Derived Geometric Satake Equivalence

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1 Notations

In this section, we introduce the notations which will be used in this note following [1].

Assume k to be the algebraically closed characteristic zero coefficient field. Let G be a semisimple complex algebraic group, and $G_{\mathcal{O}} = G(\mathbb{C}[[t]]), G_F = G(\mathbb{C}((t)))$ be the positive jet group and loop group, respectively. The affine Grassmannian $Gr_G = G_F/G_{\mathcal{O}}$ carries the category $\mathrm{P}_{G_{\mathcal{O}}}(Gr_G)$ (resp. $\mathrm{P}_{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(Gr_G)$) of $G_{\mathcal{O}}$ (resp. $G_{\mathcal{O}} \rtimes \mathbb{G}_m$)-equivariant perverse constructible sheaves. Write $G^\vee$ for the Langlands dual group of G over k, and $\mathrm{Rep}(G^\vee)$ the category of finite dimensional representations of $G^\vee$ over k. We have

$$\tilde{\mathcal{S}} : \mathrm{Rep}(G^\vee) \longrightarrow \mathrm{P}_{G_{\mathcal{O}}}(Gr_G),$$
$$\mathcal{S} : \mathrm{Rep}(G^\vee) \longrightarrow \mathrm{P}_{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(Gr_G)$$

the geometric Satake equivalence and its extension to the category $\mathrm{P}_{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(Gr_G)$ (cf.[2]).

Denote by $\mathfrak{g}^\vee$ the Lie algebra of $G^\vee$, and choose a Cartan subalgebra $\mathfrak{t}^\vee \subset \mathfrak{g}^\vee$. Write the torus which corresponds to $\mathfrak{t}^\vee$ as $T^\vee \subset G^\vee$. Choose a Borel and its opposite Borel subalgebra $\mathfrak{b}^\vee_\pm$ containing $\mathfrak{t}^\vee$ and write their nilpotent radicals as $\mathfrak{n}^\vee_\pm$. Let $N^\vee_\pm \subset G^\vee$ be the corresponding unipotent subgroups. Denote the weyl group of $G^\vee$ by W. We write $\mathrm{T}(\mathfrak{t}^{\vee *}/W)$ for the total space of the tangent bundle of $\mathfrak{t}^{\vee *}/W$.

Let $\mathbb{X}_\bullet(T)$ and $\mathbb{X}^+_\bullet(T)$ denote the group of characters and semigroup of dominant characters of $T^\vee$. Let $V \in \mathrm{Rep}(G^\vee)$, denote by V_λ the λ-weight space of V. Let ρ denotes the half of the sum of the positive roots of $\mathfrak{g}^\vee$. For any $\lambda \in \mathbb{X}_\bullet(T)$, denote by t^λ the corresponding fixed point in Gr_G, and T_λ the semi-infinite orbit through t^λ. For any $\lambda \in \mathbb{X}^+_\bullet(T)$, denote by $Gr_{\leqslant\lambda}$ the corresponding Schur variety.

We recall the following notations in [1, §2.2]. Let U denote the universal enveloping algebra of $\mathfrak{g}^\vee$. Denote by $U_\hbar$ the graded universal enveloping algebra which arises from U by taking the Rees construction. Let $\widetilde{\mathcal{HC}}_\hbar$ denote the category of asymtotic $\mathfrak{g}^\vee$-modules, and $\mathcal{HC}_\hbar \subset \widetilde{\mathcal{HC}}_\hbar$ be the full subcategory of objects which are finitely generated as $U_\hbar \otimes 1$ modules (equivalently, as $1 \otimes U_\hbar$ modules).

We recall the following notations in [1, §2.3, §2.6]. Write $\kappa_\hbar : \mathcal{HC}_\hbar \to \mathrm{QCoh}^{\mathbb{G}_m}((\mathfrak{t}^{\vee *}/W)^2 \times \mathbb{A}^1)$ and $\kappa : \mathrm{Coh}^{G^\vee \times \mathbb{G}_m}(\mathfrak{g}^{\vee *}) \longrightarrow \mathrm{Coh}^{\mathbb{G}_m}(\mathrm{T}(\mathfrak{t}^{\vee *}/W))$ for the Kostant functors. For any scheme X and its closed subscheme $Z \subset X$, denote by $N_X Z$ the deformation to the normal cone. Define $\mathcal{HC}^{\mathrm{fr}} \subset \mathcal{HC}_\hbar$ to be the full subcategory consisting of free objects.

