ROUND–OFF SENSITIVITY IN THE N–BODY PROBLEM

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Abstract

The solutions to the equations of motion of the gravitational N-body problem are extremely sensitive to very small changes in initial conditions, resulting in a near-exponential growth of deviations between neighboring trajectories in the system's $6N$-dimensional global phase space. We have started to investigate the character of this instability, and the relative contributions to the exponential growth given by two-body, three-body, and higher-order encounters. Here we present our first results on 3–body scattering, where we measured the total amplification factor of small perturbations in the initial conditions.

The method

The degree of intrinsic instability in the orbits of the stars in a three–body encounter can be determined by monitoring the divergence of two systems with only slightly different initial conditions. The algorithm follows an original (parent) system and a close replica (child). We start with two systems, parent and child, originally ϵ apart (according to a particular norm, see below). Both systems are integrated simultaneously until $\epsilon^* f$ apart, with f some chosen treshold (but not too high, in order to stay in the linear regime). This is essentially a parallel computation. Subsequently, the child system is called back home to a distance ϵ from the parent system, by pulling it back along the rope of shortest distance in the space associated with the adopted norm. The time and amplification is recorded. This is repeated until at least one of the stars escapes. The Birkhoff (1927) escape criterion, see e.g. Szebehely (1973), has been used. Finally, the total amplification factor is computed.

The instability of a gravitational interaction is also apparent in the numerical instability of the corresponding equations of motion, a fact noted first by Miller (1964). Recently, techniques have beed developed to cope with this problem by transforming the equations of motion. In the case of a 2–body scattering, the Kustaanheimo–Stiefel (KS) transformation (*cf.* Stiefel and Scheifele, 1971) can be used to regularize the numerical problem. A more general formulation, which regularizes each of the $\frac{1}{2}N(N-1)$ pairs in an N–body interaction, has been derived by Heggie (1974). We implemented this method in a form given by Mikkola

(1985). Regularization comes at a price however, and one finds oneself integrating $4N(N-1)+1$ differential equations, instead of the $6(N-1)$ original Hamiltonian equations.

The integrator is the IMSL implementation of the Bulirsch & Stoer integrator. The integration step is chosen according to accuracy requirements in the extrapolation process. For a detailed comparison between various integration methods, see Alexander (1986).

As a illustrative case, we integrated the Pythagorean problem (e.g. Szebehely & Peters, 1967) forwards until the three body interaction was over, reversed the velocities, integrated backwards and recovered the original initial conditions with a relative accuracy of 10^{-5} (figure 1).

Results

The figures give an overview of the time evolution of a few representative three–body encounters, in particular the detailed behaviour of the growing amplification factor. In our calculations $G=1$ and all masses are equal to 0.5. Time is plotted along the abscissa. The left scale is the $\log_{10}$ of the minimum of the three distances between each of the three possible pairs of bodies, indicated by the full curve drawn in normal weight. The right scale is the $\log_{10}$ of the total amplification, relative to the initial separation, indicated by the heavy, approximately monotonically rising curve. Figure 1 shows those curves for the Pythagorean problem. While the behaviour appears to be rather chaotic, the formation of the binary at $t \approx 60$ is clearly visible. The following three figures include an additional curve which represents the mean distance between the bodies (the associated linear scale is not indicated, but is larger than that used to represent the minimum pairwise distance; this extra curve is dotted although the dots are not always clearly visible).

We have experimented with two different norms for expressing the distance between two three-particle systems: the Euclidean norm in $4N(N-1)$ dimensional regularized phase space (KS–norm) and the Euclidean norm in $6N$ dimensional ordinary phase space (PS–norm). The KS and the PS norms behave fundamentally different in the case of a two–body encounter, in a way which can be understood from a detailed analysis of these encounters in real and in regularized phase space. Emperically, there seems to be less difference between the behavior of the two norms in a three-body encounter. In the remainder of this paper, we only report the results from using the PS norm.

