THE IWAHORI-WHITTAKER CATEGORY AND GEOMETRIC CASSELMAN-SHALIKA EQUIVALENCE

TONY FENG

1. Equivariant sheaves

1.1. **Motivation.** Let k be a coefficient ring. We will mostly be interested in $k \in \{\mathbf{Q}_\ell, \mathbf{Z}_\ell, \mathbf{F}_\ell\}$ where ℓ is invertible in our ground field, and the talk will focus on these cases.

Suppose a group H acts on a variety X over $\mathbf{F}_q$. Then $H(\mathbf{F}_q)$ acts on $X(\mathbf{F}_q)$ and hence on $\mathrm{Fun}(X(\mathbf{F}_q); k)$. We contemplate decomposing the space of functions according to eigenfunctions of this action. By this we mean a function f on $X(\mathbf{F}_q)$ such that

$$f(hx) = \psi(h) f(x) \tag{1.1}$$

for some character $\psi \colon H(\mathbf{F}_q) \to k^\times$.

According to Grothendieck's function-sheaf dictionary, important functions often arises as the trace functions associated to natural sheaves. Hence we are motivated to formulate the notion of an "equivariant sheaf" which categorifies the property (1.1).

1.2. **Character sheaves.** First we will geometrize the character ψ.

Definition 1.1. A *character sheaf over k* on a group H is a rank one k-local system $\mathcal{L}$ on H, plus an isomorphism $u \colon \underline{k} \xrightarrow{\sim} e^*\mathcal{L}$, along with an isomorphism

$$\mu \colon m^*\mathcal{L} \xrightarrow{\sim} \mathcal{L} \boxtimes \mathcal{L}$$

on $H \times H$, which is required to be compatible with u in the sense that we have an identity of maps:

$$\mu|_{H \times \{e\}} = \mathrm{Id} \otimes u \colon \mathcal{L} \cong \underline{k} \otimes_{\underline{k}} \mathcal{L} \xrightarrow{\sim} e^*\mathcal{L} \otimes \mathcal{L},$$

$$\mu|_{\{e\} \times H} = u \otimes \mathrm{Id} \colon \mathcal{L} \cong \mathcal{L} \otimes_{\underline{k}} \underline{k} \xrightarrow{\sim} \mathcal{L} \otimes e^*\mathcal{L}.$$

Example 1.2. Pick an additive character $\psi \colon \mathbf{F}_q \to k^\times$. Recall that the Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbf{A}^1_{\mathbf{F}_q}$ is the character sheaf whose associated trace function is ψ. This can be constructed as a direct summand of the Artin-Schreier cover $\mathbf{A}^1_{\mathbf{F}_q} \xrightarrow{x^q - x} \mathbf{A}^1_{\mathbf{F}_q}$.

Example 1.3. More generally, using the Lang isogeny $G \xrightarrow{F(g)g^{-1}} G$, one can show that if G is a connected *abelian* group scheme over $\mathbf{F}_q$, then any character χ of $G(\mathbf{F}_q)$ arises from a character sheaf $\mathcal{L}_\chi$.

1.3. **Equivariance for character sheaves.** Let $\mathcal{L}$ be a character sheaf on H.

Roughly speaky, an $(H,\mathcal{L})$-equivariant sheaf on X should be a sheaf $\mathcal{F}$ with isomorphisms "$\mathcal{F}_{h\cdot x} = \mathcal{L}_h \otimes \mathcal{F}_x$", such that the two induced identifications $\mathcal{F}_{h_1 h_2 \cdot x} \approx \mathcal{L}_{h_1} \otimes \mathcal{L}_{h_2} \otimes \mathcal{F}_x$ coincide. Now, geometrically we should think of h and x are varying through ring-valued points of H and X, and that motivates us to consider the following universal case.

Definition 1.4. We define an $(H,\mathcal{L})$-*equivariant sheaf* on X to be a sheaf $\mathcal{F}$, plus an isomorphism

$$m^*\mathcal{F} \cong \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{F},$$

satisfying the cocycle condition on $H \times H \times X$.

We define $(H,\mathcal{L})$-*equivariant perverse sheaves* in an analogous way, asking $\mathcal{F}$ to be perverse.

1.4. **Equivariant derived categories.** Defining the derived category of equivariant sheaves is subtler. Indeed, when $\mathcal{L}$ is the trivial character sheaf $\underline{k}$, the notion of $(H,\underline{k})$-equivariant derived category should specialize to the usual equivariant derived category, and we know that this is tricky to define.

Morally, we want to start as before with an isomorphism $m^*\mathcal{K} \cong \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{K}$, but then instead of asking that it satisfies a cocycle condition, we should ask for an infinite sequence of coherent homotopies.

However, when the group H is *acyclic*, meaning that $H^*(H,k) = k$, then the definition simplifies, and we can define an equivariant complex in the same way as above: $\mathcal{K} \in D^b(X;k)$ with an isomorphism $m^*\mathcal{K} \cong \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{K}$ satisfying the cocycle condition. Notably, this is a *condition* rather than a structure, so that the category of $(H,\mathcal{L})$-equivariant sheaves is a subcategory of the ordinary derived category. We emphasize that this is a special feature of the acyclic situation.

