(Equivariant) Cohomology of the affine Grassmannian and Ginzburg's picture.

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I Ginzburg's Questions

G a reductive algebraic group over $\mathbb{C}$.

k a field, $\check{G}$ the Langlands dual group over k

(i.e. the split reductive group with dual root datum)

Recall: Geometric Satake Equivalence

$$\begin{tikzcd}
\mathrm{Perv}_{G_{\mathcal{O}}}(Gr, k) \arrow[rr, "H^\bullet(Gr,-)"] & \circlearrowright & \mathrm{Vect}_k \\
& \mathrm{Rep}(\check{G}) \arrow[ul, "\sim", "\mathcal{S}"'] \arrow[ur, "\text{forget}"'] &
\end{tikzcd}$$

$\mathcal{S}$ is an equivalence of monoidal category.

There is also another commutative diagramme

$$\begin{tikzcd}
& \text{graded } H^\bullet(Gr, k)\text{-modules} \arrow[dr, "\text{forget}"] & \\
\mathrm{Perv}_{G_{\mathcal{O}}}(Gr, k) \arrow[ur, "H^\bullet(Gr,-)"] & & \mathrm{Vect}_k . \\
& \mathrm{Rep}(\check{G}) \arrow[ul, "\mathcal{S}", "\sim"'] \arrow[uu, bend left] \arrow[ur, "\text{forget}"'] &
\end{tikzcd}$$

Ginzburg's Questions:

① What's the representation-theoretic meaning of the grading on $H^\bullet(Gr, \mathcal{S}(V)) \cong V$?

② What's the representation-theoretic meaning of

$$\mathrm{Rep}(\check{G}) \longrightarrow \text{graded } H^{\cdot}(Gr, k)\text{-module}$$

Try to answer $\boxed{1}$: $\mathrm{Rep}\,\check{G} \xrightarrow{(*)} \mathbb{Z}\text{-graded-vector space} \xrightarrow{\text{forget}} \mathrm{Vect}_k$

$V \longmapsto H^{\cdot}(Gr, \mathcal{S}(V))$

$\mathbb{Z} = X^*(\mathbb{G}_m)$ Tannakian formalism $\Rightarrow \exists$ a 1-dim torus $\mathbb{G}_m \longrightarrow \check{G}$

a theory that allows one to recover algebraic groups from their category of representations

s.t. $(*)$ is just the restriction to $\mathbb{G}_m$.

Hence this grading corresponds to the eigenspace decomposition of a semisimple element h

Ginzburg: h is a regular semisimple element.

From now on, assume $k = \mathbb{C}$

II Cohomology ring of Gr

Principal sl_2-triple:

A principal sl_2-triple in $\check{g}$ is a triple (e, h, f)

s.t. $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = -h$

and e is a regular nilpotent element

Then we know that ① $Z_{\check{g}}(e) := \{x \in \check{g} \mid [x, e] = 0\}$

is an abelian Lie sub-algebra of $\check{g}$

(because the centraliser of a regular semisimple is a Cartan subalgebra, and $Z_{\check{g}}(e)$ is a "limit" of these Cartans)

② $ad(h)(Z_{\check{g}}(e)) \subseteq Z_{\check{g}}(e)$ (Jacobi identity)

To answer Ginzburg's question ①, first we need to compute $H^{*}(Gr, k)$

Recall: $K \subseteq G$ a maximal compact subgroup,

then $Gr \underset{\text{homotopy}}{\simeq} \Omega K$ the based loop space of K

Algebraic topology:

$H^*(\Omega K, \mathbb{C})$ is a commutative, cocommutative graded Hopf algebra generated by primitive elements

x s.t. $\Delta(x) = 1 \otimes x + x \otimes 1$

multiplication

Δ induced by $\Omega K \times \Omega K \to \Omega K$

Proposition (Ginzburg): $H^*(Gr, \mathbb{C}) \cong U(Z_{\mathfrak{g}^\vee}(e))$

as graded Hopf algebras, where the grading on RHS is induced by eigenvalues of ad h $\curvearrowright$ $Z_{\mathfrak{g}^\vee}(e)$.

semi-simple

idea of the proof: Tannakian formalism $\Rightarrow$ a morphism of Hopf algebras $H^*(Gr, \mathbb{C}) \longrightarrow U(\mathfrak{g}^\vee)$

Ginzburg proved that ① the image of the first Chern class of the determinant line bundle is a principal nilpotent element e. Since LHS is commutative, its image is in $U(Z_{\mathfrak{g}^\vee}(e))$

② $H^*(Gr, \mathbb{C}) \longrightarrow U(Z_{\mathfrak{g}^\vee}(e))$ is an iso.

