SOLITONS & SYMMETRY

IAS PRINCETON

HERMANN WEYL SYMMETRY

INVARIANT THEORY

PHYSICS

PETER GODDARD MAGNETIC

MONOPOLES

VON NEUMANN COMPUTERS

SOLITON A SOLUTION OF

CERTAIN KIND OF NON-LINEAR PDE

THAT "BEHAVES LIKE A PARTICLE"

INCLUDES MANY IMPORTANT

CASES THAT TURN UP IN VARIOUS

BRANCHES OF PHYSICS

+ INTERESTING MATHEMATICS

SOLITONS IN 1+1 DIMENSIONS

EX KdV EQUATION : $u(x,t)$

NON-LINEAR

$$u_t + uu_x + u_{xxx} = 0$$

SOLVED BY INTERPRETING $u(x,t)$ AS THE POTENTIAL FOR THE LINEAR OPERATOR $L(t) = \frac{d^2}{dx^2} + u$ ACTING ON FUNCTIONS OF x (AND EVOLVING IN TIME ACCORDING TO KdV) !!!

HIDDEN SYMMETRIES (CONSERVED QUANTITIES) ASSOCIATED TO LIE GROUP SL_2

EXTENSIVE THEORY

SOLITONS IN 3 AND 4 DIMENSIONS

EX dim 4 INSTANTONS : SOLUTIONS OF SELF-DUAL YANG-MILLS EQNS (EXPLOITED BY DONALDSON — ON GENERAL 4-MANIFOLDS

dim 3 MAGNETIC MONOPOLES : SOLUTIONS OF BOGOMOLNY EQN (CLOSELY RELATED TO SDYM)

SOLVED (IN R^4 OR R^3) BY USING PENROSE TWISTOR* THEORY

ASSOCIATED LINEAR OPERATOR IS DIRAC OPERATOR (NATURAL POTENTIAL)

$\Longrightarrow$ 1+1 dim THEORY BY DIMENSIONAL REDUCTION (R. WARD MASON - WOODHOUSE)

AIM TO DISCUSS MAGNETIC

<u>MONOPOLES</u> WITH <u>PLATONIC</u>

SYMMETRIES (WHY ?)

<u>DIRAC</u> MAGNETIC MONOPOLE

FIELD PRODUCED BY <u>POINT-SOURCE</u>

OF MAGNETISM

(STATIC) SOLUTION OF (<u>LINEAR</u>)

MAXWELL EQUATIONS WITH

POINT <u>SINGULARITY</u>

IN <u>NON-ABELIAN GAUGE-THEORIES</u>

<u>'t HOOFT</u> + <u>POLYAKOV</u> FOUND <u>SOLITON</u>

SOLUTION (NO SINGULARITY) LOOKED

LIKE DIRAC MONOPOLE NEAR ∞

SIMPLIFIED MODEL BPS

MORE TRACTABLE

DATA $SU(2)$ - CONNECTION (POTENTIAL) A ON $\mathbb{R}^3$: COMPONENTS $A_\mu(x) \in su(2)$

HIGGS FIELD Φ : $\Phi(x) \in su(2)$

BOGOMOLNY EQN

$$\boxed{D_A \Phi = * F_A}$$

COVARIANT DERIVATIVE

CURVATURE (FIELD)

DUALITY OPERATOR
2 FORMS $\longleftrightarrow$ 1-FORMS

BDY CONDS $F_A \to 0$ AS $|x| \to \infty$

$|\Phi| \to 1$ " " "

TOPOLOGICAL INVARIANT (INTEGER k)

DEGREE OF MAP $\Phi_\infty : S^2_\infty \to S^2_1$

IN $\mathbb{R}^3$ IN $su(2)$

k = MAGNETIC CHARGE

= "NUMBER" OF BASIC MONOPOLES ("PARTICLES")

GAUGE-TRANSFORMATIONS

/ AUTOMORPHISMS OF SU(2)-BUNDLE)

SOLUTIONS MODULO AUTOMORPHISMS

GIVE MODULI SPACE M_k

(USUAL TO USE BASE-POINT (ORIGIN)