Example 1.5. The case of interest for us in this talk is when H is (pro)-unipotent, and in this case the acyclicity is certainly satisfied. Suppose furthermore that the H-action induces a stratification of X (which is a finiteness condition on the action, namely that there are locally finitely many orbits). Then it is true that Beilinson's realization functor induces an equivalence

$$D^b(P_{(H,\mathcal{L})}(X,k)) \xrightarrow{\sim} D^b_{(H,\mathcal{L})}(X;k),$$

using the fact that the stratification on X induced by H is by affine spaces. This will be important for us.

Remark 1.6. In general, when H is not necessarily acyclic, an approach to the $(H,\mathcal{L})$-equivariant category is as follows [LY, §2.5]. By the theory of character sheaves [Yun, Remark 2.3.7(1)], there is a central isogeny $\nu\colon \widetilde{H} \to H$ such that $\mathcal{L}$ appears as a summand of $\nu_! \underline{k}$. We can view $\widetilde{H}$ as acting on X with $\ker(\nu)$ acting trivially. The induced action on $D_{\widetilde{H}}(X)$ decomposes it into a direct sum, with summands indexed by characters of $\ker(\nu)$. We define $D_{H,\mathcal{L}}(X)$ to be the summand corresponding to $\mathcal{L}$.

2. The Casselman-Shalika formula

We are now motivated to consider a situation which is fundamental in p-adic representation theory. Let $K = \mathbf{F}_q((t))$ and $\mathcal{O} = \mathbf{F}_q[[t]]$, but the results stated in this section have a version for p-adic fields too.

2.1. **The spherical Hecke algebra.** The *spherical Hecke algebra* (over k) is

$$\mathcal{H}(G) := \mathrm{Fun}_c(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}); k).$$

It is an algebra under convolution, which turns out to be commutative.

To get a feeling for what this algebra looks like, note that there is a natural way to index orbits of $G(\mathcal{O})$ on $G(K)/G(\mathcal{O})$ by the dominant cocharacters $X_*(T)_+$, with $\lambda \in X_*(T)_+$ corresponding to the orbit of t^λ, the image of $t \in K^\times$ under the cocharacter λ.

Evidently a natural basis is given by the indicator functions $\mathbb{1}_{G(\mathcal{O})t^\lambda G(\mathcal{O})}$, but from the perspective of Langlands duality there is a better basis: the one given by the irreducible representations of $\mathrm{Rep}_k(\widehat{G})$ under the Satake isomorphism

$$\mathcal{H}(G) \cong K_0(\mathrm{Rep}_k(\widehat{G})).$$

This other basis we will call $(h_\lambda \colon \lambda \in X_*(T)_+)$.

Remark 2.1. With a suitable normalization, the function h_λ can be seen as the trace function associated to the IC sheaf of Gr^λ. Namely, this IC sheaf should be normalized to be perverse of weight zero; however this requires choosing a square root of the cyclotomic character, so it is a bit subtle to pin down correctly.

2.2. **Unramified Whittaker functions.** Choose a non-trivial additive character $\psi \colon \mathbf{F}_q \to k^\times$. We have a residue map $N(K) \to N(\mathbf{F}_q)$, induced by sending $\sum a_i t^i \in K$ to $a_{-1} \in \mathbf{F}_q$. We also have a map

$$\chi \colon N(\mathbf{F}_q)/[N(\mathbf{F}_q), N(\mathbf{F}_q)] \approx \prod_{\Delta_s} \mathbf{F}_q \xrightarrow{+} \mathbf{F}_q.$$

Abusing notation, we also denote by ψ the composition $N(K) \xrightarrow{\chi} \mathbf{F}_q \xrightarrow{\psi} k^\times$. This is called a *non-degenerate Whittaker character*.

Remark 2.2. Although the structure theory of smooth representations is quite similar over p-adic and function fields, the Whittaker characters look very different in these two situations! Since the addition on a p-adic field is divisible, additive characters on p-adic fields never have finite order.

Definition 2.3. The space of *unramified Whittaker functions* (over k) is

$$\mathcal{W}(G) := \mathrm{Fun}_c(N(K), \psi\backslash G(K)/G(\mathcal{O}); k).$$

Explicitly, this is the space of compactly supported functions $f \colon G(K)/G(\mathcal{O}) \to \mathbf{C}$ such that $f(ng) = \psi(n)f(g)$ for all $n \in N(K)$.

To get a feel for what this space looks like, let's first understand the double coset space $N(K)\backslash G(K)/G(\mathcal{O})$. The orbits of $N(K)$ on $G(K)/G(\mathcal{O})$ are indexed by $X_*(T)$, and represented by t^λ.

However, an orbit $N(K)t^\lambda G(\mathcal{O})$ only supports a ψ-equivariant function if ψ is trivial when restricted to the stabilizer of one (equivalently, every) point in the orbit. The stabilizer of t^λ in $N(K)$ is $t^\lambda N(\mathcal{O})t^{-\lambda}$, and we need this to be contained in $\ker(\chi)$. This is only the case when λ pairs non-negatively with all the positive roots, i.e. when λ is dominant.

Let $\phi_\lambda \in \mathcal{W}(G)$ be the unique function supported on the double coset $N(K)t^\lambda G(\mathcal{O})$ with value 1 at t^λ.

2.3. **The Casselman-Shalika formula.** Note that there is a right action of $\mathcal{H}(G)$ on $\mathcal{W}(G)$ by convolution on the right.

Assuming that k has **characteristic zero**, it is a fact that

$$\boxed{\phi_\eta * h_\lambda = q^{\langle \lambda, \rho \rangle} \phi_{\zeta+\lambda}}$$

where ρ is the usual half sum of positive roots for G. This is essentially equivalent to the *Casselman-Shalika formula*, which gives a formula for the Fourier expansion of the Whittaker function of an unramified representation. (This formula is crucial in number theory to relate local zeta integrals of Whittaker functions to local L-factors.)