III Ginzburg's miracle and a "prototype" of derived Satake

Answer to Ginzburg's questions:

- The grading on $H^\cdot(Gr, \mathcal{S}(V))$ corresponds to the eigenspace grading of the action of h on V
- The action of $H^*(Gr, k)$ on $H^*(Gr, \mathcal{S}(V)) \simeq V$ for $V \in Rep(\check{G})$ is just the restriction to $U(Z_{\check{g}}(e))$.

Moreover, the functor

$$Perv_{G_{\mathcal{O}}}(Gr, k) \xrightarrow{H^\cdot(Gr,-)} \text{graded } H^\cdot(Gr,k)\text{-modules}$$

is fully faithful!

(i.e., "a small piece of $\check{G}$ can determine the Hom spaces")

<u>Rmk/Sanity check:</u> $\check{G} = SL_2$. it's just the restriction to the Borel.

<u>Rmk:</u> "graded" is necessary, otherwise, $Z_{\check{g}}(e)$ is contained in the nilpotent radical, and the restriction is not full.

Rmk: In general, this fact is quite mysterious to me on the representation theory side

e.g. $\check{G} = SL_3$ $\quad e = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$ $\quad h = \begin{bmatrix} 2 & & \\ & 0 & \\ & & -2 \end{bmatrix}$

$Z_{\check{g}}(e) = \begin{bmatrix} 0 & a & b \\ & 0 & -a \\ & & 0 \end{bmatrix}$, why are these enough to determine the Hom spaces?

How about "higher Hom's"?

$\mathrm{Perv}_{G_{\mathcal{O}}}(Gr, k)$ is semi-simple, so no higher extensions.

But $\mathrm{Hom}^i_{\underline{D^b_{(G_{\mathcal{O}})}(Gr,k)}}(\mathcal{F}, \mathcal{G})$ are non-trivial.

sheaves constructible for $G_{\mathcal{O}}$-orbits

Thm (Ginzburg) $\forall \lambda, \mu \in X_*^+(T)$ such that $t^\lambda, t^\mu \in Gr$ are in the same connected component, $\forall i \in \mathbb{Z}$

we have

$$\mathrm{Hom}^i_{D^b_{(G_{\mathcal{O}})}(Gr,k)}(IC_\lambda, IC_\mu) \cong \mathrm{Hom}^i_{H^\bullet(Gr)}(H^\bullet(IC_\lambda), H^\bullet(IC_\mu))$$

where $\mathrm{Hom}^i_{H^\bullet(Gr,k)}$ means the space of morphisms shifting grading by i.

Using this result, Ginzburg gave an "answer" to a " prototype" of the derived Satake

Ginzburg's picture

In the following diagramme, those 🟨 parts are the "players" that we already know

$$\begin{tikzcd}
V \arrow[d, mapsto] & \mathrm{Rep}\,\check{G} \arrow[r, "\overset{\mathcal{S}}{\sim}"] \arrow[d, "Fr"'] & \mathrm{Perv}_{G_{\mathcal{O}}}(Gr, k) \arrow[d, hook] \\
V \otimes \mathcal{O}(\mathcal{N}) & Coh^{\check{G} \times \mathbb{C}^*}(\mathcal{N}) \arrow[d, "\text{take fibre over } e"'] \arrow[dr, "\kappa^{\circ}"] & D^b_{(G_{\mathcal{O}})}(Gr, k) \arrow[d, "H^{\bullet}(Gr, -)"] \\
& \text{graded } \check{G}(e)\text{-modules} \arrow[r, "\sim"] & \text{graded } H^{\bullet}(Gr)\text{-modules} \arrow[d, equal] \\
& & U(\mathcal{Z}_{\check{\mathfrak{g}}}(e))
\end{tikzcd}$$

$\hookrightarrow$ the centraliser of e in $\check{G}$

Ginzburg then introduce a " mysterious player" $Coh^{\check{G} \times \mathbb{C}^*}(\mathcal{N})$ where $\mathcal{N} \subseteq \check{\mathfrak{g}}$ is the nilpotent cone.