AND AUTS $g(x)$ WITH $g(0) = 1$)

RESULTS

M_k IS MANIFOLD OF DIMENSION

$4k\ (+2)$ (k POSITIONS + k PHASES)

THEOREM (DONALDSON - JARVIS) $k > 0$

M_k = SPACE OF HOLOMORPHIC (RATIONAL)

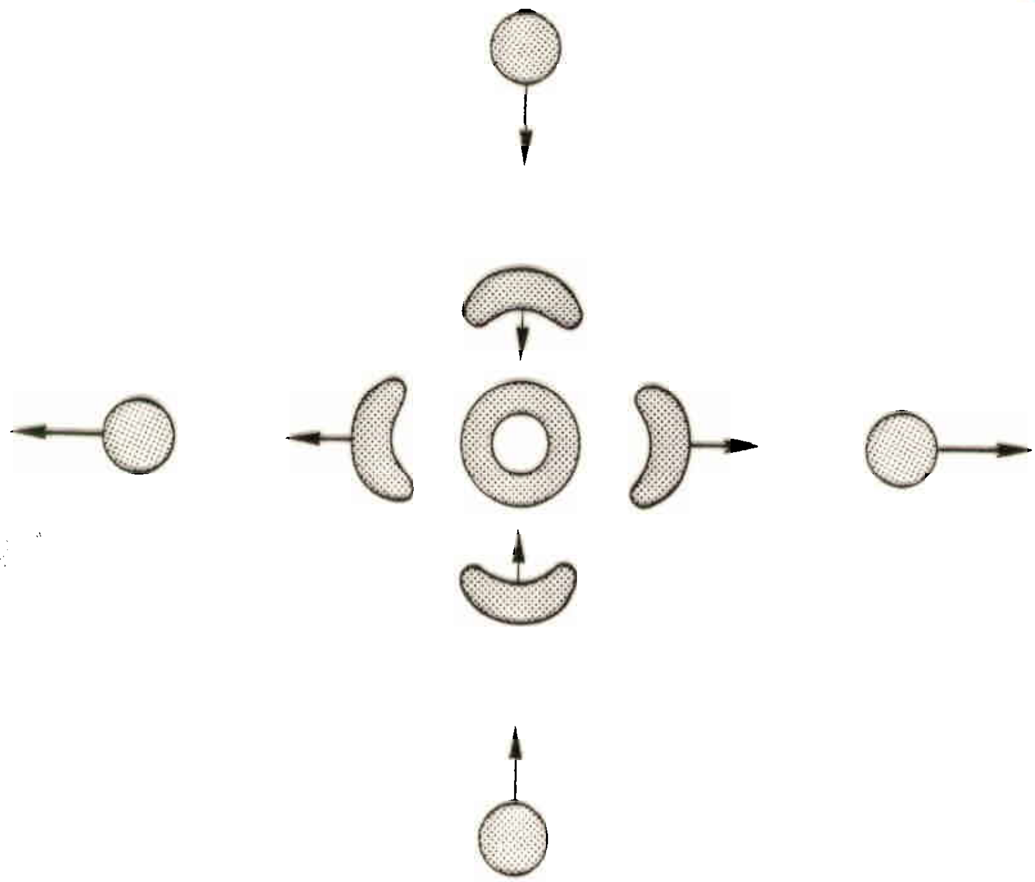
MAPS $f : S^2_\infty \to S^2_1$

$S^2_\infty = CP_1$, $S^2_1 = CP_1$

OF DEGREE k

NOTE INVERSE $(f \longmapsto A, \Phi)$

NOT EXPLICIT



$k=1$ $\quad f(z)=z$ "IDENTITY MAP"

$\Rightarrow$ BASIC 1-MONOPOLE

SPHERICALLY SYMMETRIC ABOUT ORIGIN

(ESSENTIALLY UNIQUE)

SOLUTION EXPLICIT + SIMPLE

$k=2$

SPECIAL CASE $f(z)=z^2$

HAS AXIAL SYMMETRY

"COLLISION STATE"

