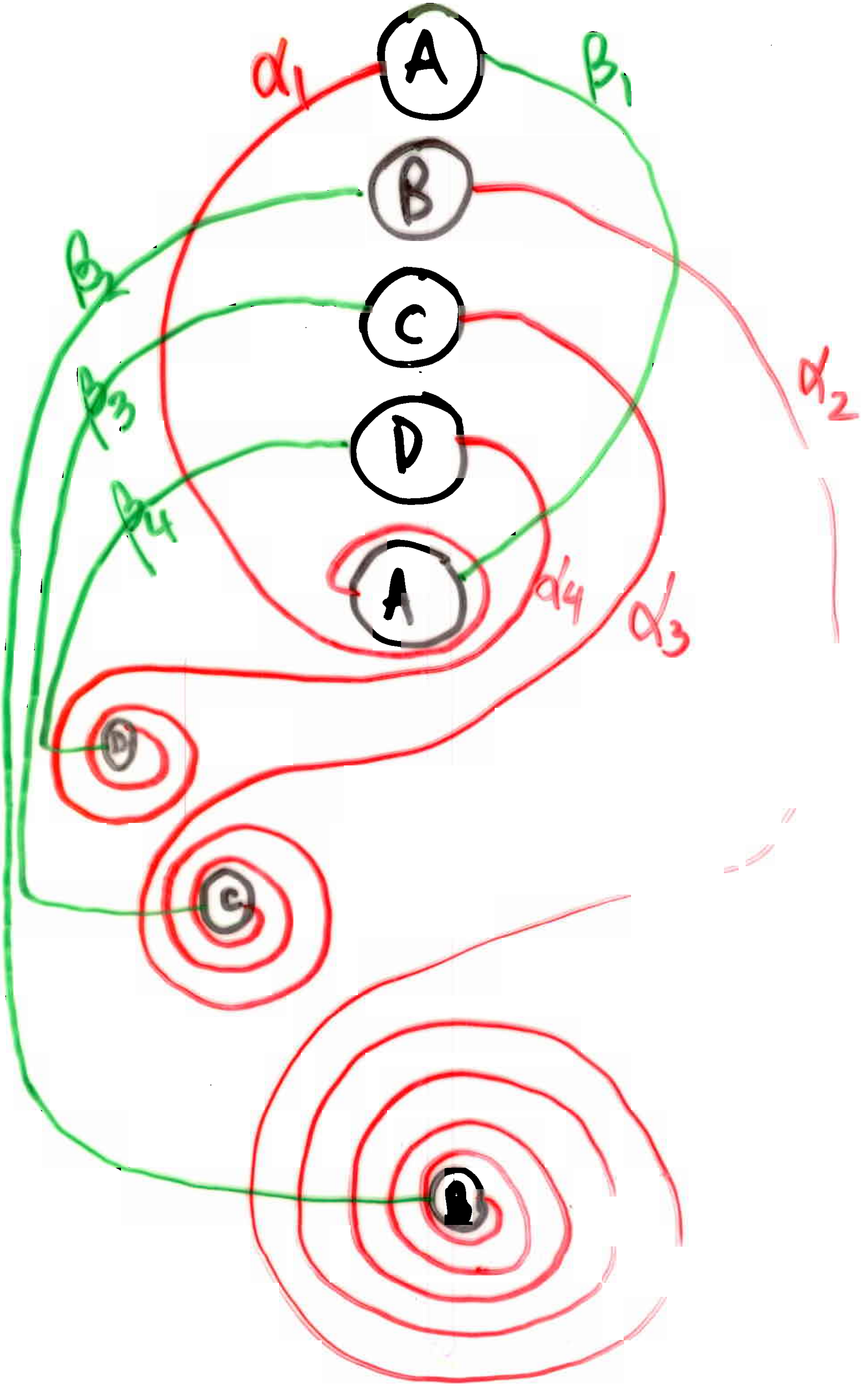
$K \subset Y \rightsquigarrow$ "Knot Floer Homology"

O-Szabó

Rasmussen.

K induces a filtration of $CF(Y)$. Chain homotopy type of filtered complex is knot invariant.

$$(Y,K) \rightsquigarrow \left\{ \begin{tikzcd} & A_s \arrow[dl, "v"'] \arrow[dr, "h"] & \\ B & & B \end{tikzcd} \right\}_{s \in \mathbb{Z}}.$$



A
B
C
D
A
α_1
β_1
β_2
α_2
β_3
β_4
α_4
α_3
D
C
B

Atiyah-Floer Picture:

One should be able to construct interesting 3-manifolds invariants as follows:

$$\Sigma_g \rightsquigarrow V_{\Sigma_g} = \text{symp mfd}$$

$$U \text{ s.t. } \partial U = \Sigma \rightsquigarrow L_U \subset V_\Sigma.$$

$$Y = U_0 \cup_\Sigma U_1 \rightsquigarrow HF^{Lag}(L_{U_0}, L_{U_1})$$

Lagrangian Floer Homology

(V^{2n}, ω) = symplectic manifold

- $d\omega = 0$, $\omega \in \Omega^2(V;\mathbb{R})$
- $\omega^n > 0$.