2 Overview

Recall that in the previous talk, we gave a quick review of the geometric Satake equivalence which states that there exists an equivalence of symmetric monoidal categories

$$\tilde{\mathcal{S}} : \mathrm{Rep}(G^\vee) \longrightarrow \mathrm{P}_{G_{\mathcal{O}}}(Gr_G).$$

In fact, before the above equivalence was established, Drinfeld proposed the following question:

Can we describe the derived category $\mathrm{D}_{L^+G}(Gr_G)$?

Here are some motivations of Drinfeld's question:

- in Langlands program, we have an important family of commuting operators, *Hecke operators*. They act on the space of automorphic forms and their eigenvalues and eigenforms give crucial arithmetic information.
- in geometric Langlands program, Hecke operators (at a point) corresponds to the equivariant derived category $\mathrm{D}_{G_\mathcal{O}}(Gr_G)$.

It is then natural to ask for a description of the category $\mathrm{D}_{G_\mathcal{O}}(Gr_G)$. The first step towards answer Drinfeld's question is as follows. We can define a natural t-structure on $\mathrm{D}_{G_\mathcal{O}}(Gr_G)$, and then describe its heart. This is exactly the geometric Satake equivalence which we recall at the beginning of this section.

The goal of this notes is to give a quick introduction to the main theorem of [1]

Theorem 2.1. *There exists a canonical equivalence of monoidal triangulated categories*

(1) $\mathrm{D}^{G^\vee}_{\mathrm{perf}}(U^\square_\hbar) \simeq \mathrm{D}_{G_\mathcal{O}\rtimes\mathbb{G}_m}(Gr_G)$.

(2) $\mathrm{D}^{G^\vee}_{\mathrm{perf}}(\mathrm{Sym}^\square\mathfrak{g}^\vee) \simeq \mathrm{D}_{G_\mathcal{O}}(Gr_G)$.

where,

- $\square$ *denotes that we consider the algebra which it decorates as a* ***dg-algebra*** *with trivial differentials,*
- *the cohomological grading of* $U^\square_\hbar$ *and* $\mathrm{Sym}^\square\mathfrak{g}^\vee$ *is defined so that elements* $\mathfrak{g}^\vee$ *and* $\hbar$ *have degree two, and*
- perf *denotes* ***perfect complexes*** *i.e. those which are generated by free objects by taking direct sums and cones.*

3 Topological Filtrations

By our knowledge of equivariant cohomology, for any $\mathcal{F} \in \mathrm{D}_{G_\mathcal{O}\rtimes\mathbb{G}_m}(Gr_G)$ we have the following isomorphism

$$\mathrm{H}^*_{T\times\mathbb{G}_m}(\mathcal{F}) \simeq \mathrm{H}^*_{G_\mathcal{O}\rtimes\mathbb{G}_m}(\mathcal{F}) \otimes_{\mathcal{O}(\mathfrak{t}/W)} \mathcal{O}(\mathfrak{t}).$$

In this section, we will endow the equivariant cohomology $\mathrm{H}^*_{T\times\mathbb{G}_m}(\mathcal{F})$ with a filtration which

- comes from the equivariant MV-filtration (cf.[2])
- has a canonical splitting with associated graded equals $\oplus_{\lambda\in\mathbb{X}_\bullet}\mathrm{H}^*_{T\times\mathbb{G}_m}(t^\lambda)\otimes\mathrm{H}^*_{T_\lambda}(Gr_G,\mathcal{F})$.

Recall notations in §1. For any $\lambda \in \mathbb{X}_\bullet(T)$, denote by $\bar{T}_\lambda$ the closure of T_λ. Filter $\mathrm{H}^*_{T\times\mathbb{G}_m}(\mathcal{F})$ by the image of the natural map $r_\lambda : \mathrm{H}^*_{\bar{T}_\lambda, T\times\mathbb{G}_m}(\mathcal{F}) \to \mathrm{H}^*_{T\times\mathbb{G}_m}(\mathcal{F})$. It follows from the property of the MV-weight functor (cf.[2]) that the associated graded of this filtration equals $\oplus_{\lambda\in\mathbb{X}_\bullet}\mathrm{H}^*_{T\times\mathbb{G}_m}(t^\lambda)\otimes\mathrm{H}^*_{T_\lambda}(Gr_G,\mathcal{F})$.