"TOROIDAL" SHAPE

GENERAL CASE $f(z)=z^2-c$

$|c|$ LARGE SPLITS APPROX. INTO

2 BASIC MONOPOLES FAR APART

$90°$ SCATTERING UNDER DIRECT COLLISION

M_k HAS NATURAL RIEMANNIAN METRIC

(EXPLICIT FOR $k=2$) AND GEODESICS

DESCRIBE SLOW DYNAMICS OF

INTERACTING MONOPOLES

JACOBIAN $J(f)$ "BRANCH POINTS"

POINTS ON S^2_∞ WHERE $df = 0$

IF $f = \dfrac{p}{q}$ $\quad p(t_0, t_1)$, $q(t_0, t_1)$ HOMOGENEOUS POLYN. DEG k

$$J(f) = \det \begin{pmatrix} \frac{\partial f}{\partial t_0} & \frac{\partial f}{\partial t_1} \\ \frac{\partial g}{\partial t_0} & \frac{\partial g}{\partial t_1} \end{pmatrix}$$

HOMOG. POLYN DEGREE $2k-2$

CLASSICAL INVARIANT

IF $\Gamma \subset SO(3)$ (OR $\Gamma' \subset SU(2)$)

FINITE SUBGROUP, CAN LOOK FOR

Γ-INVARIANT MAPS f (Γ ACTING ON BOTH SIDES)

QUESTION GIVEN J DEGREE $2k-2$ IS THERE ONE (OR MORE) f DEG k WITH $J(f) = J$?