$L^n \subset V^{2n}$ = Lagrangian submanifold

- $\omega|_L \equiv 0$.

$L_0, L_1 \subset V^{2n}$, an obstruction to separating L_0 + L_1.

$HF(L_0, L_1)$ s.t.

$\chi(HF(L_0, L_1)) = \#(L_0 \cap L_1)$.

Q: Is there a more geometric construction of such invariants?

Thm (Kronheimer-Mrowka-O-Szabó)

If $S^3_r(K) \cong S^3_r(\text{Unknot})$

$\Rightarrow K = \text{Unknot}$.

Cor: $S^3_r(K) \cong \mathbb{RP}^3 \Rightarrow$ $S^3_2(U)$

$K = \text{Unknot}$.

Remarks:

1. For $r=0$, this is a theorem of Gabai.
2. For $r=\frac{p}{q}$ $q \neq \pm 1$, it is a theorem of Culler Gordon Luecke Shalen
3. For $p=\pm 1$ it is a theorem of Gordon-Luecke.

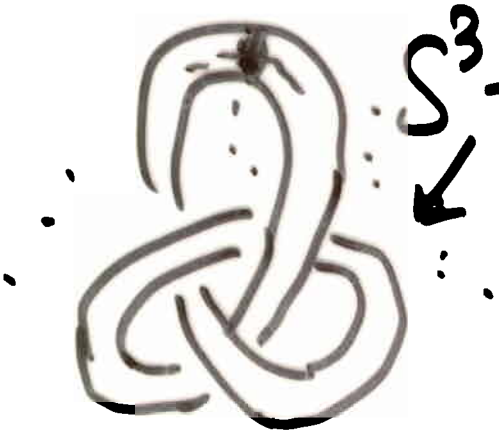
Another Problem:

Dehn surgery.

Given $K \subset S^3$ and $r \in \mathbb{Q}$, we can construct a new 3-manifold

$D^2 \times S^1$

$S^3 - nbd(K)$



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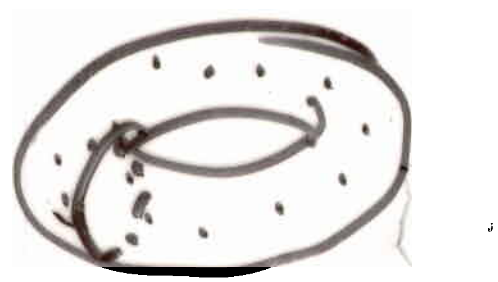

The "gluing" is traditionally specified by $r = \frac{p}{q}$ via the convention

$$\mu' = p\mu + q\lambda$$ where

μ & λ are meridian & longitude of K.

Why do we want this?

- Understand 4-manifolds.
- There are 3-D problems which can be phrased from 4-D point of view

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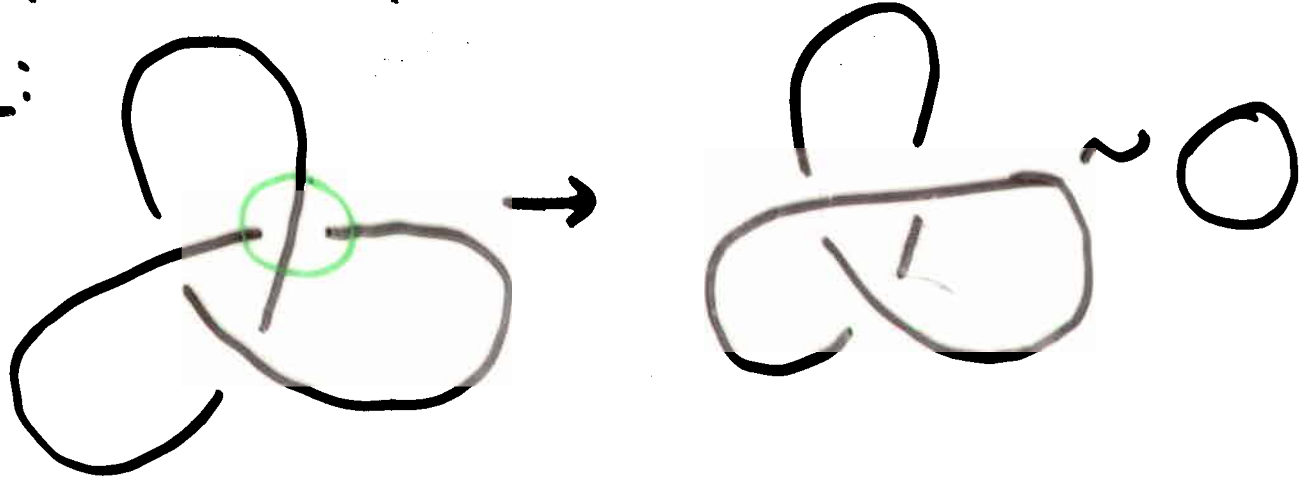
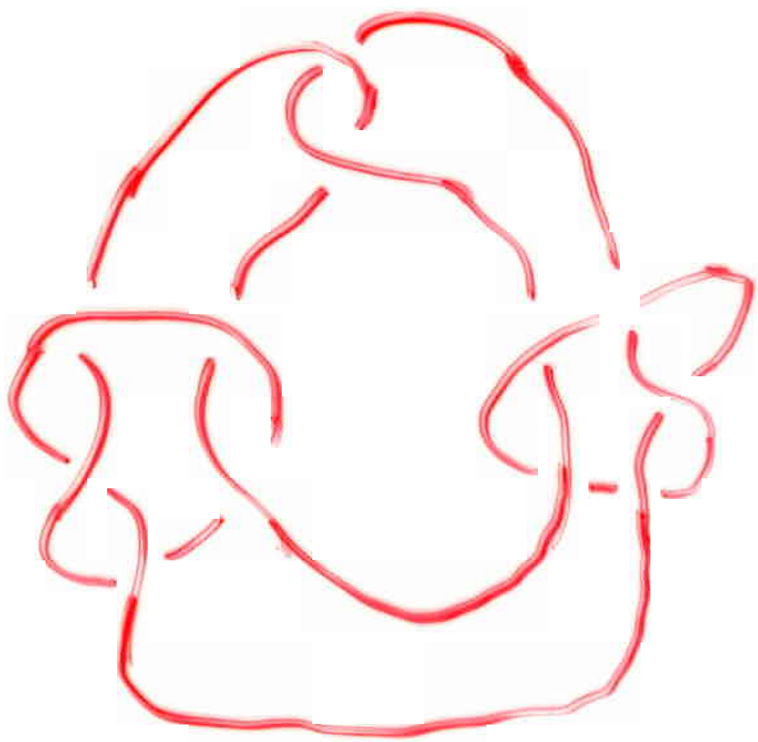
Ex: Unknotting Number.

