# Derived equivariant cohomology of the affine Grassmannian and Bezrukavnikov and Finkelberg's equivalences

or, recovering triangulated cats from additive ones

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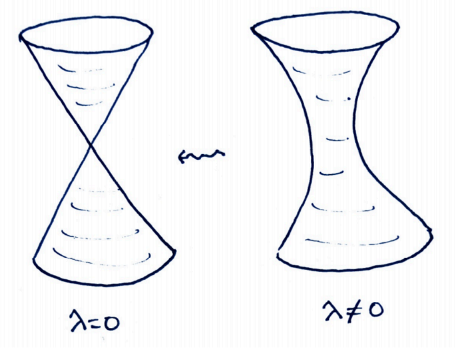
IAS

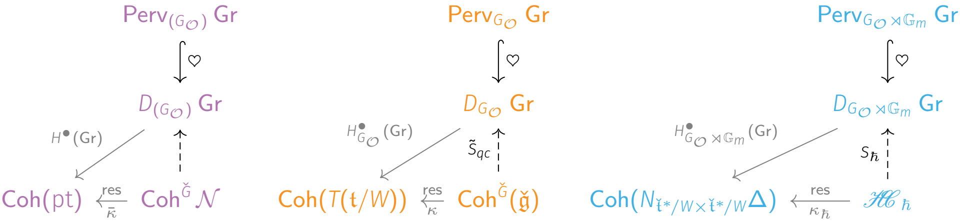
# Motto

It admits all deformation is justified by the search for the invariant.

—Ozenfant and Le Corbusier



# Picking up where Linyuan left off




$\{0\}$ $\qquad$ $\check{\mathfrak{t}}^*/W$ $\qquad$ $\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1$

Here $D_H$ denotes derived category of $H$-equivariant sheaves, and $D_{(H)}$ denotes derived category of constructible sheaves for $H$-orbits.

Left to middle is hard; Linyuan gave us the idea: gluing Ginzburg's fibres, upon identifying $H^\bullet(Gr)$ and $\mathcal{U}(\mathfrak{g}^e)$; considering graded modules over $\mathcal{U}(\mathfrak{g}^e)$ as fibres of $G$-equivariant coherent sheaves on $\mathfrak{g}$ restricted to the principal nilpotent $e$. (In the figure above we are suppressing the $\mathbb{G}_m$ gradings on Coh's of fibres coming from the Cartan $h$.) Right to middle is easy (assuming some hard facts).

# Goal of derived Satake

'In the geometric Langlands program, one has triangulated categories, dg-categories and Hecke operators and Hecke operators at a point aka $D_{G_O}$ Gr. A fundamental question (first asked by Drinfeld) is:'

*What is the monoidal category of Hecke operators? i.e. Describe $D_{G_\mathcal{O}} \mathrm{Gr}_G$...in terms of the dual group.*

From Ginzburg BF deduce that the former can be linked to conjugation equivariant coherent sheaves on the Langlands dual Lie algebra.

Unlike Ginzburg, they rely on "asymptotics" coming from added $\mathbb{G}_m$-equivariance. $D_{G_\mathcal{O} \rtimes \mathbb{G}_m} \mathrm{Gr}_G$ is related to a quantization of $\mathrm{Coh}^{\check{G}}\, \check{\mathfrak{g}}^*$ (to ad-integrable $G$-equivariant $\mathcal{U}\mathfrak{g}$-bimodules $=:$ Harish-Chandra bimodules).

They show that the the full subcategory of semisimple complexes in $D_{G_\mathcal{O} \rtimes \mathbb{G}_m} \mathrm{Gr}_G$ embeds into the category of certain coherent sheaves on a deformation to the normal cone of $\check{\mathfrak{t}}^*/W$.

# Triangulated cats from additive ones

Finally, they express equivariant cohomology of arbitrary (not necessarily semisimple) equivariant complexes in terms of Harish-Chandra bimodules using the fact that there is an equivalence between deformed $\mathsf{Coh}^G \mathfrak{g}$ and semisimple complexes in $D_{G_\mathcal{O} \rtimes \mathbb{G}_m} \mathsf{Gr}_G$ which generate the latter as a triangulated category.

