Lefschetz operators, Hodge-Riemann forms, and representations

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Motivation

(M, ω) compact Kähler manifold.

$L = L_\omega : H^*(M, \mathbb{R}) \longrightarrow H^{*+2}(M, \mathbb{R})$

Hard Lefschetz

For all $k \geq 0$ we have

$$L^k : H^{n-k}(M, \mathbb{R}) \xrightarrow{\sim} H^{n+k}(M, \mathbb{R})$$

$\left(n = \dim_{\mathbb{C}} M = \frac{1}{2} \dim_{\mathbb{R}} M \right)$

$\Longleftrightarrow$ "$H^*(M, \mathbb{R})$ is a $\mathbb{Z}$-graded representation of $sl_2(\mathbb{R})$ such that

$e.? = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.? = L(?)$"

$\mathrm{Gr}^\vee = G^\vee_{\mathcal{K}} / G^\vee_{\mathcal{O}}$ affine Grassmannian $/\mathbb{C}$

$\mu \in X \rightsquigarrow t_\mu \in \mathrm{Gr}^\vee$ torus fixed pt.

Iwasawa decomposition "semi-infinite orbit"

$$\mathrm{Gr}^\vee = \bigsqcup_{\mu \in X} S^\vee_\mu \,, \quad S^\vee_\mu = N^\vee_{\mathcal{K}} \cdot t_\mu$$

Weight functors

a field

$\mathcal{F} \in \mathrm{Per}_{G^\vee_{\mathcal{O}}}(\mathrm{Gr}^\vee, k)$:

$$H^*(\mathrm{Gr}^\vee, \mathcal{F}) \cong \bigoplus_{\mu \in X} \underbrace{H_c^{2\langle \rho, \mu \rangle}(S^\vee_\mu, \mathcal{F})}_{\overset{!!}{H^*(\mathrm{Gr}^\vee, \mathcal{F})_\mu}}$$

Action of the determinant line bdl

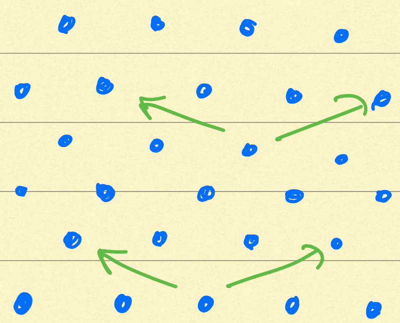
$$L : H^*(Gr^\vee, \mathcal{F}) \longrightarrow H^*(Gr^\vee, \mathcal{F})$$

and $$L|_{H^*(Gr^\vee, \mathcal{F})_\mu} \subseteq \bigoplus_{\alpha \in \Pi} H^*(Gr^\vee, \mathcal{F})_{\mu+\alpha}$$

simple roots.

$$\rightsquigarrow \quad L = \sum_{\alpha \in \Pi} L_\alpha \quad (\text{degree } L_\alpha = \alpha)$$

X- grading $\qquad\qquad\qquad$ $\mathbb{Z}$ - grading



Geometric Satake equivalence

$\Rightarrow$ " there is a $\mathfrak{g}_k$ - module structure

on $H^*(\mathrm{Gr}^\vee, \mathcal{F})$ such that

$L_\alpha(?) = e_\alpha \cdot ?$ for all $\alpha \in \Pi$ "

Serre generator

Motivating Question

Given $M = \bigoplus_{\mu \in X} M_\mu \circlearrowleft F_\alpha \ (\alpha \in \Pi)$,

(of degree $-\alpha$)

when is that a representation of $\mathfrak{g}_k$?

Setup

$M = \bigoplus_{\mu \in X} M_\mu$ X-graded k-v.sp.
(finite dimens.)

$\alpha \in \Pi$: $F_\alpha : M \to M$ of degree $-\alpha$

Def: $I \subseteq X$ is called closed, if
$\mu \in I$, $\alpha \in \Pi$ imply $\mu + \alpha \in I$.