The amplification factor is not sensitive to the direction of the initial offset between parent and child systems. This is a consequence of the fact that the divergence is totally dominated by the largest Lyapunov coefficient at that moment. The subspace of all initial conditions that are orthogonal to the eigenvector associated with the largest eigenvalue has measure zero.

The largest amplification factors occur in scattering events in which all three stars remain bound together for several orbits before at least one single star escapes: so-called resonance scattering events. We have found that resonant three–body scattering has total amplification factors of $> 10^6$, with typical values in the range $10^8 \sim 10^{20}$. Much higher values occur occasionally, the highest recorded so far being 10^{150}. This implies that three-body scattering calculations are severely limited by the finite wordlength of computers. Worse still, in the more extreme cases even octuple precision would not be sufficient. Fig. 2 shows a typical resonance scattering event. For comparison, fig. 3 displays a non-resonant scattering, with a much smaller amplification factor. Figure 4 shows one of the more extreme cases of longer-lasting resonances. The black strips are actually oscillations that had to be compressed in order to show the whole scattering. The accuracy of the integration is reflected in the constancy of the amplitude of the binary whenever the third body is far away.

Conclusions

We have measured the amplification of perturbations in initial conditions for resonant three–body scattering. The amplification factors are typically in the range $10^8 \sim 10^{20}$, though much larger values occur occasionally, with measured values exceeding 10^{150}. A reliable integration of a three–body scattering encounter is therefore hardly feasible: no matter how large the wordlength used in a computation, there will always be some scattering events with an amplification factor which is so large that the result cannot be faithfully represented with the maximal number of significant digits available at that wordlength. This implies that the outcome of the experiment can be off by a very large amount, due to the round-off errors. Conversely, the initial conditions of the experiments could be shifted in a very small (albeit unknown) amount in order to yield the observed outcome when calculated to infinite accuracy. This difference between the *a priori* and the *a posteriori* initial conditions is often less than what can be resolved with the finite wordlength used.

This implies that imperfect scattering calculations are useful after all, even though the computations randomize the true initial conditions corresponding with the observed results. Fortunately, we are generally indeed more interested in statistical information concerning scattering experiments, expressed in terms of cross sections and reaction rates. To obtain these, initial conditions are chosen at random in Monte Carlo fashion, thereby overshadowing the extra randomization introduced by the calculation. (*cf.* Hut and Bahcall 1983).

More complete statistical results and their consequences for N–body calculations will be presented elsewhere.

Acknowledgments

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References

Alexander, M.E., 1986. *Journal Comp. Phys.*, **64**, p 195.
Birkhoff, G.D., 1927. *Dynamical Systems*, Am. Math. Soc. Publ., Providence, R.I.
Heggie, D.C., 1973. *Celestial Mech.*, **10**, p 217.
Hut, P. and Bahcall, J.N., 1983, *Astrophys. J.* **268**, p 319.
Mikkola, S., 1985. *Mon. Not. R. Astr. Soc.*, **215**, p 271.
Miller, R.H., 1964. *Ap. J.*, **140**, p 250.
Stiefel, E.L. and Scheifele, G., 1971. *Linear and Regular Celestial Mechanics*, Springer Verlag, Berlin.
Szebehely, V., 1973. *Recent Advances in Dynamical Astronomy*, Tapley B.D. and Szebehely, V. eds, p 75.
Szebehely, V. and Peters F., 1976. *Astron. J.*, **72**, p 876.