The **Unknotting Number** of K is the minimal number of crossing changes required to unknot K.

E.g.:





$\Delta_K(T) = 1.$

BUT: $\widehat{HFK}(K, 2) \cong \mathbb{Z} \oplus \mathbb{Z}.$

Recall: $\deg \Delta_K \leq g(K).$

Not sharp
in general.

"Knot Floer Homology":

Recall Alexander Polynomial.

$\Delta_L(T) \leftarrow$ invariant of oriented links.

$$\Delta(\bigcirc) = 1$$

$$\Delta_{\nearrow\!\!\!\nwarrow}(T) - \Delta_{\nwarrow\!\!\!\nearrow}(T) = (T^{1/2} - T^{-1/2})\,\Delta_{)(}(T)$$

Knot Floer Homology:
Bigraded Abelian group

$$K \longmapsto \widehat{HFK}_d(K, s) \qquad d, s \in \mathbb{Z}.$$

$$\Delta_K(T) = \sum (-1)^d \, rk\left(\widehat{HFK}_d(K,s)\right) T^s$$

$(\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\}, z)$

Consider $\mathrm{Sym}^g(\Sigma)$

$$\left.\begin{array}{l} \mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \\ \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g \end{array}\right\} \subset \boxed{\mathrm{Sym}^g(\Sigma)}$$

$$CF(Y) = \bigoplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}[U]x$$

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in [x,y]} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) U^{n_z(\phi)} \cdot y$$

<u>Thm</u>: (O-Szabó)

$HF(Y) = H_*(CF(Y))$ is a 3-manifold invariant.

Thm: (O-Szabó)

$$\max\{s \mid \widehat{HFK}(K,s) \neq 0\} \overset{!}{=} g(K)$$

Proof uses:

Gabai

$\downarrow$

Eliasbberg. Thurston.

$\downarrow$

Eliashberg, Etnyre

$\downarrow$

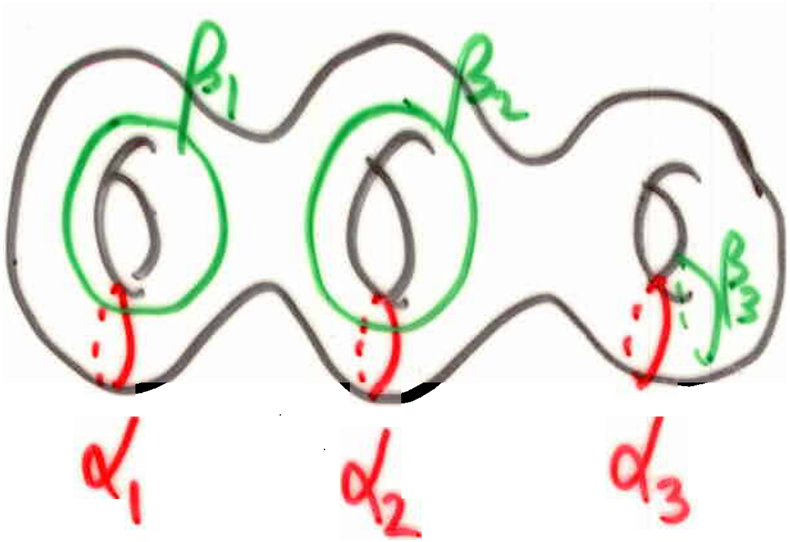
Donaldson (Lefschetz Pencils)

$\downarrow$

Properties of Floer Homology

Heegaard Floer Homology.