Denote by $M_I \subseteq M$ the smallest subvectorspace containing $\bigoplus_{\mu \in I} M_\mu$ that is F_α-stable for all $\alpha \in \Pi$.

Bilinear forms

<u>Def</u> A bilinear form $b : M \times M \to k$ is called an <u>HR-form</u> if:

- symmetric and non-degenerate
- $M_\mu \perp M_\nu$ if $\mu \neq \nu$
- $[F_\alpha, F_\beta^*]\big|_{M_\mu} = \begin{cases} 0, & \alpha \neq \beta \\ \langle \mu, \alpha^\vee \rangle \cdot \mathrm{id}_{M_\mu}, & \alpha = \beta. \end{cases}$
- I closed $\Rightarrow$ $b|_{M_I \times M_I}$ non-degen.

($F_\beta^* : M \to M$ adjoint to F_β wrt b)

Theorem: (char $k \neq 2$)

The following are equivalent:

1) there exists an HR-form on M.

2) there exists a structure of a

semisimple X-graded representation

of $\mathfrak{g}_k$ on M such that

$f_\alpha \cdot ? = F_\alpha(?)$ for all $\alpha \in \Pi$.

Serre generator

(X-graded means: Cartan acts on

M_μ via μ)

A "group" version

$$U_{\mathbb{Z}} = \left\langle e_\alpha^{(n)} := \frac{e_\alpha^n}{n!},\ f_\alpha^{(n)} := \frac{f_\alpha^n}{n!} \,\middle|\, \alpha \in \Pi,\ n \geq 0 \right\rangle_{\mathbb{Z}}$$

$$\subseteq U(\mathfrak{g}_{\mathbb{C}})$$

fancy sl_2 relations

$$[e_\alpha^{(m)}, f_\beta^{(n)}] = 0 \qquad \alpha \neq \beta$$

$$e_\alpha^{(m)} f_\alpha^{(n)} = \sum_{0 \leq r \leq m,n} f_\alpha^{(n-r)} \binom{h_\alpha + 2r - (m+n)}{r} e_\alpha^{(m-r)}$$

For a ring R set $U_R := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$.

Setup

$$M = \bigoplus_{\mu \in X} M_\mu \quad (X\text{-graded } R\text{-module})$$

$$F_{\alpha,n} : M \longrightarrow M \quad (\text{homog. of degree } -n\alpha)$$

$$(\alpha \in \Pi,\ n > 0)$$

<u>**Def:**</u> $I \subseteq X$ closed :

$M_I :=$ smallest submodule of M containing $\bigoplus_{\mu \in I} M_\mu$ & stable undo $F_{\alpha,n}$ for all $\alpha \in \Pi$, $n > 0$.

The field case

<u>Def</u>: A bilinear form $b: M \times M \to k$

is called an <u>HR-form</u> if:

- it is symmetric & non-degenerate
- $M_\mu \perp M_\nu$ if $\mu \neq \nu$
- $[F_{\alpha,m}, F^*_{\beta,n}] = 0 \qquad \alpha \neq \beta$

$$F^*_{\alpha,m} F_{\alpha,n} = \sum_{0 \le r \le m,n} \binom{\langle \lambda, \alpha^\vee \rangle + m - n}{r} F_{\alpha,n-r} F^*_{\alpha,m-r}$$

(on M_λ) (fancy sl_2 relations)

- $I \subseteq X$ closed $\Rightarrow b|_{M_I \times M_I}$ non-deg.

Theorem The following are equivalent:

1) there exists an HR-form on M

2) there is a structure of an X-graded

semisimple U_k-module on M

such that $F_{\alpha,n}(?) = f_\alpha^{(n)} . ?$.