3. The Iwahori-Whittaker category

3.1. **The Whittaker category.** Let LN be the loop group of N, whose functor of points is $LN(R) = N(R((t)))$. Pick an Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbf{G}_a$. Analogously to above, we have a map

$$LN \xrightarrow{\text{Res}} N \twoheadrightarrow N/[N,N] \approx \prod_{\alpha \in \Delta_s} \mathbf{G}_a \xrightarrow{+} \mathbf{G}_a$$

through which we pull back the Artin-Schreier sheaf, to get a character sheaf also denoted $\mathcal{L}_\psi$ on LN.

If we want to geometrize the Whittaker functions, we should consider a *Whittaker category* "$P_{(LN,\mathcal{L})}(\mathrm{Gr}_G)$". However, this turns out to be a somewhat inaccessible object, because the N-orbits on Gr_G are too large (they are all infinite-dimensional). In particular, it is not clear how to define a meaningful notion of perverse sheaves on such an object.

Remark 3.1. There is a natural way to define a *derived* category $D_{(LN,\psi)}(\mathrm{Gr}_G; k)$ using infinity-categorical constructions, as explained in [GaW].

3.2. **The Iwahori-Whittaker category.** To get around this, we will consider a different category called the "Iwahori-Whittaker model".

Let I be the (positive) Iwahori subgroup of L^+G, i.e. the preimage of the Borel subgroup under $L^+G \twoheadrightarrow G$. It is a pro-algebraic group. Let I_u be the pro-unipotent radical of the Iwahori subgroup, i.e. the preimage of the unipotent radical of the Borel under $I \twoheadrightarrow B$.

Example 3.2. For $G = \mathrm{PGL}_2$, the *Iwahori subgroup* is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^+G \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod t \right\}.$$

The pro-unipotent radical is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^+G \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \pmod t \right\}.$$

As before we have a surjection $I_u \twoheadrightarrow N \twoheadrightarrow \mathbf{G}_a$, and we let $\mathcal{L}_\psi$ be the Artin-Schreier sheaf inflated to I_u.

The orbits of I_u on Gr_G are of course finite-dimensional, since $I_u \subset L^+G$. The Cartan decomposition implies that the orbits of I_u on Gr_G are also represented by t^λ for $\lambda \in X_*(T)$. For $\lambda \in X_*(T)$, we let $X_\lambda := I t^\lambda L^+G$.

Example 3.3. For $\lambda \in X_*(T)_+$, X_λ is dense open in $\mathrm{Gr}^{\leq \lambda}$, so $\dim X_\lambda = \langle 2\rho, \lambda \rangle$. Our choice of Iwahori (using the positive Borel subgroup instead of the negative one, say) was arranged to make this true.

We consider the category of equivariant perverse sheaves, $P_{(I_u, \mathcal{L}_\psi)}(\mathrm{Gr}_G)$. A consequence of Example 1.5 is that:

Proposition 3.4. *We have* $D^b(P_{(I_u,\mathcal{L}_\psi)}(\mathrm{Gr}_G; k)) \xrightarrow{\sim} D^b_{(I_u,\mathcal{L}_\psi)}(\mathrm{Gr}_G; k)$.

To get a feel for this category, let's see which orbits can support Iwahori-Whittaker equivariant sheaves. The stabilizer of t^λ in I_u is $I_u \cap t^\lambda I_u t^{-\lambda}$. For χ to be trivial on this subgroups, we need $\langle \lambda, \alpha \rangle \geq 1$ for each simple root α of G.

Assume from now on that G is adjoint. Let $\widehat{\rho}$ be the half sum of positive roots for $\widehat{G}$, which is an integral weight because $\widehat{G}$ is simply connected. When regarded as a coweight of G, $\widehat{\rho}$ has the property that $\langle \widehat{\rho}, \alpha \rangle = 1$ for each simple root $\alpha \in G$, hence it is a "minimal" dominant coweight supporting a non-zero Iwahori-Whittaker equivariant sheaf.

Remark 3.5. To get a statement that works for arbitrary groups, it is better to replace I_u by "$\mathrm{Ad}_{\widehat{\rho}(t)}(I_u)$", which makes sense even if $\widehat{\rho}(t)$ doesn't itself make sense. In this convention the t^λ for $\lambda \in X_*(T)_+$ are the representatives of orbits supporting a non-zero Iwahori-Whittaker equivariant sheaf. There is a canonical equivalence [Ra] between $D_{(\text{“}\mathrm{Ad}_{\widehat{\rho}(t)}(I_u)\text{”}, \mathcal{L}_\psi)}(\mathrm{Gr}_G; k)$ and $D_{(N, \mathcal{L}_\psi)}(\mathrm{Gr}_G; k)$, justifying in what sense the Iwahori-Whittaker model is indeed a model for the Whittaker category.

Definition 3.6. We abbreviate $P_{\mathcal{IW}}(\mathrm{Gr}_G, k) := P_{(I_u,\mathcal{L}_\psi)}(\mathrm{Gr}_G; k)$. We define the following objects in $P_{\mathcal{IW}}(\mathrm{Gr}_G; k)$ for each $\lambda \in X_*(T)_{++}$. Let $j_\lambda \colon X_\lambda \hookrightarrow \mathrm{Gr}_G$.