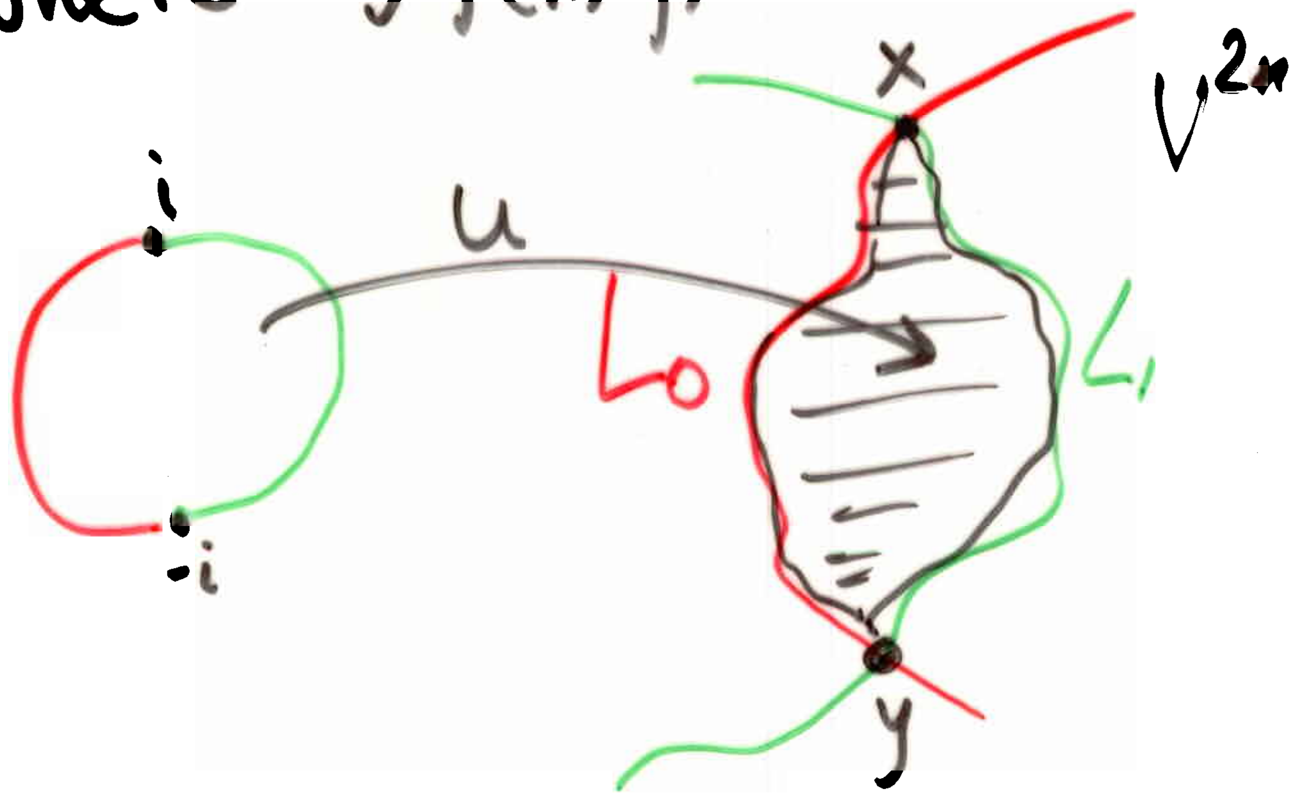
Heegaard diagram



$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{Z}\, x.$$

$$\partial x = \sum_{y \in L_0 \cap L_1} \#\left(\frac{\mathcal{M}(x,y)}{\mathbb{R}}\right) \cdot y.$$

where $\mathcal{M}(x, y) =$



s.t. $\bar{\partial} u = 0.$

<u>Thm</u>: (Floer/Gromov) $\partial^2 = 0.$

Gauge Theory Invariants

$$\underset{\text{Riemannian metric}}{(X^4, g)} \rightsquigarrow \underbrace{\mathcal{M}^{ASD}(X,g)}_{\text{algebraic topology}} \rightsquigarrow \mathcal{D}_X$$

1st order, elliptic PDE

Donaldson 80's

Floer

Inst

$$Y^3 \rightsquigarrow HF(Y)$$

Defined by doing ∞-dimensional Morse theory on a functional whose "gradient flowlines" $\longleftrightarrow$ ASD instantons on $\mathbb{R} \times Y$.