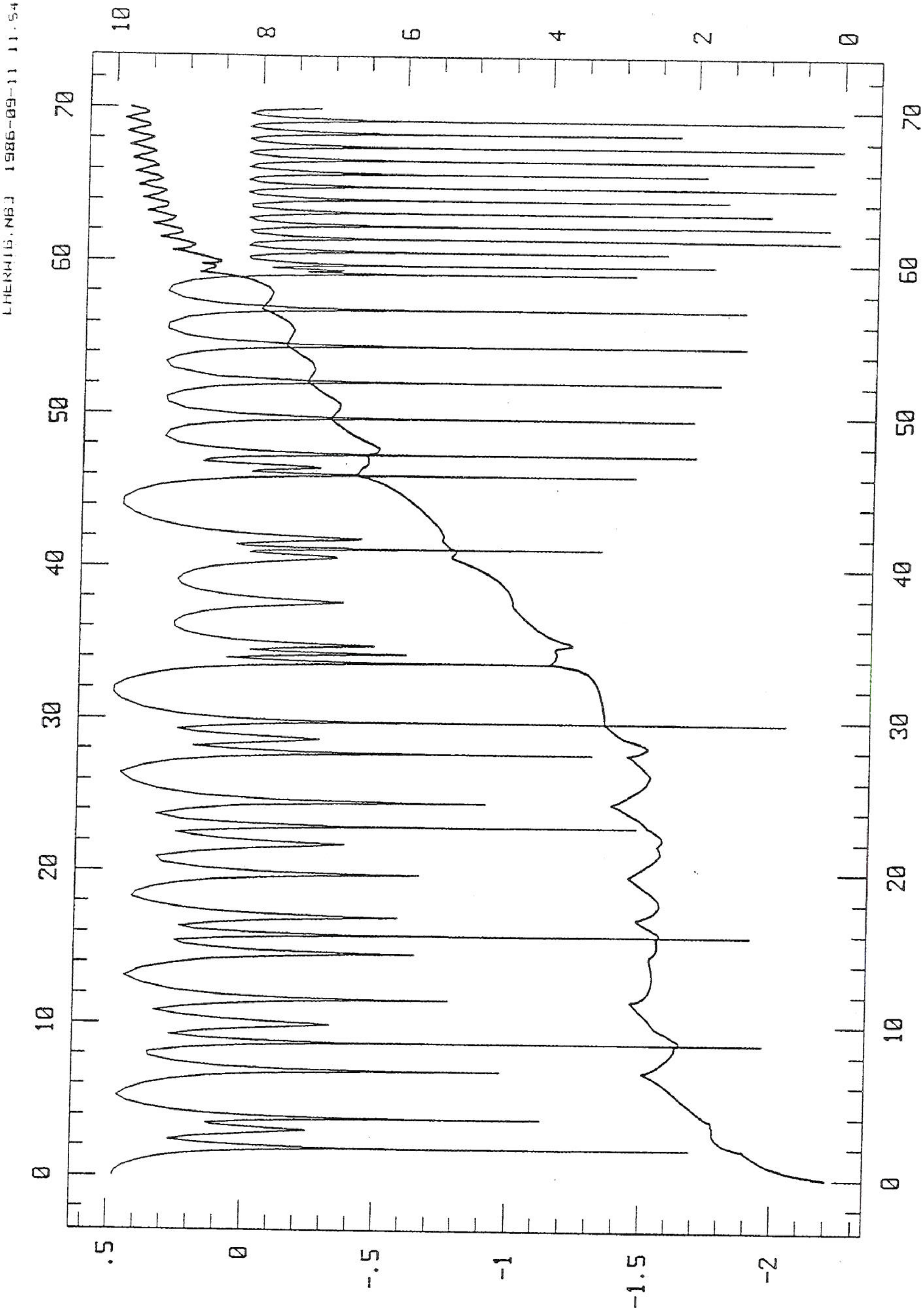

Fig. 1: The Pythagorean problem (see text)