Lemma 3.1. *Let* $\pi : \mathfrak{t} \to \mathfrak{t}/W$ *be the natural projection and* $\Gamma_\lambda := \{(x_1, x_2, a) \in \mathfrak{t}\times\mathfrak{t}\times\mathbb{A}^1 | x_2 = x_1 + \lambda a\}$. *For any* $V \in \mathrm{Rep}(G^\vee)$, *the associated graded of the filtration we discuss in the above is isomorphic to* $\oplus_{\lambda\in\mathbb{X}_\bullet}(\mathrm{id},\pi,\mathrm{id})_*\mathcal{O}(\Gamma_\lambda)\otimes V_\lambda$.

The filtration varies compatibly when we restrict our G to its Levi subgroup. Let $T \subset L \subset G$ be a Levi subgroup. Denote by P_L (resp. P_L^-) the parabolic subgroup generated by L and the positive (resp. negative) Borel subgroup B (resp. B). Write $W_L \subset W$ for the Weyl group of L. Let $\pi_L : \mathfrak{t}/W_L \to \mathfrak{t}/W$ be the natural projection. We denote by X_L^+ the set of highest weights of irreducible $L^\vee$-modules, where $L^\vee \subset G^\vee$ stands for the Langlands dual Levi subgroup. For any $L^\vee$-module V we denote by $\mathcal{S}_L(V)$ the corresponding $L(\mathcal{O}) \rtimes \mathbb{G}_m$-equivariant perverse sheaf on Gr_L.

Lemma 3.2. *For any $V \in \mathrm{Rep}(G^\vee)$, the $\mathcal{O}(\mathfrak{t}/W_L \times \mathfrak{t}/W \times \mathbb{A}^1)$-module $(\pi_L, \mathrm{id}, \mathrm{id})\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathcal{S}(V))$ admits a canonical filtration such that the associated graded is equipped with a canonical isomorphism*

$$^{\mathrm{top}}\Xi_L : \mathrm{gr}(\pi_L, \mathrm{id}, \mathrm{id})^*\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V)) \simeq (\mathrm{id}, \pi_L, \mathrm{id})_*\mathrm{H}^*_{L(\mathcal{O}) \rtimes \mathbb{G}_m}(Gr_L, \mathcal{S}(V|_{L^\vee})).$$

We recall the monoidal structure in $\mathrm{D}_{G_\mathcal{O}}(Gr_G)$. Consider the following diagram

$$\begin{array}{ccc}
(G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m)^2 & \xleftarrow{\ p\ } & G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m \times G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m \\
 & & \downarrow q \\
 & & G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m \times^{G_\mathcal{O} \rtimes \mathbb{G}_m} G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m \\
 & & \downarrow m \\
 & & G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m,
\end{array}$$

where p and q are natural morphisms and m is the convolution morphism. For any $\mathcal{F}, \mathcal{G} \in \mathrm{D}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G)$, there exists a unique $\mathcal{F} \tilde{\boxtimes} \mathcal{G} \in \mathrm{D}_{(G_\mathcal{O} \rtimes \mathbb{G}_m)^2}(G_F \rtimes \mathbb{G}_m \times^{G_\mathcal{O} \rtimes \mathbb{G}_m} G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m)$ such that $q^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \simeq p^*(\mathcal{F} \boxtimes \mathcal{G})$. Then the **convolution product** of $\mathcal{F}$ and $\mathcal{G}$ is defined as $\mathcal{F} * \mathcal{G} := Rm_!(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$.