- Define $\Delta^{\mathcal{IW}}_\lambda(k)$ be the !-extension through j_λ of the rank 1 $(I_u, \mathcal{L}_\psi)$-equivariant k-local system on X_λ. Note that it is perverse, as j_λ is an affine embedding.
- Define $\nabla^{\mathcal{IW}}_\lambda(k)$ be the $*$-extension through j_λ of the free rank 1 $(I_u, \mathcal{L}_\psi)$-equivariant k-local system on X_λ. Note that it is perverse, as j_λ is an affine embedding.
- Define $\mathrm{IC}^{\mathcal{IW}}_\lambda(k) = \mathrm{Im}\,(\Delta^{\mathcal{IW}}_\lambda(k) \to \nabla^{\mathcal{IW}}_\lambda(k))$ to be the intermediate extension of the free rank 1 $(I_u, \mathcal{L}_\psi)$-equivariant k-local system on X_λ.

Proposition 3.7. *Suppose k is a field. Then $P_{\mathcal{IW}}(\mathrm{Gr}_G; k)$ forms a highest weight category in the sense of* [Ri, Definition 7.1]*, with standard objects $\Delta^{\mathcal{IW}}_\lambda(k)$ and costandard objects $\nabla^{\mathcal{IW}}_\lambda(k)$ and simple objects $\mathrm{IC}^{\mathcal{IW}}_\lambda(k)$.*

Proof sketch. Invoking Proposition 3.4 to compute Ext groups, the properties are all formal to check, just using that the stratification is by affine spaces. □

Remark 3.8. If k is a field, then the usual Satake category $P_{G(\mathcal{O})}(\mathrm{Gr}_G; k)$ is also a highest weight category, with standard objects $\mathcal{J}_!(\lambda, k)$, costandard objects denoted $\mathcal{J}_*(\lambda, k)$, and simple objects $\mathcal{J}_{!*}(\lambda, k)$, but this is much more difficult to prove. If k has characteristic 0, then the semi-simplicity of $P_{G(\mathcal{O})}(\mathrm{Gr}_G; k)$ implies

$$\mathcal{J}_!(\lambda, k) \xrightarrow{\sim} \mathcal{J}_{!*}(\lambda, k) \xrightarrow{\sim} \mathcal{J}_*(\lambda, k). \tag{3.1}$$

Remark 3.9. If k is a field of characteristic zero, then the usual parity considerations show that $P_{\mathcal{IW}}(\mathrm{Gr}_G; k)$ is also semi-simple, so we have

$$\Delta^{\mathcal{IW}}_\lambda(k) \xrightarrow{\sim} \mathrm{IC}^{\mathcal{IW}}_\lambda(k) \xrightarrow{\sim} \nabla^{\mathcal{IW}}_\lambda(k) \quad \text{ if char } k = 0\ .$$

3.3. **Convolution action.** We have a right action of $D_{L^+G}(\mathrm{Gr}_G;k)$ on $D_{\mathcal{IW}}(\mathrm{Gr}_G;k)$ by convolution. We denote by

$$\Phi\colon D_{L^+G}(\mathrm{Gr}_G;k) \to D_{\mathcal{IW}}(\mathrm{Gr}_G;k)$$

the functor obtained by convolving against $\Delta^{\mathcal{IW}}_{\widehat{\rho}}(k)$.

Proposition 3.10. *For $k \in \{\mathbf{Q}_\ell, \mathbf{Z}_\ell, \mathbf{F}_\ell\}$, the functor Φ is t-exact for the perverse t-structure. We denote by Φ^0 the induced functor*

$$\Phi^0\colon P_{L^+G}(\mathrm{Gr}_G;k) \to P_{\mathcal{IW}}(\mathrm{Gr}_G;k).$$

The key input is the following result of Gaitsgory.

Theorem 3.11 (Gaitsgory)**.** *If k is a field, then the action of $P_{L^+G}(\mathrm{Gr}_G;k)$ on $D(\mathrm{Gr}_G,k)$ is t-exact, i.e. preserves $P_{L^+G}(\mathrm{Gr}_G;k)$.*

Proof sketch. We will give a different formula for the convolution in terms of nearby cycles; the result then immediately follows from the fact that nearby cycles (suitably normalized) preserves perversity.

The nearby cycles is calculated by the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,X}$ over a curve X, which is a degeneration from $\mathrm{Gr}_G \times \mathrm{Gr}_G$ over $X-0$ to Gr_G over 0. (Here perversity is used to get the sheaves to live over a version of the Grassmannian spread out over a curve, as all $G(\mathcal{O})$-equivariant perverse sheaves are invariant under the automorphism group of the formal disk by [Ga01, Proposition 1].)

We will write down a proper family $\widetilde{\mathrm{Gr}}_{G,X} \to \mathrm{Gr}_{G,X}$ which degenerates from $\mathrm{Gr}_G \times \mathrm{Gr}_G$ over $X-0$ to the convolution Grassmannian $\widetilde{\mathrm{Gr}}_G$ over 0, as depicted below.

$$\begin{tikzcd}
\mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[r, rightsquigarrow] \arrow[d, "\sim"] & \widetilde{\mathrm{Gr}}_G \arrow[d] \\
\mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[r, rightsquigarrow] & \mathrm{Gr}_G \\
X-0 \arrow[r, rightsquigarrow] & 0
\end{tikzcd}$$

By compatibility of nearby cycles with proper pushforward, this will prove the Theorem. It remains to construct these families.

The Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,X}$ parametrizes a G-bundle $\mathcal{E}_G$ plus a trivialization away from $0,x$. The family $\widetilde{\mathrm{Gr}}_{G,X}$ parametrizes two G-bundles $\mathcal{E}_G, \mathcal{E}'_G$ plus a chain of modifications $G \times X \xrightarrow{x} \mathcal{E}'_G \xrightarrow{0} \mathcal{E}_G$. We consider the map $\widetilde{\mathrm{Gr}}_{G,X} \to \widetilde{\mathrm{Gr}}_{G,X}$ gotten by forgetting the intermediate modification bundle $\mathcal{E}'_G$, and composing the two modifications. This evidently specializes to the convolution morphism over 0, whereas away from 0 it is an isomorphism because the intermediate bundle can be uniquely reconstructed.

$\square$

Proof of Proposition 3.10. Theorem 3.11 immediately implies the result for $k = \mathbf{Q}_\ell, \mathbf{F}_\ell$. (The concern when $k = \mathbf{Z}_\ell$ is that derived tensor products don't preserve perversity.)

For $\mathcal{F} \in P_{L^+G}(\mathrm{Gr}_G, \mathbf{Z}_\ell)$, we have $\mathbf{Q}_\ell \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \Phi(\mathcal{F}) = \Phi(\mathbf{Q}_\ell \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \mathcal{F})$ is perverse, so any perverse cohomology object ${}^p\mathcal{H}^i(\Phi(\mathcal{F}))$ with $i \neq 0$ is torsion, by the result when $k = \mathbf{Q}_\ell$.

Next since $\mathbf{F}_\ell \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \Phi(\mathcal{F}) = \Phi(\mathbf{F}_\ell \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \mathcal{F})$ is perverse, any torsion ${}^p\mathcal{H}^i(\Phi(\mathcal{F}))$ would create two consecutive non-zero perverse cohomology groups for $\Phi(\mathbf{F}_\ell \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \mathcal{F})$, contradicting Theorem 3.11 for $k = \mathbf{F}_\ell$. □

4. Iwahori-Whittaker realization of $\mathrm{Rep}(\widehat{G})$

We can now state the main theorem of the talk:

Theorem 4.1 ([BGMRR, Theorem 3.9])**.** *For $k \in \{\mathbf{Q}_\ell, \mathbf{Z}_\ell, \mathbf{F}_\ell\}$, the functor*

$$\Phi^0 \colon P_{G(\mathcal{O})}(\mathrm{Gr}_G; k) \xrightarrow{\sim} P_{\mathcal{IW}}(\mathrm{Gr}_G; k)$$

is an equivalence.

Remark 4.2. If k has characteristic 0, this is an old theorem of Arkhipov-Bezrukavnikov-Braverman-Gaitsgory-Mirkovic [ABBGM]. Recent results of Riche-Williamson and Arinkin-Bezrukavnikov, which will likely make a future appearance in this seminar, rely crucially on the statement for $k = \mathbf{F}_\ell$. Our focus will be on explaining the version where k has positive characteristic, so we will assume ingredients from the characteristic 0 version.

Remark 4.3. One reason for the interest in Theorem 4.1 is that combining it with Proposition 3.4 gives an equivalence

$$D^b_{\mathcal{IW}}(\mathrm{Gr}_G; k) \cong D^b(\mathrm{Rep}_k(\widehat{G})).$$

In other words, this gives a "geometric" realization of the *derived* category of $\mathrm{Rep}_k(\widehat{G})$, whereas the equivariant derived Satake category becomes something rather more complicated.

4.1. **Geometric Casselman-Shalika formula.** Suppose k has characteristic zero. Recall we said that the Hecke function h_λ has the property that

$$\phi_0 * h_\lambda = q^{\langle \rho, \lambda \rangle} \phi_\lambda.$$

By definition, this means that

$$\int_{N(K)} \psi(\chi(x)) h_\lambda(n^{-1} t^\mu)\, dn = \begin{cases} q^{\langle \rho, \lambda \rangle} & \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

We may rewrite this as

$$\int_{N(K)t^\mu} \psi(\chi(n^{-1} t^{-\mu})) h_\lambda(n) = \begin{cases} q^{\langle \rho, \lambda \rangle} & \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Let's rename $\chi_\mu(n) = \chi(n t^{-\mu})$. Up to replace ψ by another non-degenerate character of $\mathbf{F}_q$, the natural geometric upgrade of this statement would be

$$\boxed{H^i_c(S_\mu, \mathcal{J}_{!*}(\lambda, \mathbf{Q}_\ell)|_{S_\mu} \otimes_{\mathbf{Q}_\ell} \chi_\mu^*(\mathcal{L}_\psi^{\mathbf{Q}_\ell})) = \begin{cases} \mathbf{Q}_\ell(-\langle 2\rho, \mu \rangle) & \lambda = \mu, i = \langle 2\rho, \mu \rangle \\ 0 & \text{otherwise} \end{cases}} \quad (4.1)$$

where $S_\mu = LN \cdot t^\mu L^+ G \subset \mathrm{Gr}_G$ is the "semi-infinite orbit of t^μ", and $\chi_u \colon S_\mu \to \mathbf{G}_a$ is the geometric version of the function denoted χ_μ above. Note that $\mathcal{J}_{!*}(\lambda, \mathbf{Q}_\ell)|_{S_\mu}$ is supported on $\mathrm{Gr}^{\leq \lambda} \cap S_\mu$, which is finite-dimensional (so there are no subtleties in defining cohomology). This is called the *geometric Casselman-Shalika formula*, originally proved in [FGV].