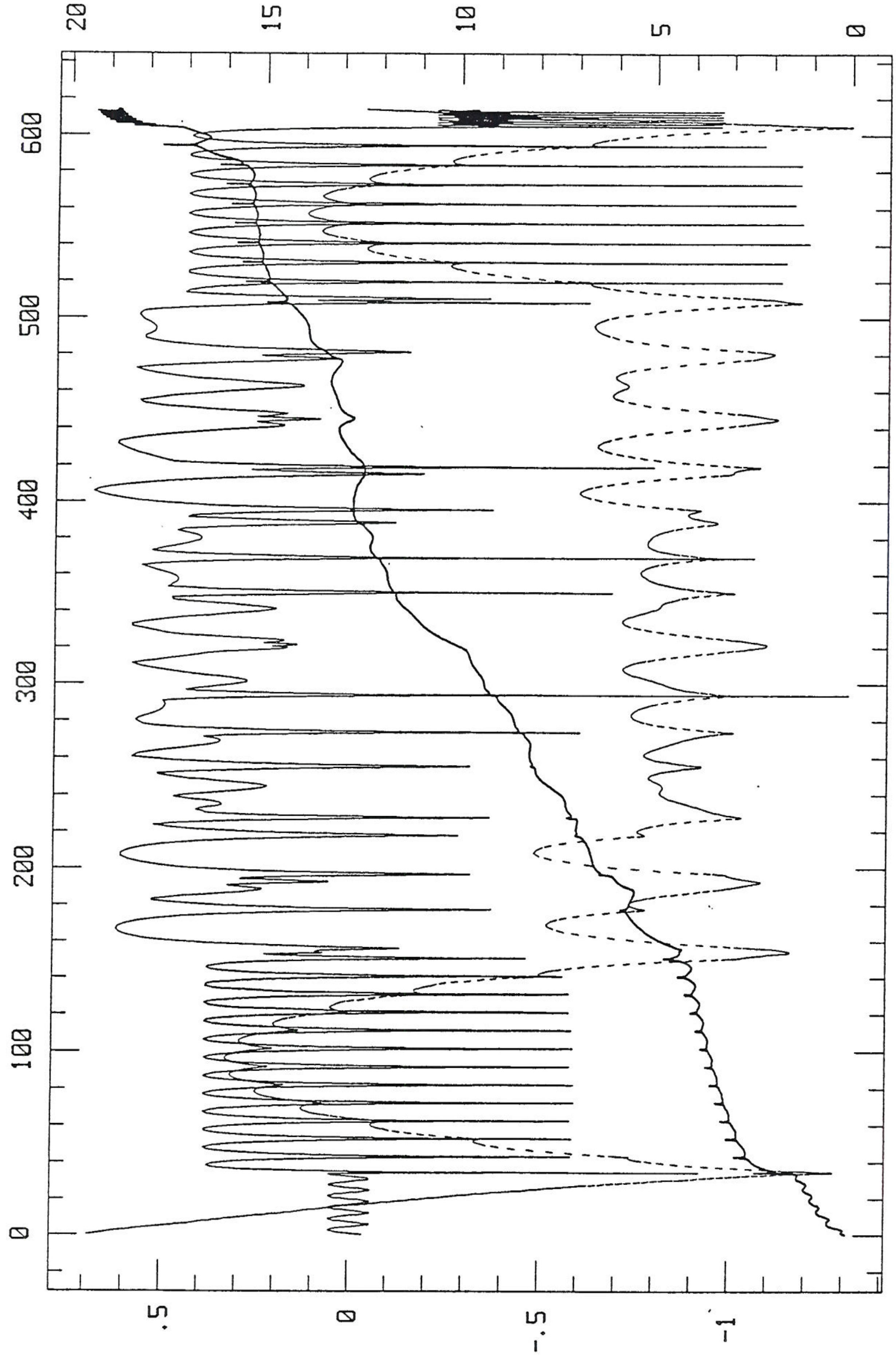

Fig. 2: A typical resonance scattering event (see text)

