PCMI Lecture 1

Rigid cocycles

§1. Singular moduli

The values of Klein's j-invariant

$$j(q) = \left(1 + 240 \sum_{n\geq 1} \frac{n^3 q^n}{1-q^n}\right)^3 \Big/ \; q\prod_{n\geq 1}(1-q^n)^{24}$$

$$= \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \qquad q = e^{2\pi i \tau}$$

at CM points $\tau \in S_\infty = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ (= "singular moduli") are arithmetically rich.

Examples: $j(\sqrt{-1}) = 1728$, $\quad j(\zeta_6) = 0$

$$j(\sqrt{-15}) = -5^2 \cdot 3^3 \, \frac{637 + 283\sqrt{5}}{2\sqrt{5}} \qquad \begin{cases} \text{Norm} = -3^6 \cdot 5^3 \cdot 11^3 \\ \text{Trace} = -3^3 \cdot 5^2 \cdot 283 \end{cases}$$

They generate ring class fields

(e.g. $K = \mathbb{Q}(\sqrt{-15})$, Hilbert class field $H = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$)

Renaissance: Work of Gross-Zagier (1985) studies

$$\mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right) \in \mathbb{Z} \qquad \begin{cases} \mathrm{disc}\ \tau_1 = D_1 < 0 \\ \mathrm{disc}\ \tau_2 = D_2 < 0 \end{cases} \quad \text{coprime + fundamental}$$

[READ LETTER]

Zagier computes

$$j\left(\tfrac{1+\sqrt{-7}}{2}\right) - j\left(\tfrac{1+\sqrt{-43}}{2}\right) = 3^6 \cdot 5^3 \cdot 7 \cdot 19 \cdot 73$$

x	$\frac{7\cdot 43 - x^2}{4}$	x	$\frac{7\cdot 43 - x^2}{4}$	x	$\frac{7\cdot 43 - x^2}{4}$
1	$3\cdot 5^2$	7	$3^2\cdot 7$	13	$3\cdot 11$
3	73	9	$5\cdot 11$	15	19
5	$3\cdot 23$	11	$3^2\cdot 5$	17	3

Results lead to progress on Birch-Swinnerton-Dyer (Gross-Zagier 1986) relating heights of Heegner points to derivatives of L-functions.

This course will discuss real quadratic analogues of singular moduli:

Key notion: Rigid cocycles (joint w. H. Darmon).

Remark. Other approaches have been explored

- Stark's conjectures on leading terms of L-functions
  - → (Archimedean) Manin: non-commutative geometry
  - → (non-Arch) p-adic Gross-Stark + refinements Dasgupta-Kakde
- Cycle integrals of j-function (Kaneko, Duke-Imamoglu-Toth)

§2. Quadratic forms

A quadratic form is an element

$$\langle a,b,c\rangle := aX^2 + bXY + cY^2 \quad \in \mathbb{Z}[X,Y]$$

Say $\langle a,b,c\rangle$ is primitive if $\gcd(a,b,c)=1$.

There is a right $SL_2(\mathbb{Z})$-action by ring automorphisms on $\mathbb{Z}[X,Y]$,

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} : \quad \begin{array}{l} X \longrightarrow pX+qY \\ Y \longrightarrow rX+sY \end{array}$$

which preserves the sets $\mathcal{F}_D$ of primitive forms (a>0 if D<0) of discriminant $D = b^2-4ac$.

When D non-square, have bijection

$$\begin{array}{rcl} \mathcal{F}_D/SL_2(\mathbb{Z}) & \longrightarrow & Pic^+\left(\mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]\right) \\ \langle a,b,c\rangle & \longrightarrow & [(2a, -b+\sqrt{D})]. \end{array}$$

We define reduced forms according to $\mathrm{sgn}(D)$

① $D<0$ definite forms

Say $\langle a,b,c\rangle$ is reduced if $|b| \le a \le c$ and $b \geq 0$ if either equality holds.

② $D>0$ indefinite forms

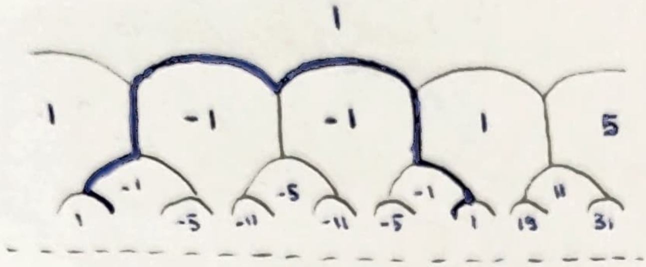
Say $\langle a,b,c\rangle$ is nearly reduced if $ac<0$
reduced if $ac<0$ and $b > |a+c|$.

Any form with D non-square has first and second roots

$$\begin{array}{rcl} r : \mathcal{F}_D & \longrightarrow & \mathbb{C} \\ \langle a,b,c\rangle & \longrightarrow & \frac{-b+\sqrt{D}}{2a} \end{array} \qquad \begin{array}{rcl} r' : \mathcal{F}_D & \longrightarrow & \mathbb{C} \\ \langle a,b,c\rangle & \longrightarrow & \frac{-b-\sqrt{D}}{2a} \end{array}$$

(Reducedness can be characterised in terms of roots, see exercises).

Topograph: Conway considers tree $\mathcal{T} = j^{-1}([0,1728])$



$F = \langle 1,1,-1 \rangle$ disc $D = 5$

Connected components of $\mathfrak{H}_\infty \setminus \mathcal{T}$ are labelled by $F(r,s)$, with $(r,s) \in \mathbb{P}^1(\mathbb{Q})$ the unique adjacent cusp, in lowest terms.

Can often guide intuition!

When $D > 0$ non-square, define for $F \in \mathfrak{F}_D$

$$\Sigma_F := \{ \langle a,b,c \rangle \sim F \; : \; ac < 0 \}$$

Can be computed efficiently: Note that

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $\langle a,b,c \rangle \mapsto \langle a, b+2a, a+b+c \rangle$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: $\langle a,b,c \rangle \mapsto \langle c, -b, a \rangle$

Algorithm: i) Apply power of T until

$$\begin{cases} -|a| < b \le |a| & \text{if } |a| \ge \sqrt{D} \\ \sqrt{D} - 2|a| < b < \sqrt{D} & \text{if } |a| < \sqrt{D} \end{cases}$$

ii) If reduced, stop.
Otherwise, apply S and go back to i).

Get reduced form after at most $\frac{1}{2}\log_2(|a|/\sqrt{D}) + 2$ steps.

From reduced form, the set Σ_F is computed (exercise).