We will **assume** (4.1) going forward.

4.2. **Iwahori-Whittaker version of Casselman-Shalika.** If we believe Theorem 4.1, then for k a field we should have $\Phi^0(\mathcal{J}_!(\lambda,k)) \cong \Delta^{\mathcal{IW}}_{\lambda+\widehat{\rho}}(k)$. We will try to prove this by showing that "$\Phi^0(\mathcal{J}_!(\lambda,k))$ maps out like a costandard". If we believe this, then the following Proposition is an immediate consequence of the structure of highest weight categories.

Proposition 4.4. *For $\lambda \neq \mu$ we have* $\operatorname{Hom}_{P_{\mathcal{IW}}(\mathrm{Gr}_G;k)}(\Phi^0(\mathcal{J}_!(\lambda,k)), \nabla^{\mathcal{IW}}_{\mu+\widehat{\rho}}(k)) = 0.$

Our next goal will be to prove this. It will be done in two steps:

(1) Rewrite the Hom in question as cohomology groups of sheaves not involving the standard/costandard objects.
(2) Show that the cohomology groups vanish, using Geometric Casselman-Shalika.

We will perform the steps in reverse order (so starting with the cohomological vanishing statement).

Recall that we defined $X_\mu = I \cdot t^\mu L^+G$. We have a function $\chi'_u \colon z^{-\widehat{\rho}} X_{\mu+\widehat{\rho}} \to \mathbf{G}_a$ such that $\chi'_u(z^{-\widehat{\rho}} u t^\mu t^{\widehat{\rho}}) = \chi(u)$. This is the Iwahori-Whittaker version of χ_μ. The following is essentially the Iwahori-Whittaker analog of the vanishing statement in Geometric Casselman-Shalika.

Lemma 4.5. *For $\lambda \neq \mu$, $H_c^{\langle \lambda+\mu, 2\rho\rangle}(\mathrm{Gr}^\lambda \cap z^{-\widehat{\rho}} X_{\mu+\widehat{\rho}}, (\chi'_u)^*\mathcal{L}_\psi(k)) = 0$ for $k \in \{\mathbf{Q}_\ell, \mathbf{Z}_\ell, \mathbf{F}_\ell\}$.*

Proof sketch. We have $t^{-\widehat{\rho}} X_{\mu+\widehat{\rho}} \subset S_\mu$, and the restriction of χ_μ to $t^{-\widehat{\rho}} X_{\mu+\widehat{\rho}}$ is χ'_μ. Hence we have $\dim(\mathrm{Gr}^\lambda \cap t^{-\widehat{\rho}} X_{\mu+\widehat{\rho}}) \leq \dim(\mathrm{Gr}^\lambda \cap S_\mu) = \langle \lambda+\mu, \rho\rangle$. If the inequality is strict, then the conclusion is obvious.

So that leaves us with the case where the inclusion is an open immersion. Hence we reduce to showing that

$$H_c^{\langle \lambda+\mu, 2\rho\rangle}(\mathrm{Gr}^\lambda \cap S_\mu, \chi_\mu^*(\mathcal{L}_\psi)|_{\mathrm{Gr}^\lambda \cap S_\mu}) = 0 \tag{4.2}$$

if $\lambda \neq \mu$. Since this is the top dimensional cohomology, the vanishing of (4.2) follows in all cases from the case $k = \mathbf{Q}_\ell$, where we can apply Geometric Casselman-Shalika. So we assume going forward that $k = \mathbf{Q}_\ell$.

By base change, the group (4.2) is identified with $H_c^{\langle \lambda+\mu, 2\rho\rangle}(S_\mu, j_!(\underline{\mathbf{Q}_\ell})_{\mathrm{Gr}^\lambda} \otimes \chi_\mu^*(\mathcal{L}_\psi))$. This looks a similar to Geometric Casselman-Shalika, except that we must replace $j_!(\underline{\mathbf{Q}_\ell})_{\mathrm{Gr}^\lambda}$ by the intermediate extension.

For $\mathcal{F} \in P_{G(\mathcal{O})}(\mathrm{Gr}_G, \mathbf{Q}_\ell)$, (4.1) implies (by an inductive argument on the Jordan-Hólder filtration of $\mathcal{F}$) that $H^i_c(S_\mu; \mathcal{F} \otimes_{\mathbf{Q}_\ell} \chi_\mu^*(\mathcal{L}_\psi^{\mathbf{Q}_\ell})) = 0$ unless $i = \langle 2\mu, \rho\rangle$. Since the map $(j_\lambda)_!(\underline{\mathbf{Q}_\ell})_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\rho\rangle] \to \mathcal{J}_!(\lambda, \mathbf{Q}_\ell)$ tautologically induces an isomorphism of degree-0 perverse cohomology sheaves, on cohomology it induces

$$H_c^{\langle \lambda+\mu, 2\rho\rangle}(S_\mu, j_!(\underline{\mathbf{Q}_\ell})_{\mathrm{Gr}^\lambda} \otimes \chi_\mu^*(\mathcal{L}_\psi)) \xrightarrow{\sim} H_c^{\langle \lambda+\mu, 2\rho\rangle}(S_\mu, \mathcal{J}_!(\lambda, \mathbf{Q}_\ell) \otimes \chi_\mu^*(\mathcal{L}_\psi)).$$