Local ring case

$$R = \mathbb{Z}_p = \left\{ \frac{a}{b} \in \mathbb{Q} \,\middle|\, p \text{ does not div. } b \right\}$$

$M = \bigoplus M_\mu \hookleftarrow F_{\alpha,n}$ (as before)

Assume: • M free of finite rank

• $M_I \subseteq M$ splits (in $\mathbb{Z}_p$-mod)

<u>Def:</u> A bilinear form $b : M \times M \to \mathbb{Z}_p$ is an <u>HR-form</u> if

- it is symmetric & non-degenerate
- $M_\mu \perp M_\nu \quad (\mu \neq \nu)$
- $[F_{\alpha,m}, F^*_{\beta,n}] = 0 \quad (\alpha \neq \beta)$

$$F^*_{\alpha,m} F_{\alpha,n} = \sum_{0 \leq r \leq m,n} \binom{\langle \lambda, \alpha^\vee \rangle + m - n}{r} F_{\alpha,n-r} F^*_{\alpha,m-r}$$

(on M_λ)

- $I \subseteq X$ closed $\Rightarrow b|_{M_I \times M_I}$ faithful

Theorem $(p \neq 2)$

The following are equivalent:

1) there exists an HR-form on M,

2) there exists a $\mathcal{U}_{\mathbb{Z}_p}$-module structure on M such that

$F_\alpha(?) = f_\alpha . ? \quad \forall \alpha \in \Pi$

and with this structure, M is a tilting module.

A different take

Data: $M = \bigoplus_{\mu \in X} M_\mu$

$\alpha \in \Pi$, $n > 0$: $F_{\alpha,n}, E_{\alpha,n} : M \longrightarrow M$

of degree $-n\alpha$, $+n\alpha$

We assume the fancy sl_2 relations.

For $\mu \in X$ set $M_{\delta_\mu} := \bigoplus_{\alpha, n > 0} M_{\mu + n\alpha}$

$$\begin{tikzcd} M_\mu \arrow[r, bend left, "F_\mu = (F_{\alpha,n})"] & M_{\delta_\mu} \arrow[l, bend left, "E_\mu = (E_{\alpha,n})^T"] \end{tikzcd}$$

$b : M \times M \to \mathbb{Z}_p$ HR-form

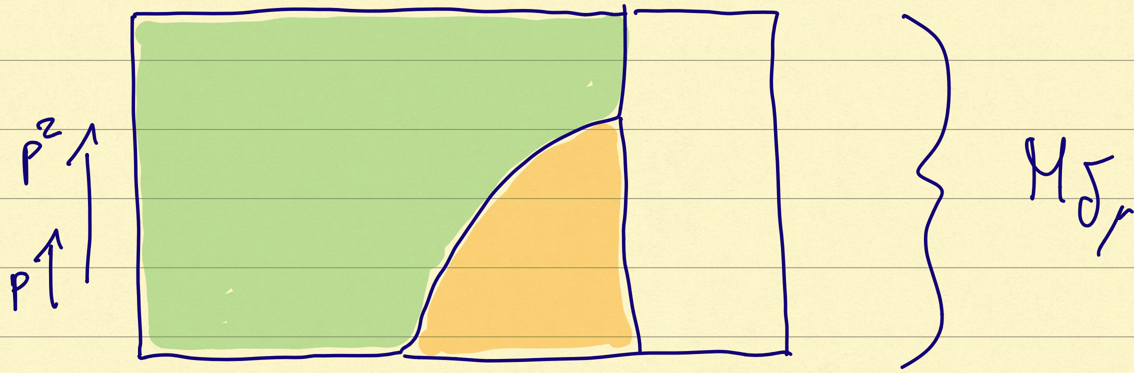
- $b|_{M_I \times M_I}$ faithful

  $\hat{=}$ $E_\mu|_{\operatorname{im} T_\mu}$ injective

- b non-degenerate

  $\hat{=}$ $E_\mu(\operatorname{im} T_\mu) \subseteq E_\mu(M_\mu) \subseteq M\delta_\mu$