The monoidal structure of the equivariant cohomology functor $\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\bullet)$ can be seen as follows

$$\begin{aligned}
\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{F} * \mathcal{G}) &\simeq \mathrm{H}^*(G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m, \mathcal{F} * \mathcal{G}) \\
&\simeq \mathrm{H}^*(G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m \times^{G_\mathcal{O} \rtimes \mathbb{G}_m} G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m, \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \\
&\simeq \mathrm{H}^*(G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m, \mathcal{F}) \otimes_{\mathrm{H}^*(G_\mathcal{O} \rtimes \mathbb{G}_m \backslash \mathrm{pt})} \mathrm{H}^*(G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_F \rtimes \mathbb{G}_m / G_\mathcal{O} \rtimes \mathbb{G}_m, \mathcal{G}) \\
&\simeq \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{F}) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{G}) \\
&=: \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{F}) * \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{G}).
\end{aligned}$$

In the rest of this section, we sketch the proof of the fact that the filtration is compatible with the monoidal structure of $\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\bullet)$.

For any $V_1, V_2 \in \mathrm{Rep}(G^\vee)$, we have a canonical isomorphism

$$^{\mathrm{top}}\omega_{V_1, V_2} : \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathcal{S}(V_1 \otimes V_2)) \simeq \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathcal{S}(V_1)) * \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathcal{S}(V_2)).$$

Then Lemma 3.2 implies that

$$^{\mathrm{top}}\Xi_V : \mathrm{gr}(\pi, \mathrm{id}, \mathrm{id})^*\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V)) \simeq (\mathrm{id}, \pi, \mathrm{id})_*\mathrm{H}^*_{T \times \mathbb{G}_m}(Gr_T, \mathcal{S}(V|_{T^\vee})) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} k(\mathfrak{t} \times \mathbb{A}^1).$$

For the rest of this notes, we follow the conventions in [1] and denote $(\mathrm{id}, \pi, \mathrm{id})_*\mathrm{H}^*_{T \times \mathbb{G}_m}(Gr_T, \mathcal{S}(V|_{T^\vee}))$ by $\mathrm{H}^*_{T \times \mathbb{G}_m}(Gr_T, \mathcal{S}(V|_{T^\vee}))$ for simplicity. Tensoring the above isomorphism with $k(\mathfrak{t} \times \mathbb{A}^1)$ over $\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)$, we get by Lemma 3.1 an isomorphism

$$\begin{aligned}
^{\mathrm{top}}\Xi_V^{\mathrm{gen}} :&\mathrm{gr}(\pi, \mathrm{id}, \mathrm{id})^*\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V)) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} k(\mathfrak{t} \times \mathbb{A}^1) \\
&\simeq \oplus_{\lambda \in \mathbb{X}_\bullet}(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} k(\mathfrak{t} \times \mathbb{A}^1)) \otimes V_\lambda
\end{aligned}$$

Proposition 3.3. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{H}^*_{G_{\mathcal{O}}\rtimes\mathbb{G}_m}(Gr_G, \mathcal{S}(V_1\otimes V_2))\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} k(\mathfrak{t}\times\mathbb{A}^1) & \xrightarrow{{}^{\mathrm{top}}\omega_{V_1,V_2}} & \mathrm{H}^*_{G_{\mathcal{O}}\rtimes\mathbb{G}_m}(\mathcal{S}(V_1)) * \mathrm{H}^*_{G_{\mathcal{O}}\rtimes\mathbb{G}_m}(\mathcal{S}(V_2))\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} k(\mathfrak{t}\times\mathbb{A}^1) \\ \downarrow {}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_1\otimes V_2} & & \downarrow {}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_1} * {}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_2} \\ \oplus_{\lambda\in\mathbb{X}_\bullet}(\mathcal{O}(\Gamma_\lambda)\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} k(\mathfrak{t}\times\mathbb{A}^1))\otimes(V_1\otimes V_2)_\lambda & \xrightarrow{\simeq} & (\mathcal{V}_1 * \mathcal{V}_2)\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} k(\mathfrak{t}\times\mathbb{A}^1), \end{array}$$

where

$$\mathcal{V}_1 := \oplus_{\lambda\in\mathbb{X}_\bullet}(\mathcal{O}(\Gamma_\lambda)\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} k(\mathfrak{t}\times\mathbb{A}^1))\otimes(V_1)_\lambda$$
$$\mathcal{V}_2 : \oplus_{\lambda'\in\mathbb{X}_\bullet}(\mathcal{O}(\Gamma'_\lambda)\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} k(\mathfrak{t}\times\mathbb{A}^1))\otimes(V_2)_{\lambda'}.$$

4 Algebraic Filtration

The first goal of this section is to define a functor $\chi_\hbar$.