Then, since $\mathcal{J}_!(\lambda, \mathbf{Q}_\ell) \cong \mathcal{J}_{!*}(\lambda, \mathbf{Q}_\ell)$ by (3.1), we can apply Geometric Casselman-Shalika to deduce that the RHS vanishes for $\lambda \neq \mu$. $\square$

Proof sketch of Proposition 4.4. The difficulty in understanding the objects in the statement is that $\Phi^0(\mathcal{J}_!(\lambda,k))$ involves a perverse truncation. The **trick** to overcome this is to observe that by exactness of Φ, and the fact that $j_!$ is right exact in the perverse t-structure, we can replace $\Phi^0(\mathcal{J}_!(\lambda,k))$ with $(j_\lambda)_! k$. Recall also from Definition 3.6 that $\nabla^{\mathcal{IW}}_{\mu+\widehat{\rho}}(k)$ was simply $(j_{\mu+\widehat{\rho}})_*((\chi'_\mu)^*\mathcal{L}_\psi[\langle \mu+\widehat{\rho}, 2\rho\rangle])$.

In particular, base change lets us rewrite (using Proposition 3.4 again)

$$\mathrm{Hom}_{P_{\mathcal{IW}}(\mathrm{Gr}_G,k)}(\Phi^0(\mathcal{J}_!(\lambda,k)),\nabla^{\mathcal{IW}}_{\mu+\widehat{\rho}}(k)) = \mathrm{Hom}_{D(\mathrm{Gr}_G,k)}(\Phi((j_\lambda)_!\underline{k},),\nabla^{\mathcal{IW}}_{\mu+\widehat{\rho}}(k)).$$

as $H^{\langle\mu+2\widehat{\rho}-\lambda,2\rho\rangle}(\mathrm{Gr}^\lambda \cap t^{-\widehat{\rho}}X_{\mu+\widehat{\rho}},\ldots)$ – that is to say, cohomology of a certain !-pullback. By Verdier duality, this is dual to compactly supported cohomology of a certain $*$-pullback, which after some manipulation, is seen to be exactly the cohomology group which is analyzed in Lemma 4.5. □

4.3. **Conclusion of the proof of Theorem 4.1.** The main point is to show that the functor Φ^0 takes the standard (resp. costandard) objects to the standard (resp. costandard) objects.

By considering the geometry of the convolution morphism, we have that $\Phi^0(\mathcal{J}_!(\lambda,k))$ is (1) supported on $\mathrm{Gr}^{\leq\lambda+\widehat{\rho}}$, and (2) its restriction to $X_{\lambda+\widehat{\rho}}$ is perverse, of rank 1. These result from (1) convolution takes $\mathrm{Gr}^{\leq\lambda}\mathbin{\widetilde{\times}}\mathrm{Gr}^{\leq\widehat{\rho}}$ to $\mathrm{Gr}^{\leq\lambda+\widehat{\rho}}$, and (2) its restriction to the pre-image of the open orbit in the codomain is an isomorphism.

Hence adjunction gives for formal reasons a map

$$f^k_\lambda\colon \Delta^{\mathcal{IW}}_{\lambda+\widehat{\rho}}(k)\to\Phi^0(\mathcal{J}_!(\lambda,k))$$

which is an isomorphism after restriction to $X_{\lambda+\widehat{\rho}}$, which is the open stratum in the support of both sides.

Proposition 4.4 shows that $\Phi^0(\mathcal{J}_!(\lambda,k))$ has no quotients supported on lower dimensional strata. Hence the map f_λ must be surjective. It is also injective in characteristic 0 because in that case $\Delta^{\mathcal{IW}}_{\lambda+\widehat{\rho}}(k)$ is simple by Remark 3.9, so $f^{\mathbf{Q}_\ell}_\lambda$ is an isomorphism and $\ker(f^{\mathbf{Z}_\ell}_\lambda)$ is a torsion object. But it is a sub-object of $\Delta^{\mathcal{IW}}_{\lambda+\widehat{\rho}}(\mathbf{Z}_\ell)$, and this is torsion-free, as the injectivity of multiplication by ℓ on $\mathbf{Z}_\ell$ is preserved by the left exact functor of !-extension (which is already perverse in this case, cf. Definition 3.6). So $f^{\mathbf{Z}_\ell}_\lambda$ is also an isomorphism.

We will use this to show that $f^{\mathbf{F}_\ell}_\lambda$ is also an isomorphism. Taking $-\otimes_{\mathbf{Z}_\ell}\mathbf{F}_\ell$, we deduce that

$$f^{\mathbf{Z}_\ell}_\lambda\otimes_{\mathbf{Z}_\ell}\mathbf{F}_\ell\colon \Delta^{\mathcal{IW}}_{\lambda+\widehat{\rho}}(\mathbf{F}_\ell)\xrightarrow{\sim}\Phi^0(\mathcal{J}_!(\lambda,\mathbf{Z}_\ell))\otimes_{\mathbf{Z}_\ell}\mathbf{F}_\ell$$

is an isomorphism. Now, since Φ is exact for the perverse t-structure, we have $\Phi^0(\mathcal{J}_!(\lambda,\mathbf{Z}_\ell))\otimes_{\mathbf{Z}_\ell}\mathbf{F}_\ell\xleftarrow{\sim}\Phi^0(\mathcal{J}_!(\lambda,\mathbf{Z}_\ell)\overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell}\mathbf{F}_\ell)$. Finally, we recall from [MV, Proposition 8.1] that $\mathcal{J}_?(\lambda,\mathbf{Z}_\ell)\overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell}\mathbf{F}_\ell\cong\mathcal{J}_?(\lambda,\mathbf{F}_\ell)$. (The proof is based on the interpretation of the weight functors as cohomology of MV cycles.) This completes the argument.