NOTE dimensions of spaces EQUAL

$f \leftrightarrow P_1 \subset P_k$ $\quad$ dim $2k-2$

J : dim $2k-2$

Projective dims

CONSIDER CASE $k=3$

$2k-2 = 4$ $J(f)$ IS QUARTIC

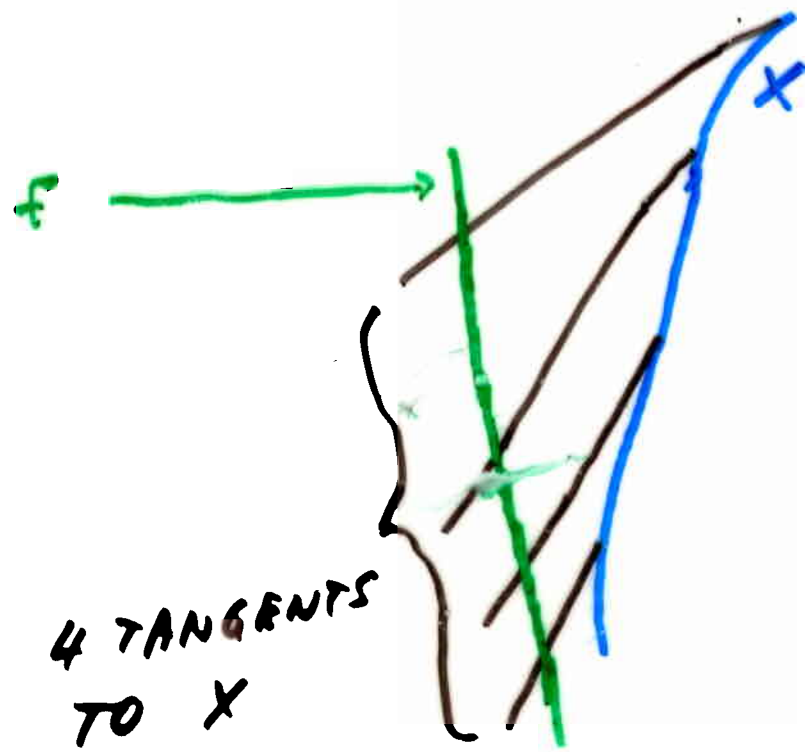
INTERPRET PROBLEM GEOMETRICALLY

f CORRESPONDS TO CP_1 IN

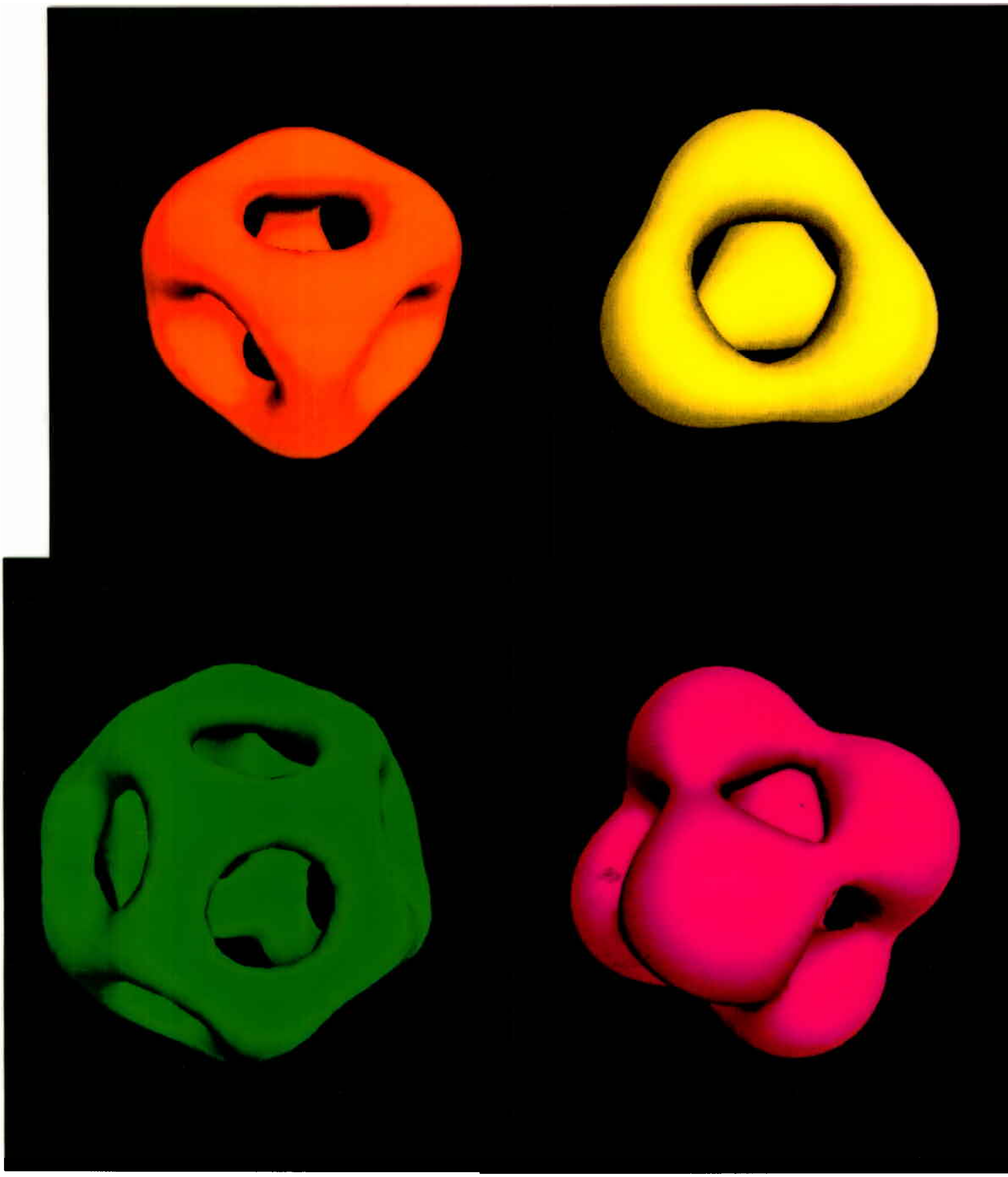
CP_3 (CUBICS UP TO SCALE)