Definition 4.1. The **universal Verma module** is defined to be

$$\mathcal{M}_\hbar(-\rho) := U_\hbar\otimes_{U_\hbar(\mathfrak{b}^\vee)} k[\hbar][\mathfrak{t}^\vee](-\rho),$$

where the action of $U_\hbar(\mathfrak{b}^\vee)$ on $k[\hbar][\mathfrak{t}^\vee](-\rho)$ factors through that of $k[\hbar][\mathfrak{t}^\vee]$ and the later action is defined as $t\cdot x := (t-\hbar\rho(t))x$ for any $t\in\mathfrak{t}$ and $x\in k[\hbar][\mathfrak{t}^\vee](-\rho)$.

Definition 4.2. For any $M\in\widetilde{\mathcal{HC}}_\hbar$, we define

$$\chi_\hbar(M) := \mathcal{M}_\hbar(-\rho)\otimes^L_{U_\hbar} M\otimes^L_{U_\hbar(\mathfrak{n}^\vee)_2}\psi,$$

where $\psi : U_\hbar(\mathfrak{n}^\vee_-)\to k[\hbar]$ is a homomorphism such that $\psi(f_\alpha)=1$ for all simple root α and root generator $f_\alpha\in\mathfrak{n}^\vee_-\subset U_\hbar(\mathfrak{n}^\vee_-)$. Define $U_\hbar(\mathfrak{n}^\vee_-)^2 := U_\hbar(\mathfrak{n}^\vee_-)\otimes U(\mathfrak{n}^\vee_-)$. Extend ψ to a character $\psi_{(2)} : U_\hbar(\mathfrak{n}^\vee_-)^2\to k[\hbar]$ which is trivial on the second multiple of $U_\hbar(\mathfrak{n}^\vee_-)$. The restriction of $\psi_{(2)}$ to the second multiple equals $-\psi$ and this restriction is used in Definition 4.2.

Note that $\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)=U_\hbar$ acts on $\chi_\hbar(M)$ since $U_\hbar$ normalizes $U_\hbar(\mathfrak{b}^\vee)$, and $\mathcal{O}(\mathfrak{t}^{\vee *}/W\times\mathbb{A}^1)=Z_\hbar$ embeds into the second copy of $U_\hbar$ in $U_\hbar\otimes U_\hbar$. Thus $\chi_\hbar(M)$ is an $\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathfrak{t}^{\vee *}/W\times\mathbb{A}^1)$-module.

Definition 4.3. For any $V\in\mathrm{Rep}(G^\vee)$, define $\varphi(V) := \chi_\hbar(U_\hbar\otimes V)$, and $\phi(V) := \kappa_\hbar(U_\hbar\otimes V)$.

The functors defined in the above are closely related.

Lemma 4.4. *For any $V\in\mathrm{Rep}(G^\vee)$, $\varphi(V)=(\pi,\mathrm{id},\mathrm{id})^*\phi(V)$.*

The above lemma allows us to induce a filtration on $\phi(V)$ from that of $\varphi(V)$. Note that $\mathcal{M}_\hbar(-\rho)\otimes V$ has a canonical filtration with associated graded $\oplus_{\lambda\in\mathbb{X}_\bullet}\mathcal{M}_\hbar(\lambda-\rho)\otimes V_\lambda$, where

$$\mathcal{M}_\hbar(\lambda-\rho) := U_\hbar\otimes_{U_\hbar(\mathfrak{b}^\vee)} k[\hbar][\mathfrak{t}^\vee](\lambda-\rho).$$

In the above definition, we view $k[\hbar][\mathfrak{t}^\vee](\lambda-\rho)$ as a $U_\hbar(\mathfrak{b}^\vee)$-module which factors through the $U_\hbar(\mathfrak{t}^\vee)=k[\hbar][\mathfrak{t}]$-module where for any $t\in\mathfrak{t}^\vee$, t acts by multiplication by $t+\hbar\lambda(t)-\hbar\rho(t)$.