  torsion torsion free



$E_\mu(\operatorname{im} F_\mu)$

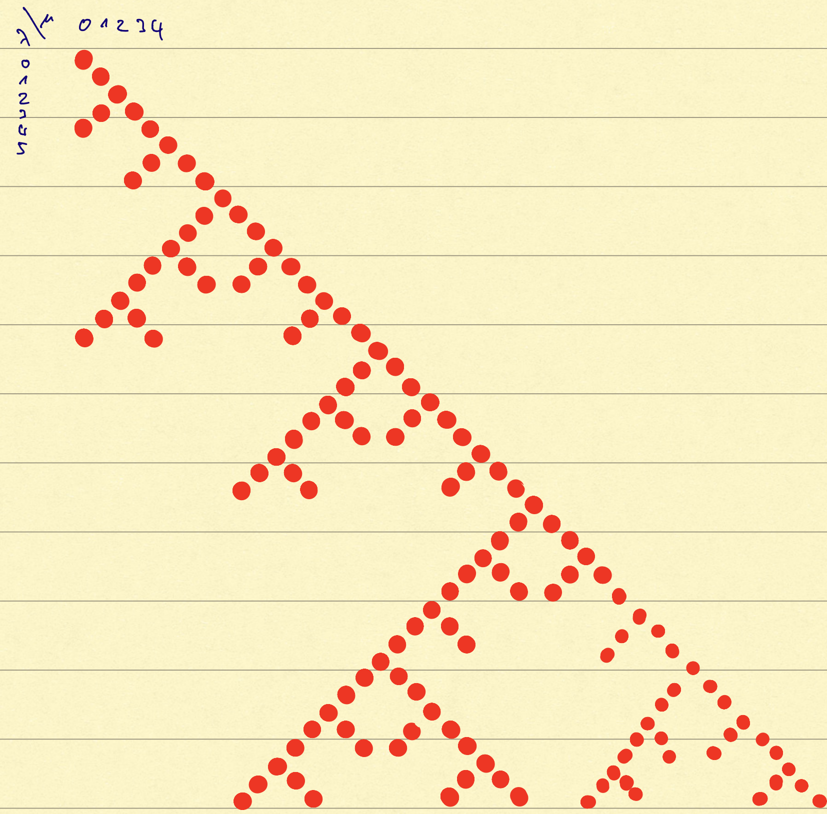
$+$ $E_\mu(M_\mu)$

Can we "filter" and obtain "approximations" to tilting modules?