LET $X \subset CP_3$ BE RATIONAL CUBIC

CURVE REPRESENTING PERFECT CUBES



4 POINTS ON X $\longrightarrow$ 4 TANGENTS $\longrightarrow$ 2 TRANSVERSAL LINES



THEOREM (MFA 1951)

4 POINTS ON X HAVE AN

EQUIANHARMONIC CROSS-RATIO

(VERTICES OF A REGULAR TETRAHEDRON)

$\Longleftrightarrow$ THE 2 TRANSVERSALS COINCIDE

$\Rightarrow$ $\exists$ UNIQUE 3-MONOPOLE WITH

TETRAHEDRAL SYMMETRY !!!

$$f(z) = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1} \qquad k=3 \quad \text{TETRAHEDRAL}$$

$$f(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1} \qquad k=4 \quad \text{OCTAHEDRAL}$$

$$f(z) = \frac{z^7 - 7z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1} \qquad k=7 \quad \text{ICOSAHEDRAL !}$$

$J(f) = 0$ GIVES MID-POINTS OF FACES

POINTS OF MINIMUM ENERGY DENSITY

SKYRMIONS (1961)

CLASSICAL MODEL OF NUCLEUS

DATA MAP $g : R^3 \to SU(2) = S^3$

BDY COND. $g(x) \to 1$ AS $|x| \to \infty$

TOPOLOGICAL INVARIANT INTEGER k

DEGREE $\bar{g} : S^3 \to S^3$

INTERPRETED AS NUMBER OF

PROTONS / NEUTRONS

ENERGY $E(g) = \int_{R^3} \|dg\|^2 + \int_{R^3} \|\Lambda^2 dg\|^2$

(STATIC) SKYRMION IS SOLUTION

MINIMIZING E (GIVEN BY EULER-LAGRANGE

EQUATION)

ESSENTIALLY UNIQUE SOLUTION EXPECTED

ANALYTICAL SOLUTIONS NOT KNOWN

ONLY NUMERICAL WORK

DIFFICULT PROBLEM *

$k=1$ SPHERICALLY SYMMETRIC SOLUTION (SOLVE ODE NUMERICALLY)

GIVES BASIC SKYRMION (HYDROGEN NUCLEUS)

$k>1$ k FAR SEPARATED BASIC SKYRMIONS "ASYMPTOTIC SOLUTION"

BUT WHEN CLOSE TOGETHER ?

ANALOGY ?? WITH MONOPOLES

SUGGESTS LOOKING FOR SYMMETRIC SOLUTIONS (MANTON)

REMARKABLE FACT !!