Lemma 4.5. *For any $V\in\mathrm{Rep}(G^\vee)$, the $\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathfrak{t}^{\vee *}/W\times\mathbb{A}^1)$-module $\varphi(V)$ has a natural filtration induced from the filtration of $\mathcal{M}_\hbar(-\rho)\otimes V$, which we discuss in the above, with associated graded*

$$\oplus_{\lambda\in\mathbb{X}_\bullet}(\mathrm{id},\pi,\mathrm{id})_*(\mathcal{O}(\Gamma_\lambda)\otimes V_\lambda).$$

Like the case for $\mathrm{H}^*_{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(\bullet)$, the filtration on $\varphi(V)$ also varies compatibly when restricting to Levi subgroups. Let $T^\vee \subset L^\vee \subset G^\vee$ be a Levi subgroup with the Lie algebra $\mathfrak{t}^\vee \subset \mathfrak{l}^\vee \subset \mathfrak{g}^\vee$. We denote by $\mathfrak{p}_L^\vee$ (resp. $\mathfrak{p}_L^-$) the parabolic subalgebra generated by $\mathfrak{l}^\vee$ and the positive (resp. negative) Borel subalgebra $\mathfrak{b}^\vee$ (resp. $\mathfrak{b}_-^\vee$). We denote by $\pi_L : \mathfrak{t}^{\vee *}/W_L \to \mathfrak{t}^{\vee *}/W$ the natural projection.

Lemma 4.6. *For any $V \in \mathrm{Rep}(G^\vee)$, the $\mathcal{O}(t^{\vee *}/W_L \times \mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)$-module $(\pi_L, \mathrm{id}, \mathrm{id})^* \phi(V)$ carries a canonical filtration such that its associated graded is equipped with a canonical isomorphism*

$$^{\mathrm{alg}}\Xi_L : (\pi_L, \mathrm{id}, \mathrm{id})^* \phi(V) \simeq (\mathrm{id}, \pi_L, \mathrm{id})_* \phi_L(V|_{L^\vee}).$$

In particular, the above lemma implies that there is a canonical isomorphism

$$^{\mathrm{alg}}\Xi_V : (\pi, \mathrm{id}, \mathrm{id})^* \phi(V) \simeq (\mathrm{id}, \pi, \mathrm{id})_* \phi_T(V|_{T^\vee}).$$

As in [1], we will write $\phi_T(V|_{T^\vee})$ for the RHS of $^{\mathrm{alg}}\Xi_V$ for the rest of this notes. It follows that we have a canonical isomorphism

$$\begin{aligned} ^{\mathrm{alg}}\Xi_V^{\mathrm{gen}} : &\phi(V) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} k(\mathfrak{t}^{\vee *} \times \mathbb{A}^1) \\ = &\mathrm{gr}(\pi, \mathrm{id}, \mathrm{id})^* \phi(V) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} k(\mathfrak{t}^{\vee *} \times \mathbb{A}^1) \\ \simeq &\phi_T(V|_{T^\vee}) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} k(\mathfrak{t}^{\vee *} \times \mathbb{A}^1) \\ \simeq &\oplus_{\lambda \in \mathbb{X}_\bullet} (\mathrm{id}, \pi, \mathrm{id})_* (\mathcal{O}(\Gamma_\lambda) \otimes V_\lambda). \end{aligned}$$

We move on to discuss the monoidal structure of the Kostant functor.

In the proof of Lemma 4.4, the following isomorphism is established

$$\phi(V) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} \mathcal{W}_\hbar \simeq V \otimes \mathcal{W}_\hbar,$$

where $\mathcal{W}_\hbar := U_\hbar \otimes_{U_\hbar(\mathfrak{n}_-^\vee)} \psi$ denotes the Whittaker module (cf. [1, Lemma 5]). Then it follows that

$$\phi(V_1 \otimes V_2) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} \mathcal{W}_\hbar \simeq \phi(V_1) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} \phi(V_2) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} \mathcal{W}_\hbar.$$

By Kostant's Theorem (cf.[1, Lemma 5]), we get an isomorphism

$$^{\mathrm{alg}}\omega_{V_1, V_2} : \phi(V_1 \otimes V_2) \simeq \phi(V_1) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}/W \times \mathbb{A}^1)} \phi(V_2) =: \phi(V_1) * \phi(V_2).$$