He proved that.

$$\mathrm{Hom}^i_{D^b_{(G_{\mathcal{O}})}(Gr,k)}(IC_\lambda, IC_\mu) \cong \mathrm{Hom}^i_{Coh^{\check{G}\times\mathbb{C}^*}(\mathcal{N})}(Fr(L(\lambda)), Fr(L(\mu)))$$

Almost like a derived Satake!

IV Preparation for derived Satake: equivariant cohomology of Gr.

- What is "derived Satake"?

Satake category $\mathrm{Perv}_{G_\mathcal{O}}(Gr, k) \xrightarrow[\text{forget}]{\sim} \mathrm{Perv}_{(G_\mathcal{O})}(Gr, k)$

There are two candidates for the "derived Satake"

$D^b_{G_\mathcal{O}}(Gr, k)$ and $D^b_{(G_\mathcal{O})}(Gr, k)$

But the former has a monoidal structure via

$$G_\mathcal{K}/G_\mathcal{O} \times G_\mathcal{K}/G_\mathcal{O} \xleftarrow{p} G_\mathcal{K} \times G_\mathcal{K}/G_\mathcal{O} \xrightarrow{q} G_\mathcal{K} \times^{G_\mathcal{O}} G_\mathcal{K}/G_\mathcal{O} \xrightarrow{m} G_\mathcal{K}/G_\mathcal{O}$$

while $D^b_{(G_\mathcal{O})}(Gr, k)$ doesn't (we need equivariant setting to get $(q^*)^{-1}$), which makes the Grothendieck an algebra, $\cong$ the Hecke algebra $\mathbb{C}[G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})]$ after $\otimes_\mathbb{Z} \mathbb{C}$

Drinfeld's question: What is the monoidal category of Hecke operators?

So we need to modify Ginzburg's picture

$$\begin{tikzcd}
\mathrm{Rep}\,\check{G} \arrow[rr, "\simeq"] \arrow[d] & & \mathrm{Perv}_{G_{\mathcal{O}}}(Gr,k) \arrow[d, hook] & \\
\mathrm{Coh}^{\check{G}\times\mathbb{C}^*}(\mathcal{N}) \arrow[d] \arrow[rrd, "\kappa^\circ"] & & D^b_{(G_{\mathcal{O}})}(Gr,k) \arrow[r, "\text{replace with}"] \arrow[d, "H^\bullet(Gr{,}-) \rightsquigarrow H^\bullet_{G_{\mathcal{O}}}(Gr{,}-)"] & D^b_{G_{\mathcal{O}}}(Gr,k) \\
\text{graded } \check{G}(e)\text{-modules} \arrow[rr, "\sim"] & & \text{graded } H^\bullet(Gr)\text{-modules} \arrow[r, rightsquigarrow] & H^\bullet_{G_{\mathcal{O}}}(Gr)
\end{tikzcd}$$

The first step is to compute $H^\bullet_{G_{\mathcal{O}}}(Gr)$

It is a $H^\bullet_{G_{\mathcal{O}}}(pt) \cong \mathcal{O}(\mathfrak{t}/W)$ - module

i.e. a coherent sheaf on $\mathfrak{t}/W$,

hence $\forall\ t \in \mathfrak{t}/W$, we can consider

the fibre $H^\bullet_{G_{\mathcal{O}}}(Gr)_t =: H^\bullet_t(Gr)$.

Ginzburg has described these fibres.

Consider the morphisms

$$\begin{tikzcd}(\check{\mathfrak{g}}^*)^{reg} \arrow[r, hook] \arrow[rrrrr, bend right, "pr"'] & \check{\mathfrak{g}}^* \arrow[r] & \check{\mathfrak{g}}^*/\check{G} \arrow[r, phantom, "\cong"] & \check{\mathfrak{t}}^*/W \arrow[r, phantom, "\cong"] & \mathfrak{t}/W & \end{tikzcd}$$

(via the coadjoint action)

pr provides a bijection between $\{$regular coadjoint orbits in $\check{\mathfrak{g}}^*\}$ and $\mathfrak{t}/W$

$\forall\, t \in \mathfrak{t}/W$, let $\mathcal{O}_t \subseteq (\check{\mathfrak{g}}^*)^{reg}$ be the corresponding orbit, and let $x \in \mathcal{O}_t$, the $\check{\mathfrak{g}}(x) := \mathrm{Lie}\,\check{G}(x)$ is uniquely determined by $\mathcal{O}_t$ up to canonical isomorphisms

(i.e. if $x, x' \in \mathcal{O}_t$ then $\exists\, u \in \check{G}$ s.t. $\mathrm{Ad}\,u\,(x) = x'$, which gives an iso $\check{\mathfrak{g}}(x) \xrightarrow{\simeq} \check{\mathfrak{g}}(x')$ that doesn't depend on the choice of u).