$k=2$ TOROIDAL

$k=3$ TETRAHEDRAL

$k=4$ OCTAHEDRAL

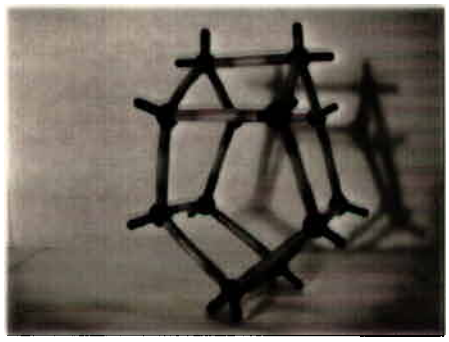
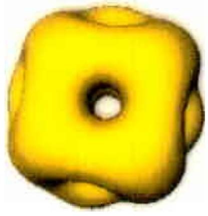
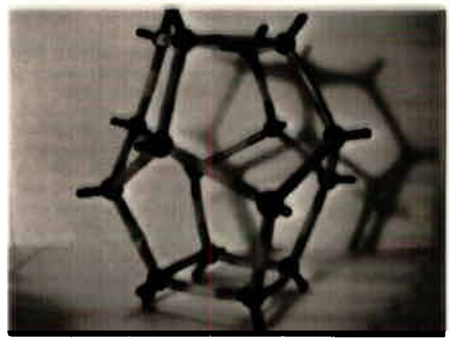
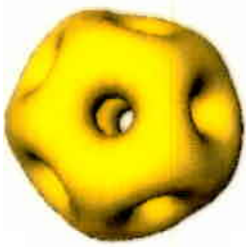
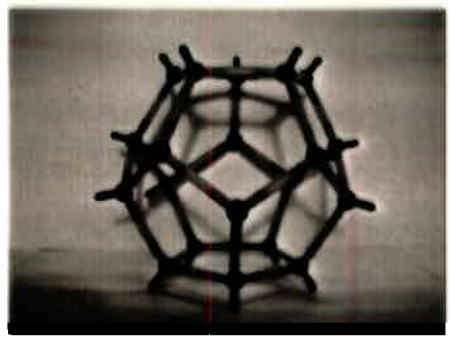
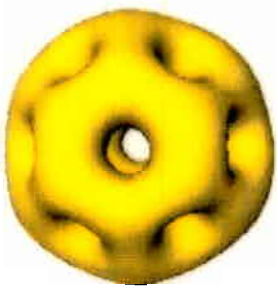
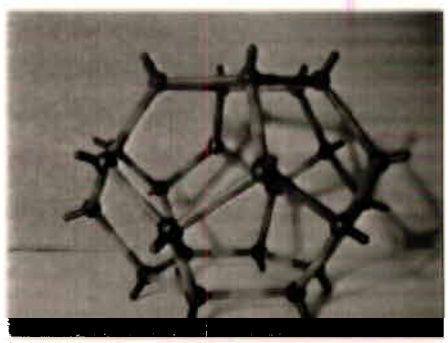
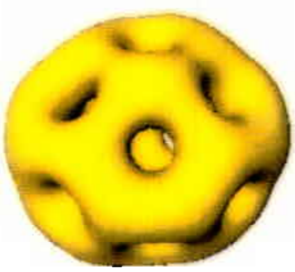
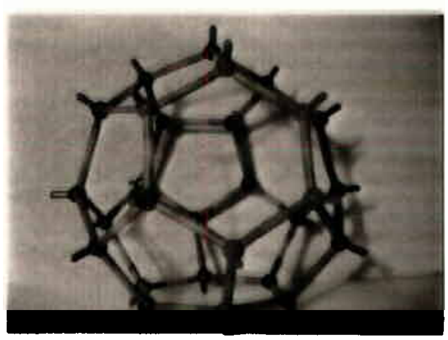
$k=7$ ICOSAHEDRAL

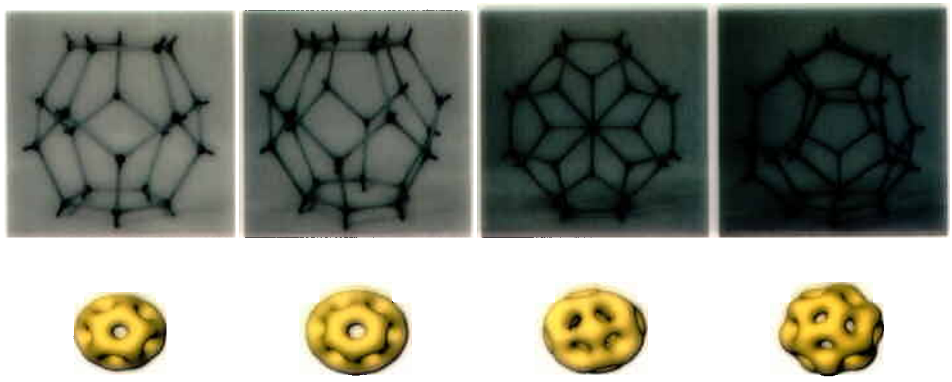
APPEAR TO BE VERY CLOSE TO THE EXACT SOLUTION