Proposition 4.7. *Keeping notations in Proposition* 3.3*, the following diagram commutes*

$$\begin{array}{ccc} \phi(V_1 \otimes V_2) & \xrightarrow{\;\;^{\mathrm{alg}}\omega_{V_1, V_2}\;\;} & \phi(V_1) * \phi(V_2) \\ \Big\downarrow {\scriptstyle ^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_1 \otimes V_2}} & & \Big\downarrow {\scriptstyle ^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_1} * {}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_2}} \\ \oplus_{\lambda \in \mathbb{X}_\bullet} (\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} k(\mathfrak{t} \times \mathbb{A}^1)) \otimes (V_1 \otimes V_2)_\lambda & \xrightarrow{\;\simeq\;} & (\mathcal{V}_1 * \mathcal{V}_2) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} k(\mathfrak{t} \times \mathbb{A}^1). \end{array}$$

5 Proof of the Main Theorem

In this section, we outline the proof of the main theorem.

Step 1: ”topology=algebra”.

Theorem 5.1. *(a) for any $V \in \mathrm{Rep}(G^\vee)$, there exists a unique $\mathcal{O}(\mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1)$-module isomorphism*

$$\eta_V : \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V)) \simeq \phi(V)$$

such that

$$\mathrm{gr}(\pi, \mathrm{id}, \mathrm{id})^* \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V)) \simeq \oplus_{\lambda \in \mathbb{X}_\bullet} (\mathrm{id}, \pi, \mathrm{id})_* (\mathcal{O}(\Gamma_\lambda) \otimes V_\lambda) \simeq \mathrm{gr}(\pi, \mathrm{id}, \mathrm{id})^* \phi(V).$$

(b) for any $V_1, V_2 \in \mathrm{Rep}(G^\vee)$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V_1)) * \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V_2)) & \xrightarrow{{}^{\mathrm{top}}\omega^{-1}_{V_1, V_2}} & \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G, \mathcal{S}(V_1 \otimes V_2)) \\ \downarrow{\scriptstyle \eta_{V_1} * \eta_{V_2}} & & \downarrow{\scriptstyle \eta_{V_1 \otimes V_2}} \\ \phi(V_1) * \phi(V_2) & \xrightarrow{{}^{\mathrm{alg}}\omega_{V_1, V_2}} & \phi(V_1 \otimes V_2). \end{array}$$

Step 2: Equivariant cohomology functor V.S. the Kostant functor

Let $\mathcal{IC}$ (resp. $\widetilde{\mathcal{IC}}$) denote the full subcategory of semi-simple complexes in $\mathrm{D}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G)$ (resp. $\mathrm{D}_{G_\mathcal{O}}(Gr_G)$). The following lemma will play an important role in the proof of the main theorem.

Lemma 5.2. *(a) The functor*

$$\mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m} : \mathcal{IC} \to \mathrm{Coh}^{\mathbb{G}_m}(N_{(\mathfrak{t}^{\vee *}/W)^2} \Delta)$$

is a full embedding.

(b) The functor

$$\mathrm{H}^*_{G_\mathcal{O}} : \widetilde{\mathcal{IC}} \longrightarrow \mathrm{Coh}^{\mathbb{G}_m}(\mathrm{T}(\mathfrak{t}^{\vee *}/W))$$

is a full embedding.

(c) The functors

$$\kappa_\hbar : \mathcal{HC}_\hbar \to \mathrm{QCoh}^{\mathbb{G}_m}((\mathfrak{t}^{\vee *}/W)^2 \times \mathbb{A}^1)$$
$$\kappa : \mathrm{Coh}^{G^\vee \times \mathbb{G}_m}(\mathfrak{g}^{\vee *}) \longrightarrow \mathrm{Coh}^{\mathbb{G}_m}(\mathrm{T}(\mathfrak{t}^{\vee *}/W))$$

are full embeddings.

(d) The functors $\kappa|_{\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}_{\mathrm{fr}}(\mathfrak{g}^{\vee *})}$ and $\kappa_\hbar|_{\mathcal{HC}^{\mathrm{fr}}_\hbar}$ are exact.