Seiberg-Witten '94

$$(X^4, g) \rightsquigarrow \mathcal{M}^{SW}(X,g) \rightsquigarrow SW_X$$

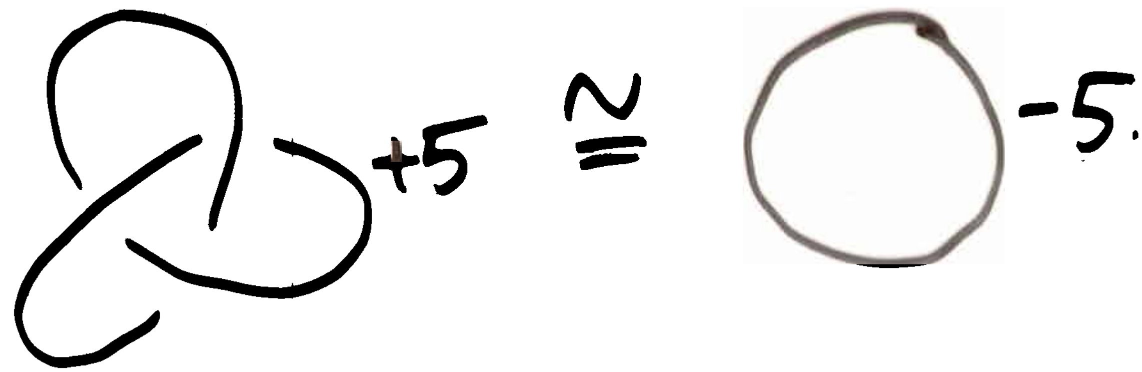
Different 1st order elliptic PDE.

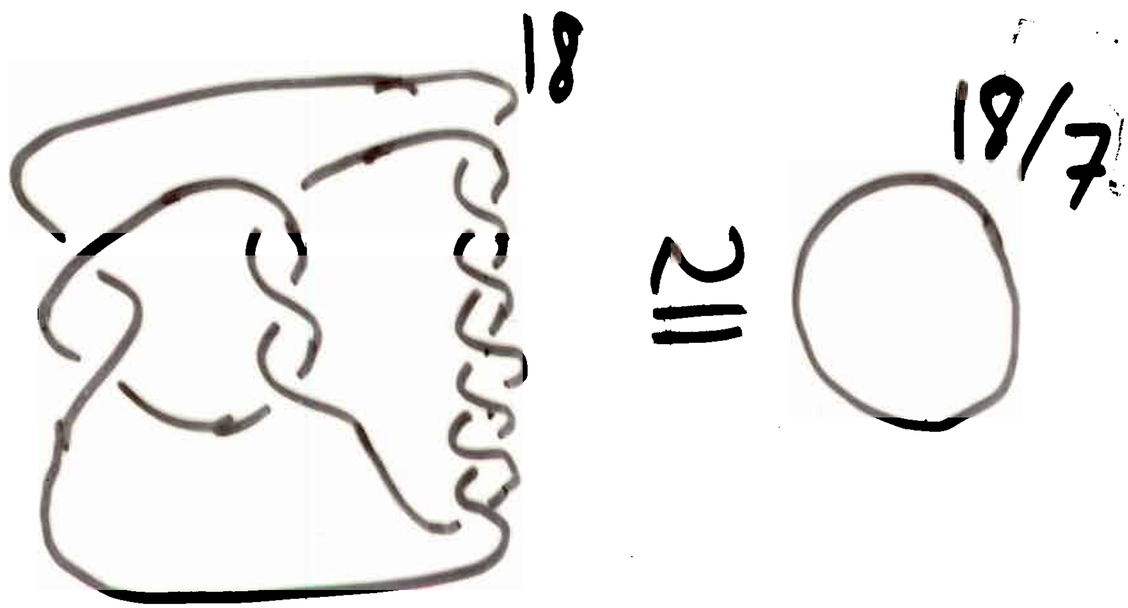
Kronheimer Mrowka '0?

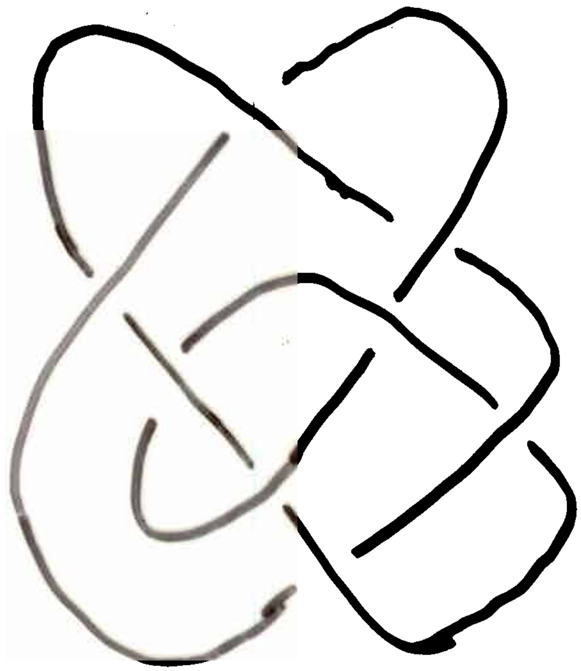
$$Y \mapsto HF^{SW}(Y)$$

Q: To what extent does $S^3_r(K)$ "remember" r & K?

Ex:







What is $u(8_{10})$?

To determine this, need

- n-step "unknotting"
- Certificate that there is no unknotting w/ $<n$ steps

Heegaard Diagrams and Holomorphic Disks.

Peter Ozsváth

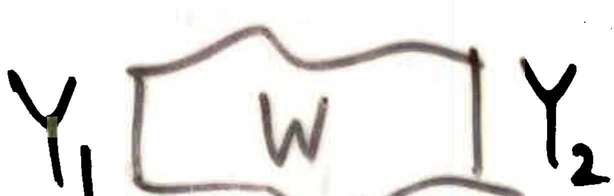
Joint w/ Zoltán Szabó.