$\Rightarrow$ "INSIGHT" FOR OTHER VALUES OF k

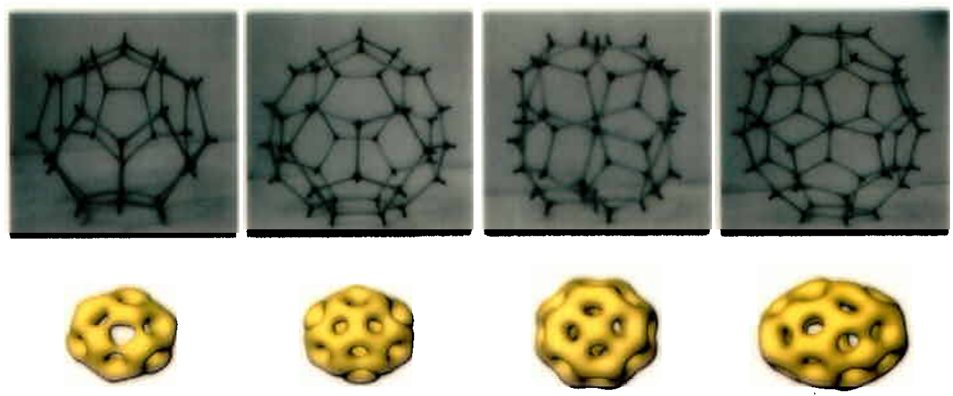
MYSTERIOUS!

SAME PICTURES

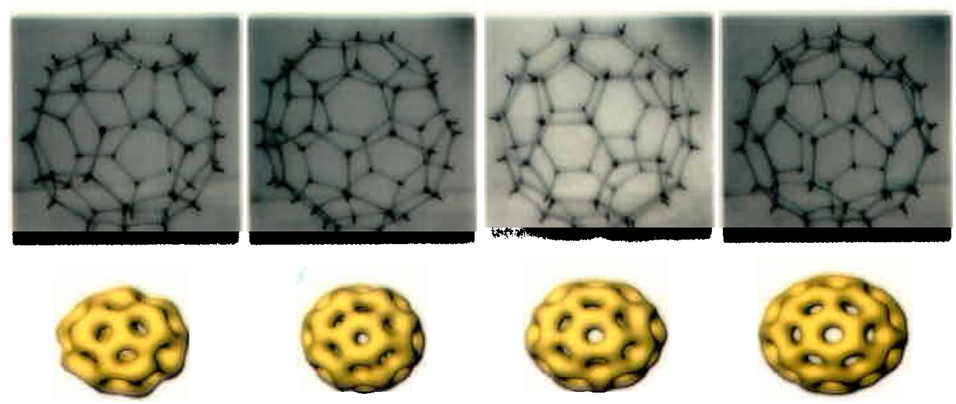













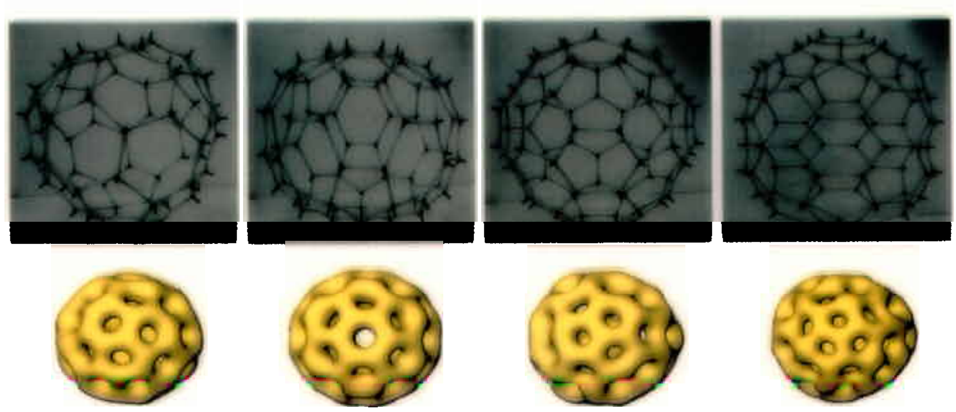
7:Y_h 8:D_{6d} 9:D_{4d} 10:D_3



11:D_{3h} 12:T_d 13:O 14:C_2



15:T 16:D_2 17:Y_h 18:D_2



19:D_3 20:D_{6d} 21:T_d 22:D_3

REFERENCE

TOPOLOGICAL SOLITONS

N. MANTON & P. SUTCLIFFE

CUP 2004