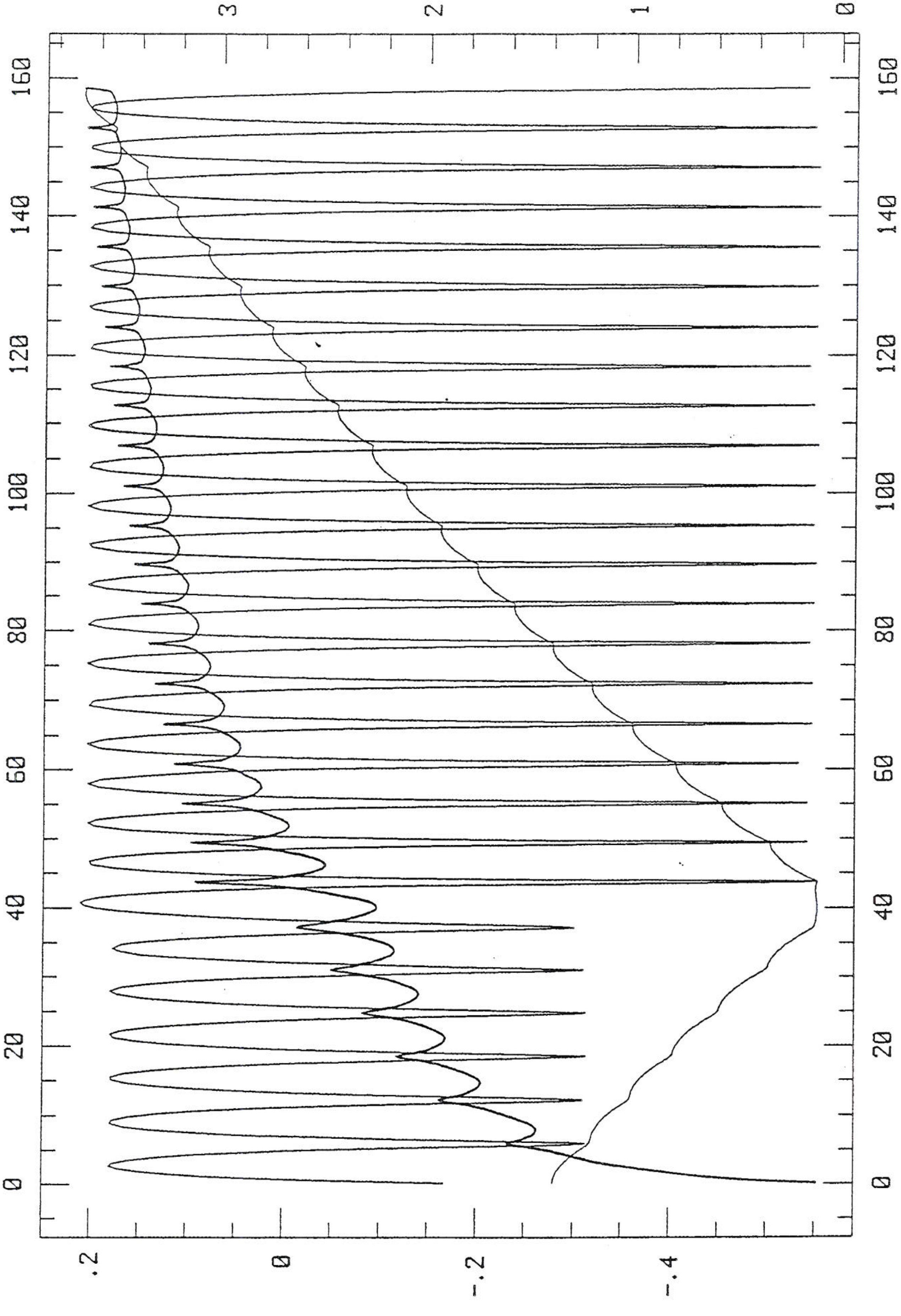

Fig. 3: A typical non-resonant scattering event (see text)