**Definition**

Suppose $B$ is a $\mathbb{Z}$ filtered ring with $\cap B_i = 0$, $\cup B_i = B$, $1 \in B_0$ and $B_i B_j \subset B_{i+j}$. Then

$$\hat{B} := \bigoplus \hbar^i B_i := \left\{ \sum{}' \hbar^i b_i \middle| b_i \in B_i \right\} \subset B[\hbar, \hbar^{-1}]$$

is called the Rees ring of $B$.

# Important properties

- $\hbar$ is central and not a zero divisor
- $\bigcup \hbar^{-i}\hat{B} = B[\hbar, \hbar^{-1}] = \hat{B}[\hbar^{-1}]$
- $\hat{B}\big|_{\hbar=0} := \hat{B}/\hbar\hat{B} = \operatorname{gr} B$ since $\sum \hbar^i b_i \bmod \hbar\hat{B} = \sum \hbar^i (b_i \bmod B_{i-1})$
- no ring map $B \to \hat{B}$
- is ring map $B_0 \to \hat{B}$
- is ring map $k[\hbar] \to \hat{B}$ at least for the filtrations which we'll consider

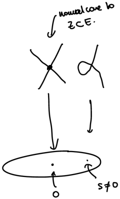
# Geometric interpretation

$Z = V(I) \subset X$; $B = \mathscr{O}_X$ is filtered by degree of vanishing along $Z$ i.e.

$$\cdots \subset I^3 \subset I^2 \subset I \subset \mathscr{O} \subset \mathscr{O} \subset \cdots$$

Then $N_X Z := \mathrm{Spec}\, \hat{\mathscr{O}}_X$ is what BF call the deformation to the normal cone of $Z$ in $X$.

In particular $N_X Z$ is fibred over $\mathbb{A}^1$ with $\hbar = 0$ fibre equal to the normal cone, and any $\hbar \neq 0$ fibre equal to $X$.




If $Z$ is smooth, then $\mathrm{gr}\, \mathscr{O}_X = \mathrm{Sym}^\bullet\, \Omega^1_X Z = T_X Z$ and the normal cone is the normal bundle.

# Deformation to the diagonal

If $Z = \Delta(Y) = V(I) \subset X = Y \times Y$, where $I = \ker(\mathscr{O}_Y \otimes \mathscr{O}_Y \xrightarrow{\Delta^*} \mathscr{O}_Y)$, then the maps

$$Y \xrightarrow{\Delta} Y \times Y \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} Y$$

yield algebra maps

$$\mathscr{O}_Y \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \mathscr{O}_Y \otimes \mathscr{O}_Y \hookrightarrow \widehat{\mathscr{O}_Y \otimes \mathscr{O}}_Y \twoheadrightarrow \mathrm{gr}\, \widehat{\mathscr{O}_Y \otimes \mathscr{O}}_Y = \underbrace{\mathscr{O}_Y \otimes \mathscr{O}_Y/I}_{\mathrm{im}\Delta^* \cong \mathscr{O}_Y} \oplus I/I^2 \oplus \cdots$$

such that the composition $\mathscr{O}_Y \otimes \mathscr{O}_Y \hookrightarrow \widehat{\mathscr{O}_Y \otimes \mathscr{O}}_Y \twoheadrightarrow \mathrm{gr}\, \widehat{\mathscr{O}_Y \otimes \mathscr{O}}_Y$ factors through $\ker \Delta^* = (f \otimes 1 - 1 \otimes f \big| f \in \mathscr{O}_Y)$.

In other words $\widehat{\mathscr{O}_Y \otimes \mathscr{O}}_Y$ admits two actions of $\mathscr{O}_Y$ which agree when we take associated graded, aka at $\hbar = 0$.