Theorem 5.1 and Lemma 5.2 yield

Theorem 5.3. *(a) The functor $\mathcal{S} : \mathrm{Rep}(G^\vee) \to \mathrm{P}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G)$ extends to a unique full embedding $\mathcal{S}_\hbar : \mathcal{HC}^{\mathrm{fr}}_\hbar \to \mathrm{D}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G)$ such that $\kappa_\hbar \simeq \mathrm{H}^*_{G_\mathcal{O} \rtimes \mathbb{G}_m} \circ \mathcal{S}_\hbar$.*

(b) The functor $\tilde{\mathcal{S}} : \mathrm{Rep}(G^\vee) \longrightarrow \mathrm{P}_{G_\mathcal{O}}(Gr_G)$ extends to a unique full embedding $\tilde{\mathcal{S}}_{\mathrm{qc}} : \mathrm{Coh}^{G^\vee \times \mathbb{G}_m}_{\mathrm{fr}}(\mathfrak{g}^{\vee *}) \to \mathrm{D}_{G_\mathcal{O}}(Gr_G)$ such that $\kappa \simeq \mathrm{H}^*_{G_\mathcal{O}} \circ \tilde{\mathcal{S}}_{\mathrm{qc}}$.

Step 3: Construction of Φ.

We hope to construct a functor

$$\Phi : \mathrm{D}^{G^\vee}_{\mathrm{perf}}(U^{\square}_\hbar) \simeq \mathrm{D}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr)$$

such that

$$\Phi|_{\mathcal{HC}_\hbar^{\mathrm{fr}}} : \mathcal{HC}_\hbar^{\mathrm{fr}} \to \mathrm{D}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_G)$$

satisfies $\Phi|_{\mathcal{HC}_\hbar^{\mathrm{fr}}} \simeq \mathcal{S}_\hbar$, where the embedding $\mathcal{HC}_\hbar^{\mathrm{fr}} \subset \mathrm{D}_{\mathrm{perf}}^{G^\vee}(U_\hbar^\square)$ sends a $G^\vee$-equivariant graded $U_\hbar$-module to the same module considered as a dg-module with zero differential. Theorem 5.3 implies that $\Phi|_{\mathcal{HC}_\hbar^{\mathrm{fr}}}$ induces an equivalence between $\mathcal{HC}_\hbar^{\mathrm{fr}}$ and $\mathcal{IC}$. If such a functor is constructed, then Theorem 5.3.(a) implies that Φ sends generators to generators and induces isomorphisms between Hom spaces of generators in all degrees. We conclude by Beilinson's Lemma that Φ is an equivalence.

Step 4: Construction of subfunctors $\{\Phi_{\leqslant\lambda}\}_{\lambda\in\mathbb{X}_\bullet^+(T)}$.

Let $\mathrm{D}_{\mathrm{perf}}^{G^\vee}(U_\hbar^\square)_{\leqslant\lambda} \subset \mathrm{D}_{\mathrm{perf}}^{G^\vee}(U_\hbar^\square)$ be the full subcategory generated by objects of the form $U_\hbar \otimes V$, where $V \in \mathrm{Rep}(G^\vee)$ is an irreducible representation of highest weight $\leqslant \lambda$.

We hope to construct a family of functors $\{\Phi_{\leqslant\lambda}\}_{\lambda\in\mathbb{X}_\bullet^+(T)}$,

$$\Phi_{\leqslant\lambda} : \mathrm{D}_{\mathrm{perf}}^{G^\vee}(U_\hbar^\square)_{\leqslant\lambda} \to \mathrm{D}_{G_\mathcal{O} \rtimes \mathbb{G}_m}(Gr_{\leqslant\lambda}).$$

such that for any $\mu \leqslant \nu$, $\Phi_\mu \simeq \Phi_\nu|_{\mathrm{D}_{\mathrm{perf}}^{G^\vee}(U_\hbar^\square)_{\leqslant\mu}}$ and satisfies obvious compatible relation for any $\lambda \leqslant \mu \leqslant \nu$.

Such functors exist by the formality of $\mathrm{End}^\bullet(V_{\leqslant\lambda} \otimes U_\hbar^\square)$ and purity of IC sheaves.

References

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