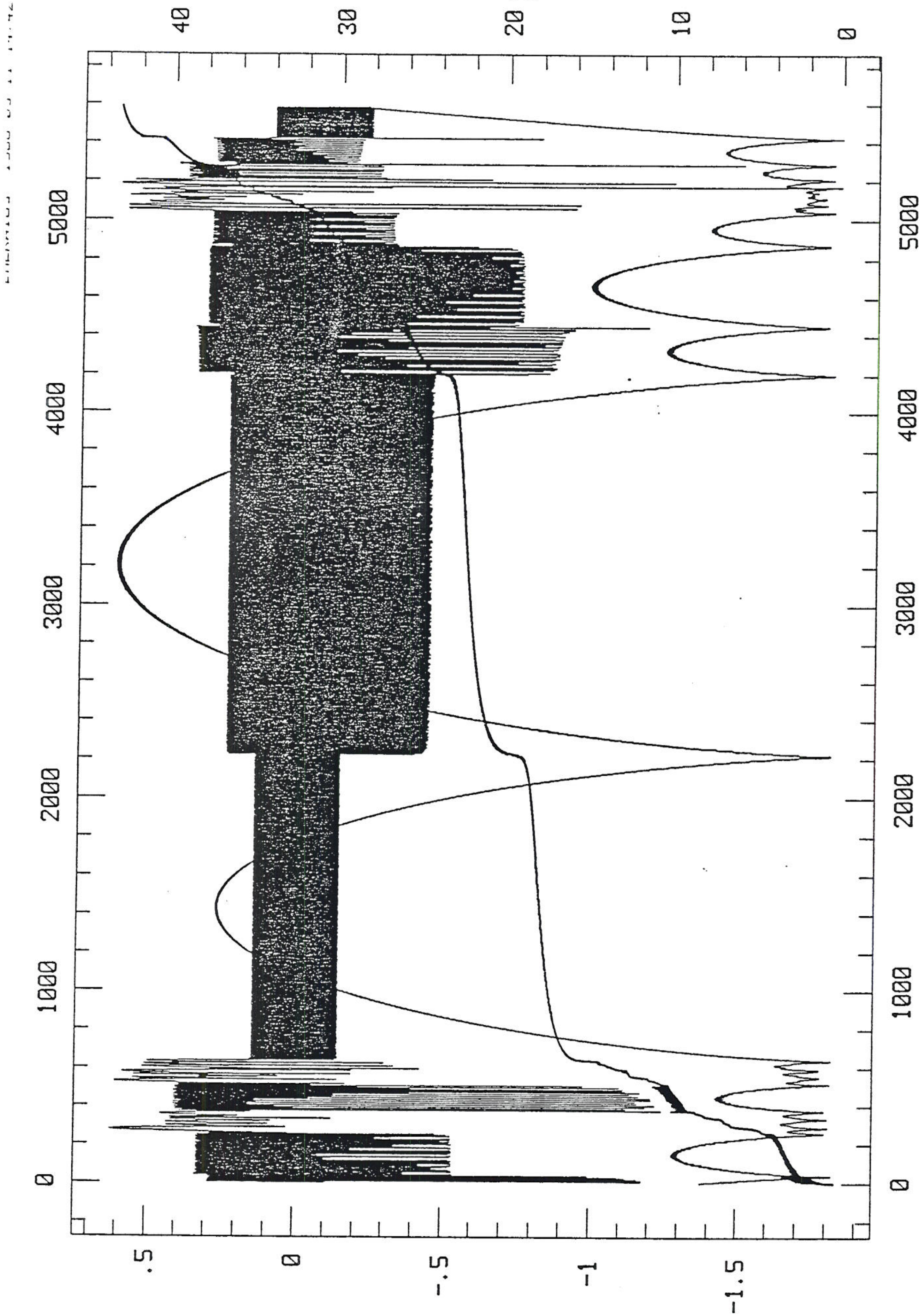

Fig. 4: One of the longer-lived resonance scattering events (see text)