# Mysteriously

$H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ can be described in this way. The mystery part lies outside of BF. It is that the two actions of $H^\bullet_{G_\mathcal{O}}(\mathrm{pt})$ coming from projections onto (torus-fixed) points

$$\begin{array}{ccccc} & & G_\mathcal{O} \rtimes \mathbb{G}_m \backslash G_\mathcal{K} / G_\mathcal{O} & & \\ & \swarrow^{\mathrm{pr}_1} & \downarrow^{\mathrm{pr}} & \searrow^{\mathrm{pr}_2} & \\ G_\mathcal{O} \rtimes \mathbb{G}_m \backslash \mathrm{pt} & & G_\mathcal{O} \rtimes \mathbb{G}_m \backslash \mathrm{pt} / G_\mathcal{O} & & \mathrm{pt}/G_\mathcal{O} \end{array}$$

agree when we turn off loop rotation, aka at $\hbar = 0$ in $H^\bullet_{\mathbb{G}_m}(\mathrm{pt})$.

# BF Theorem 1

$$\mathscr{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta) \cong H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$$

with the two actions of $H^\bullet_{G_\mathcal{O}}(\mathrm{pt})$ on the RHS corresponding to the two actions of $\mathscr{O}(\check{\mathfrak{t}}^*/W)$ on the LHS and the action of $H^\bullet_{\mathbb{G}_m}(\mathrm{pt})$ on the RHS corresponding to the action of $\mathscr{O}(\mathbb{A}^1)$ on the LHS. In particular

$$\mathscr{O}(T(\check{\mathfrak{t}}^*/W)) = H^\bullet_{G_\mathcal{O}}(\mathrm{Gr})$$

Nb. Recall that we always have a map

$$k[\hbar] \to \hat{\mathscr{O}}_X = \bigoplus_{i \geq 0} \hbar^i \mathscr{O}_X \oplus \bigoplus_{i \leq -1} \hbar^i I^{-i}$$

it's the map that takes a polynomial $\sum a_i \hbar^i$ to its image in $\bigoplus_{i \geq 0} \hbar^i \mathscr{O}_X$.[1]

[1] Multiplication by $\hbar$ shifts the filtration on $\mathscr{O}_X$ up so that $\hat{\mathscr{O}} = \cdots \oplus \hbar^{-1} I^2 \oplus I \oplus \hbar \mathscr{O} \oplus \cdots$

Intermission

# Proof sketch, preliminaries

- $H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{pt}) = H^\bullet_G(\mathrm{pt}) \otimes H^\bullet_{\mathbb{G}_m}(\mathrm{pt})$

  Roughly, $G_\mathcal{O} = G + U$ where $U$ is unipotent, so $EU = U$, and $E(G_\mathcal{O} \rtimes \mathbb{G}_m)/G_\mathcal{O} \rtimes \mathbb{G}_m = E(G \times \mathbb{G}_m)/G \times \mathbb{G}_m$

- $H^\bullet_G(\mathrm{pt}) = H^\bullet_T(\mathrm{pt})^W = \mathbb{C}[\mathfrak{t}/W]$

  Since $EG/G$ is an acyclic quotient of $EG/N_G(T)$ and $EG/T \to EG/N_G(T)$ is a $W$ cover

- $\mathsf{Gr}$ is equivariantly formal

  So[2] $G_\mathcal{O} \rtimes \mathbb{G}_m$ cohomology is torsion-free and $H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathsf{Gr}) = H^\bullet_{T \times \mathbb{G}_m}(\mathsf{Gr})^W$

[2] See this MO answer

# Proof sketch, step 1: existence

$H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ is a module over $H^\bullet_{G \times G \times \mathbb{G}_m}(\mathrm{pt}) = \mathscr{O}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W)[\hbar]$ such that the two actions of $\mathscr{O}(\check{\mathfrak{t}}^*/W)$ agree at $\hbar = 0$.