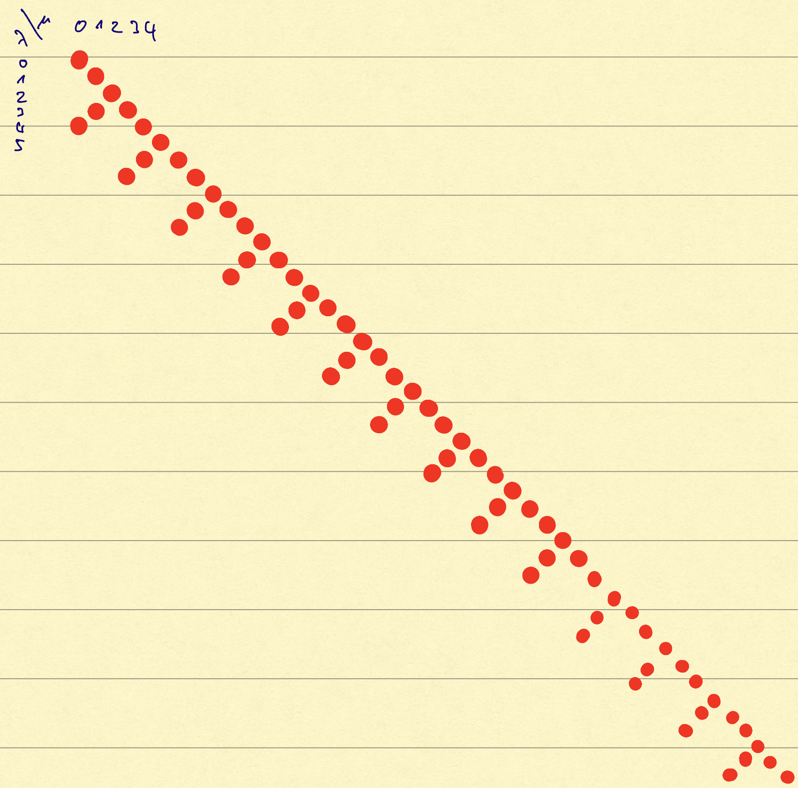
Generations of tilting modules

(after Lusztig & Williamson)

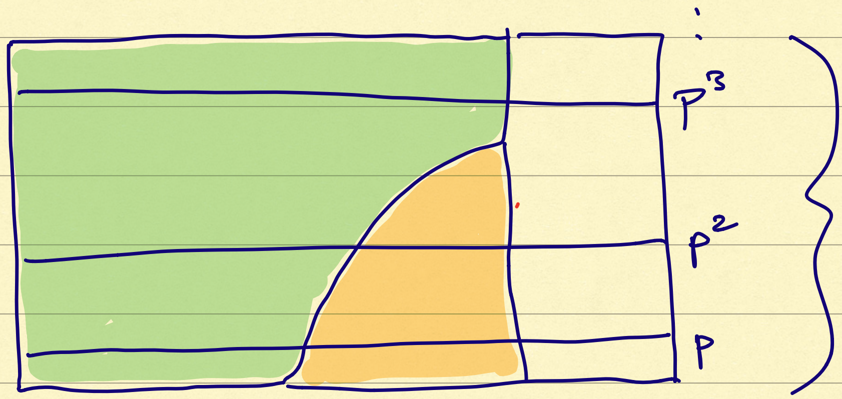
Weyl multiplicities for tilting modules for $SL_2(\overline{\mathbb{F}}_3)$



For the quantum group at a 3rd root of unity



Idea:



$E_\mu(\operatorname{im} F_\mu)$

+ $E_\mu(M_\mu)$

Naive filtration does not yield the right result (there are 1st gen. sing. with p^2 torsion and 2nd gen. sing. with p torsion)

"Quantize"

$$\begin{array}{ccc} \mathbb{Z} & \rightsquigarrow & \mathcal{Z} = \mathbb{Z}[v, v^{-1}] \\ \cap\!\!\shortmid & & \cap\!\!\shortmid \\ \mathbb{Z}_p & \rightsquigarrow & \mathcal{Z}_p = \left\{ \frac{f}{g} \;\middle|\; p \text{ does not div. } g(1) \right\} \\ \cap\!\!\shortmid & & \cap\!\!\shortmid \\ \mathbb{Q} & & \mathcal{Q} = \mathbb{Q}(v) \end{array}$$

Over $p \in \mathbb{Z}$ we have the p^e-th cyclotomic polynomials $\tau_e = \upsilon_{p^e}$ in $\mathcal{Z}_p$.

Run the same theory with

"integers replaced by quantum integers"

Advantage:

$$\begin{array}{ccc} & p\mathbb{Z}_p & \\ \swarrow & \downarrow & \searrow \\ \tau_1\mathbb{Z}_p & \tau_2\mathbb{Z}_p & \tau_3\mathbb{Z}_p \ \ldots \end{array} \qquad \begin{array}{ccc} & p^2\mathbb{Z}_p & \\ \swarrow & \downarrow & \searrow \\ \tau_1\tau_2\mathbb{Z}_p & \tau_1\tau_3\mathbb{Z}_p \ \cdots & \tau_i\tau_j\mathbb{Z}_p \end{array}$$

"p-torsion" could mean τ_i-torsion for various i

"p^2-torsion" could mean $\tau_i\tau_j$-torsion for various i,j.