An effort to understand gauge theory from a geometric/topological/(combinatorial?) point of view.

Given $Y^3 \rightsquigarrow HF(Y)$

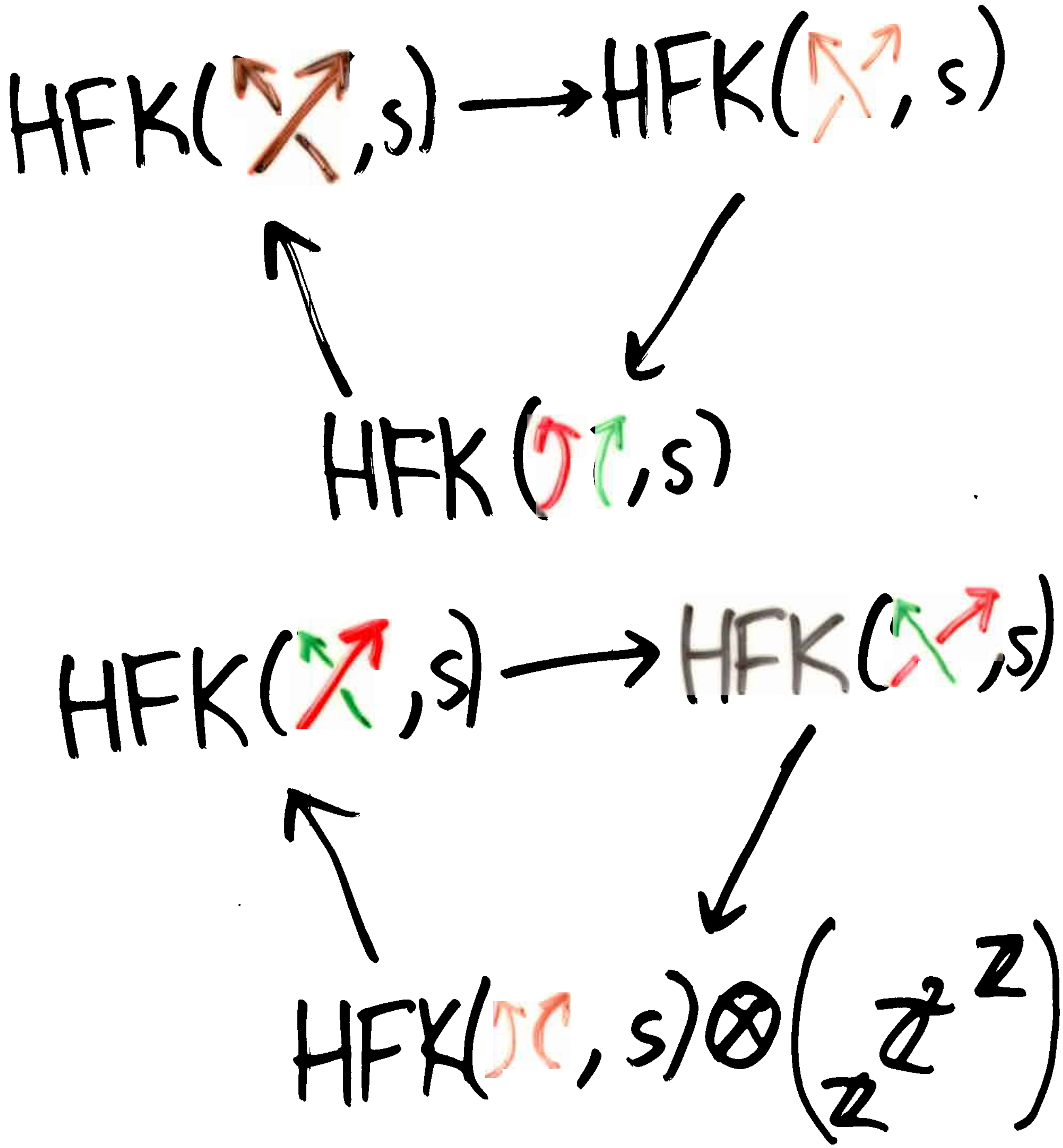
Group

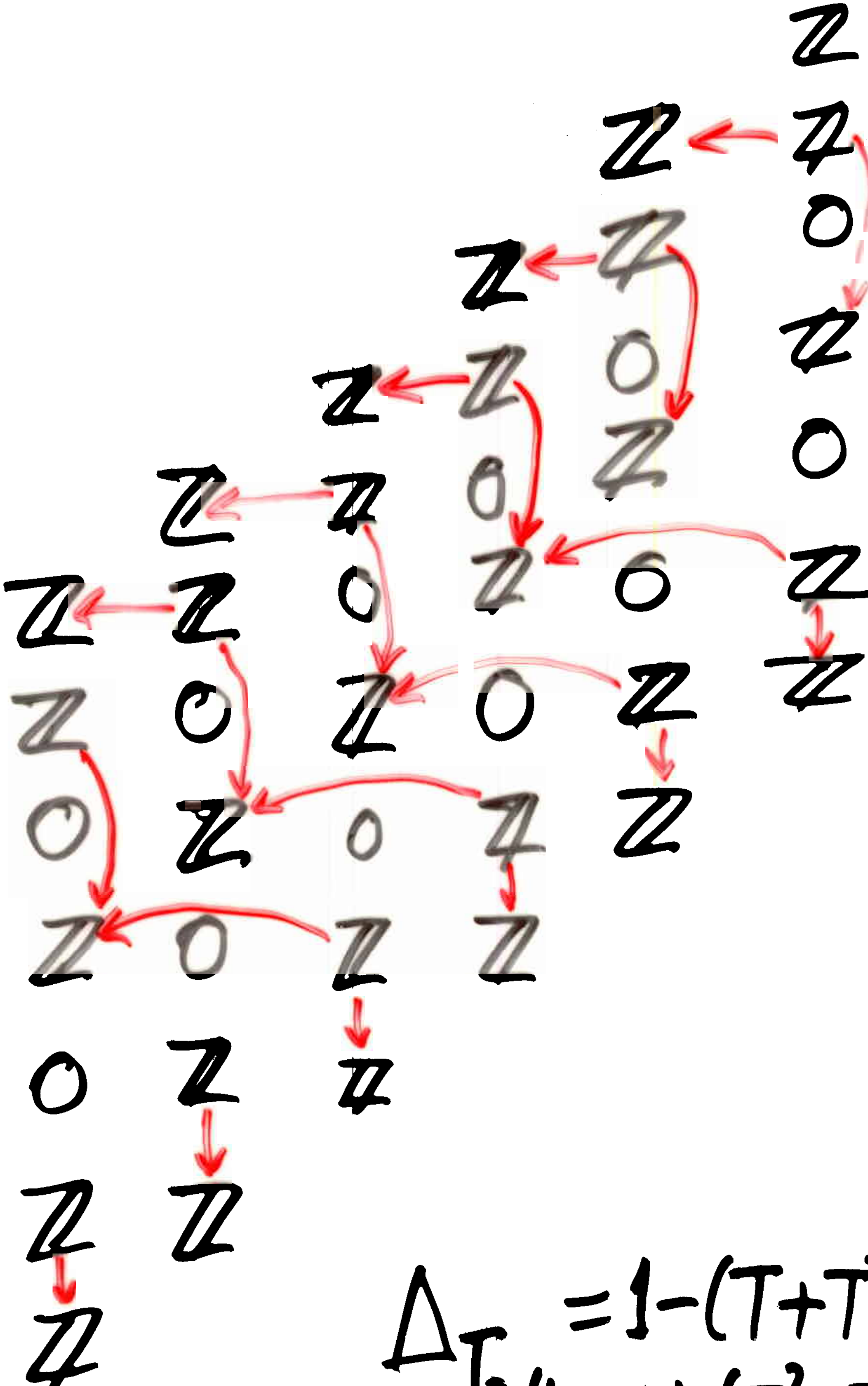
Natural under cobordisms.



$F_W : HF(Y_1) \longrightarrow HF(Y_2)$

Thm:





$$\Delta_{T_{3,4}} = 1-(T+T^{-1}) + (T^2+T^{-2})$$

Thm: $HF(Y_{p/q}(K))$ can be described in terms of $\left\{ B \xleftarrow{v} A^+_s \xrightarrow{h} B \right\}$.

E.g.