Therefore

$$(\mathrm{pr}_1^*, \mathrm{pr}_2^*, \mathrm{pr}^*) : \mathscr{O}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W)[\hbar] \to H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$$

factors through the Rees ring of $\mathscr{O}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W)$ wrt filtration by $I = \ker(\mathrm{pr}_1^* \otimes 1 - 1 \otimes \mathrm{pr}_2^*)$ i.e. $\mathscr{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W} \Delta)$.

Let $\alpha : \mathscr{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W} \Delta) \to H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$.

# Proof sketch, step 2: injectivity

Since Gr is equivariantly formal, it suffices to prove that $\alpha_{\mathrm{loc}}$ is an isomorphism.

Equivariant localization of the RHS at $T \times \mathbb{G}_m$-fixed points $\lambda \in X_\bullet(T)$ gives

$$H^\bullet_{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(\mathrm{Gr}^{G_{\mathcal{O}} \rtimes \mathbb{G}_m}) \otimes_{H^\bullet_{G_{\mathcal{O}} \rtimes \mathbb{G}_m}(\mathrm{pt})} \operatorname{Frac} \mathscr{O}(\mathfrak{t} \times \mathbb{A}^1) = \prod_\lambda \operatorname{Frac} \mathscr{O}(\mathfrak{t} \times \mathbb{A}^1)$$

Doing the same on the left

$$\mathscr{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W} \Delta) \otimes_{\mathscr{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \operatorname{Frac} \mathscr{O}(\mathfrak{t} \times \mathbb{A}^1)$$

Now, a function on $\mathfrak{t} \times \mathfrak{t} \times (\mathbb{A}^1 - 0)$, is determined by its restriction to a Zariski dense subset, e.g. to the union of the subsets

$$\Gamma_{\lambda,a} := \{(x, y, a) \big| x - y = a\lambda\} \subset \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1$$

# Proof sketch, step 3: surjectivity

Compare "$\hbar$-graded" dimensions.

# The rest of the talk

We've just described $H^\bullet_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ such that we can imagine the left side of the triangle below having nice properties (in fact, it's a full embedding on the full subcategory of semisimple complexes into the subcategory of $\mathrm{Coh}^{\mathbb{G}_m}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)$)

$$D_{G_\mathcal{O}}(\mathrm{Gr}) \subset D_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \overset{\sim}{\dashrightarrow} \mathrm{Coh}^G(\mathfrak{g}) \subset \mathscr{HC}_\hbar$$

$$D_{G_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \xrightarrow{H^\bullet} \mathscr{Q}\mathrm{Coh}^{\mathbb{G}_m}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1) \xleftarrow{\kappa_\hbar} \mathscr{HC}_\hbar$$

While what we're really after is a horizontal functor but we won't get there today. Instead we'll end by discussing the right side of the triangle. The sought after $\sim$ will fall out of the two sides?

# Ingredients

- $\mathcal{U}_\hbar = \bigoplus \hbar^i \mathcal{U}\mathfrak{g}_i$ wrt degree filtration

- $\check{G}$-modules over $\mathcal{U}_\hbar \otimes_{k[\hbar]} \mathcal{U}_\hbar$ such that actions are compatible

# Kostant–Soergel functor

BF define

$$\kappa_\hbar : \mathscr{HC}_\hbar \to \mathscr{Q}\mathrm{Coh}^{\mathbb{G}_m}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$$

by

$$M \mapsto (M \otimes_{\mathcal{U}\mathfrak{n}} \psi)^N$$

They show that $\kappa_\hbar$ is a fully-faithful embedding (on free stuff) into $\mathrm{Coh}^{\mathbb{G}_m}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)$ using transversality of the Kostant slice.

# Alternative description

One can equally define $\kappa_\hbar$ by a quotient

$$M \mapsto (M/M(n - \psi(n) \big| n \in \mathfrak{n}))^N$$

In case $M = \mathcal{U}_\hbar$ we get $Z(\mathcal{U}_\hbar) \ldots$

# Next time

Topology vs algebra? Perfect complexes? Knitty gritty filtrations.

Thank you for listening : )