Denote by $\mathfrak{a}_t := \check{\mathfrak{g}}(x)$ this Lie algebra

Prop (Ginzburg) $H^{\cdot}_t(Gr) \cong U(\mathfrak{a}_t)$

Rmk: when $t=0$, $\mathcal{O}_0$ is a regular nilpotent orbit. After identifying $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$ via killing form, we have $\mathfrak{a}_0 \cong Z_{\check{\mathfrak{g}}}(e)$ and we recover our result on non equivariant cohomology

Description using Kostant slice:

Identify $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$ via killing form

(e, h, f) principal $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}$

then the composition

$$e + Z_{\check{\mathfrak{g}}}(f) \hookrightarrow \check{\mathfrak{g}} \longrightarrow \check{\mathfrak{g}}/\check{G} \cong \mathfrak{t}/W$$

is an isomorphism. $\Rightarrow$ there is a "copy" of $\mathfrak{t}/W$ inside $(\check{\mathfrak{g}}^*)^{reg}$. Then $\forall\, t \in \mathfrak{t}/W$, let $x_t \in e + Z_{\check{\mathfrak{g}}}(f)$ be the corresponding element

then $\mathfrak{a}_t \cong Z_{\check{\mathfrak{g}}}(x_t)$

We want to "globalise" Ginzburg's result to get $H^*_{G_{\mathcal{O}}}(Gr)$: There should be a sheaf $\mathcal{I}$ over $(\check{\mathfrak{g}}^*)^{reg}$ s.t. the fibre on each ξ is $U(Z_{\check{\mathfrak{g}}}(\xi))$, and $H^*_{G_\mathcal{O}}(Gr)$ is the restriction of this sheaf to the "copy" of $\mathfrak{t}/W$

Exercise: Consider the map $(\check{\mathfrak{g}}^*)^{reg} \xrightarrow{pr} \mathfrak{t}/W$, then the $pr^* Sym(\mathcal{J}^*)$ satisfies our description for $\mathcal{I}$, where $\mathcal{J}^*$ is the cotangent bundle of $\mathfrak{t}/W$.

"Hence", we have:

Thm: (Bezrukavnikov-Finkelberg)

$$H^\cdot_{G_\mathcal{O}}(Gr) \cong \mathcal{O}(\underbrace{T(\mathfrak{t}/W)}_{\text{total space of tangent bundle}})$$

There is also an asymptotic version

$$H^\cdot_{G_\mathcal{O}\rtimes \mathbb{G}_m}(Gr) \cong \mathcal{O}(\underbrace{N_\Delta(\mathfrak{t}/W)^2}_{\text{deformation to the normal cone}})$$

V. Bezrukavnikov-Finkelberg's picture

$$\begin{tikzcd}
V \arrow[d, mapsto] & \operatorname{Rep}\check{G} \arrow[rr, "S", "\simeq"'] \arrow[d, "\mathrm{Fr}"'] & & \operatorname{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr},k) \arrow[d, hook] \\
V\otimes\mathcal{O}(\check{\mathfrak{g}}^*) & \operatorname{Coh}^{\check{G}\times\mathbb{C}^*}(\check{\mathfrak{g}}^*) \arrow[d, "\mathcal{K}"'] \arrow[rrd, "\mathcal{K}"] & & D^b_{G_{\mathcal{O}}}(\mathrm{Gr},k) \arrow[d, "H^\bullet_{G_{\mathcal{O}}}(\mathrm{Gr},-)"] \\
& \text{gd-modules of the universal centraliser sheaf} \arrow[rd, phantom, "\simeq"] & & \text{gd-}H^\bullet_{G_{\mathcal{O}}}(\mathrm{Gr})\text{-modules} \arrow[ld, phantom, "\simeq"] \\
& & \operatorname{Coh}^{\mathbb{C}^*}(T^*(\mathfrak{t}/W)) &
\end{tikzcd}$$

$\mathcal{K}$ "=" restriction to Kostant slice

By taking the fibre at e, we recover Ginzburg's $\mathcal{K}^\circ$.