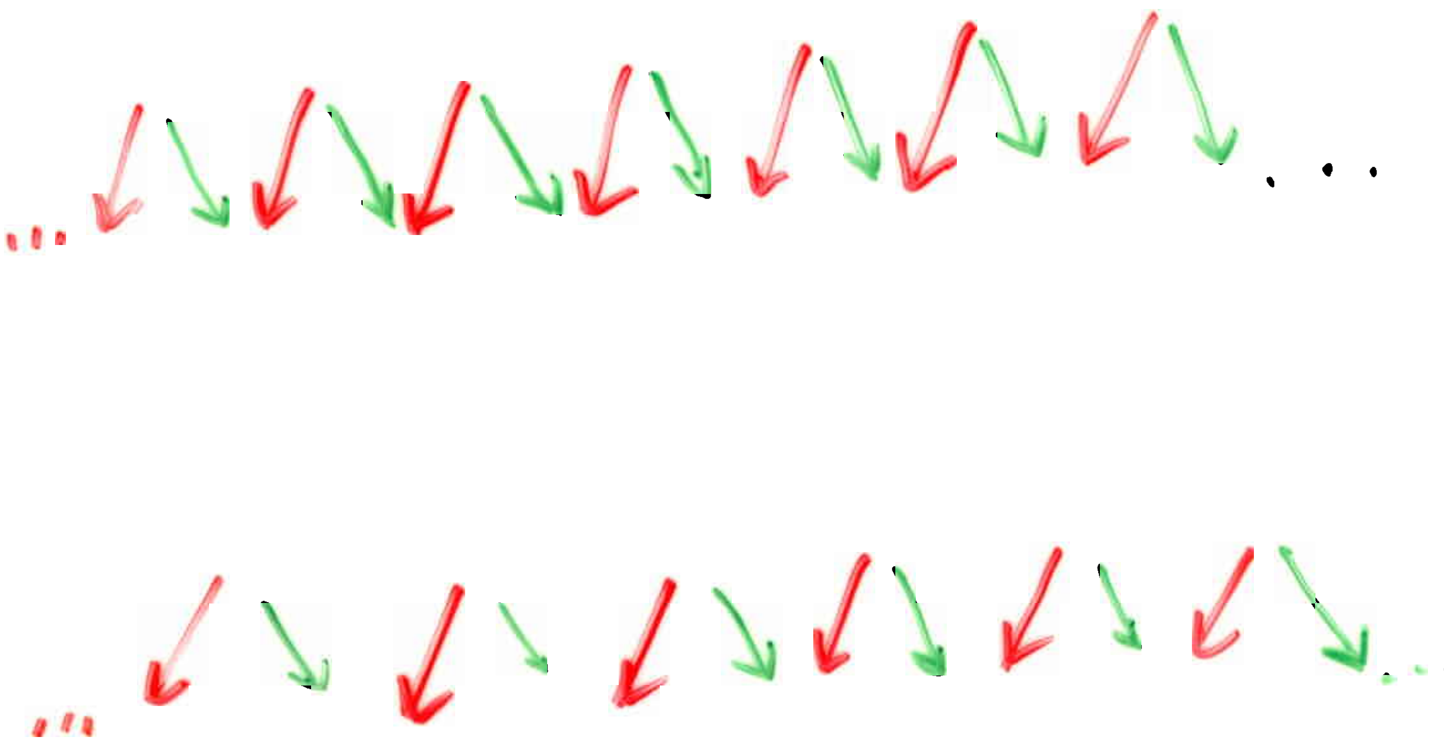
$$A_{-3} \quad A_{-2} \quad A_{-1} \quad A_0 \quad A_1 \quad A_2 \quad A_3$$

$$B \quad B \quad B \quad B \quad B \quad B$$

$$A_{-1} \quad A_{-1} \quad A_0 \quad A_0 \quad A_1 \quad A_1$$

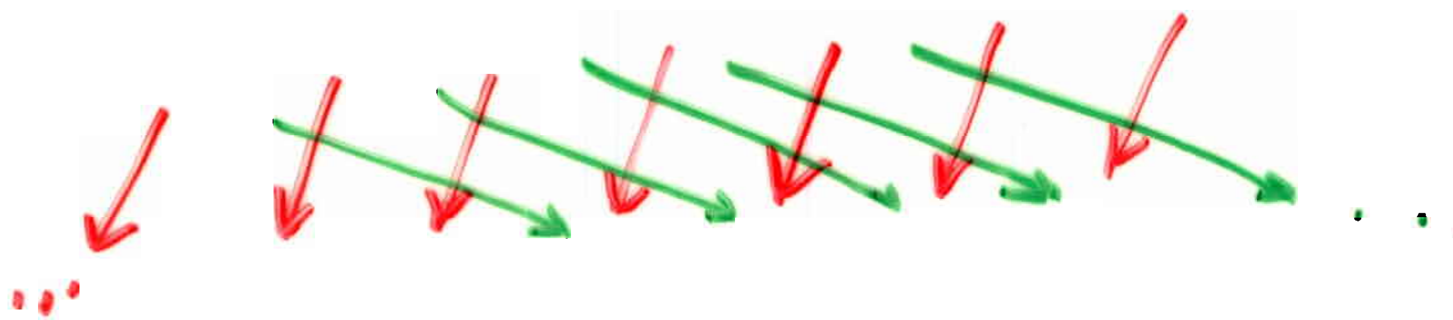
$$B \quad B \quad B \quad B \quad B$$

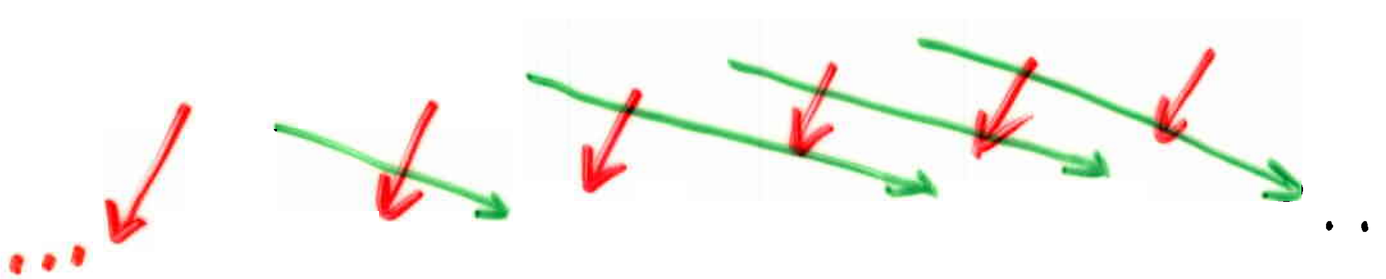
$HF^+(Y_1(K))$



$HF^+(Y_{1/2}(K))$

$HF(Y_3(K))$





$HF(Y_{3/2}(K))$