By Verdier duality, the dual map

$$g^k_\lambda\colon \Phi^0(\mathcal{J}_*(\lambda,k))\to\nabla^{\mathcal{IW}}_{\lambda+\widehat{\rho}}(k)$$

is also an isomorphism for $k\in\{\mathbf{Q}_\ell,\mathbf{Z}_\ell,\mathbf{F}_\ell\}$. So we have established the desired matching of (co)standard objects.

Now we just have to match Homs. More precisely, by exactness the functor Φ^0 induces

$$D^b(\Phi^0)\colon D^bP_{L^+G}(\mathrm{Gr}_G,k)\to D^bP_{\mathcal{IW}}(\mathrm{Gr}_G,k)$$

and it will suffice to show that $D^b(\Phi^0)$ is an equivalence.

By the usual "upper-triangularity" arguments, $D^bP_{L^+G}(\mathrm{Gr}_G,k)$ and $D^bP_{\mathcal{IW}}(\mathrm{Gr}_G,k)$ are generated as triangulated categories by the objects $\mathcal{J}_!(\lambda,k)$, respectively $\Delta^{\mathcal{IW}}_!(k)$, as well as by $\mathcal{J}_*(\lambda,k)$, respectively $\Delta^{\mathcal{IW}}_*(k)$. Since we have just verified that Φ^0 takes the standards to standards and costandards to costandards, when k is a field the matching of Homs follows from the general properties of highest weight categories, namely that [Ri, Corollary 7.6]

$$\mathrm{Hom}(\Delta_\lambda, \nabla_\mu[n]) = \begin{cases} k & n = 0, \lambda = \mu \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

It remains to analyze $k = \mathbf{Z}_\ell$. The analog of (4.3) for $k = \mathbf{Z}_\ell$ follows immediately from the same argument used to prove Proposition 3.7. So we turn our attention to $\mathrm{RHom}(\mathcal{J}_!(\lambda, \mathbf{Z}_\ell), \mathcal{J}_*(\mu, \mathbf{Z}_\ell)[n])$. Again using $\mathcal{J}_?(\lambda, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \cong \mathcal{J}_?(\lambda, \mathbf{Q}_\ell)$ and $\mathcal{J}_?(\lambda, \mathbf{Z}_\ell) \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \mathbf{F}_\ell \cong \mathcal{J}_?(\lambda, \mathbf{F}_\ell)$ yields that

$$\mathbf{F}_\ell \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}_\ell} \mathrm{RHom}(\mathcal{J}_!(\lambda, \mathbf{Z}_\ell), \mathcal{J}_*(\mu, \mathbf{Z}_\ell)[n]) \cong \mathrm{RHom}(\mathcal{J}_!(\lambda, \mathbf{F}_\ell), \mathcal{J}_*(\mu, \mathbf{F}_\ell)[n]),$$

which by (4.3) for $k = \mathbf{F}_\ell$ implies that

$$\mathrm{RHom}(\mathcal{J}_!(\lambda, \mathbf{Z}_\ell), \mathcal{J}_*(\mu, \mathbf{Z}_\ell)[n]) \cong \begin{cases} \mathbf{Z}_\ell & n = 0, \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

References

[ABBGM] Arkhipov, S.; Braverman, A.; Bezrukavnikov, R.; Gaitsgory, D.; Mirković, I. Modules over the small quantum group and semi-infinite flag manifold. Transform. Groups 10 (2005), no. 3-4, 279–362.

[BGMRR] Bezrukavnikov, Roman; Gaitsgory, Dennis; Mirković, Ivan; Riche, Simon; Rider, Laura. An Iwahori-Whittaker model for the Satake category. J. ÃĽc. polytech. Math. 6 (2019), 707–735.

[Ga01] Gaitsgory, D. Construction of central elements in the affine Hecke algebra via nearby cycles. Invent. Math. 144 (2001), no. 2, 253–280.

[FGV] Frenkel, E.; Gaitsgory, D.; Vilonen, K. Whittaker patterns in the geometry of moduli spaces of bundles on curves. Ann. of Math. (2) 153 (2001), no. 3, 699–748.

[GaW] Gaitsgory, D. The local and global versions of the Whittaker category. ArXiv preprint available at `https://arxiv.org/pdf/1811.02468.pdf`.

[LY] Lusztig, George; Yun, Zhiwei. Endoscopy for Hecke categories, character sheaves and representations. Forum Math. Pi 8 (2020), e12, 93 pp.

[MV] Mirković, I.; Vilonen, K. Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[Ra] Raskin, Sam. $\mathcal{W}$-algebras and Whittaker categories. ArXiv preprint available at `https://arxiv.org/pdf/1611.04937.pdf`.

[Ri] Riche, Simone. Geometric representation theory in positive characteristic. Available at `https://tel.archives-ouvertes.fr/tel-01431526/document`.

[Yun] Yun, Zhiwei. Rigidity in the Langlands correspondence and applications. Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. I, 199–234, Adv. Lect. Math. (ALM), 36, Int. Press, Somerville, MA, 2017.