An Introduction to Affine Grassmannian and Geometric Satake Equivalence

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Goal.
- give a quick intro. to the geometry of aff. Grass
- state the geometric Satake equiv. and survey some of its props.

§0. Notations

- $\mathbb{K}$: fld
- $\mathcal{O} := \mathbb{K}[[t]]$
- $F := \mathrm{Frac}(\mathcal{O}) = \mathbb{K}((t))$
- G: conn. split red. group $/\mathcal{O}$
- $T \subset B \subset G$

- Def (Aff. Grass.)

① as "homogeneous space"

LG: loop space, L^+G: formal jet space presheaves on $\mathrm{Aff}_{\mathbb{K}}$.

$LG(R) := G(R((t)))$, $L^+G(R) := G(R[[t]])$

$L^nG : \mathrm{Aff}_{\mathbb{K}} \to \mathrm{Set}$ $L^nG(R) := G(R[t]/t^n)$

$L^+G = \varprojlim_n L^nG$.

$\mathrm{Gr}_G := [LG/L^+G]_{fpqc}$.

② "moduli interpretation"

$\mathrm{Gr}_G : \mathrm{Aff}_{\mathbb{K}} \to \mathrm{Set}$

$\mathrm{Gr}_G(R) = \{(\mathcal{E}, \beta) \mid \mathcal{E} \to D_R \ G\text{-torsor},\ \beta : \mathcal{E}|_{D_R^\times = \mathrm{Spec}R((t))} \xrightarrow{\sim} \mathcal{E}^{triv}|_{D_R^\times}\}/\simeq$ ($D_R = \mathrm{Spec}R[[t]]$)

In particular, if $G = GL_n$,

$\mathrm{Gr}_{GL_n}(R) = \{\Lambda \subset R((t))^n \mid \Lambda$ is a lattice i.e. Λ is a f.g. proj. $R[[t]]$-mod and $\Lambda \otimes_{R[[t]]} R((t)) \simeq R((t))^n\}$

• Thm Gr_G is an ind-scheme

The proof is a bit techical and can be found in various places. So we give a very brief sketch of the arguments.

Sketch of Proof for $G = GL_n$

$\forall \Lambda \in Gr_{GL_n}(R)$, $\exists N \in \mathbb{N}_{\geq 0}$ s.t.

$$t^N R[[t]]^n \subset \Lambda \subset t^{-N} R[[t]]^n \quad (*)$$

So $Gr_G = \operatorname{colim} Gr^{(N)}$, $Gr^{(N)} := \{\Lambda \text{ lattices satisfying } (*)\}$.

Lemma 1. $Gr^{(N), n}(R) \to Gr^{(N), f}(R) = \left\{ \begin{array}{l} R[[t]]\text{-quotient of } t^{-N}R[[t]]^n / t^N R[[t]]^n, \text{ which} \\ \text{is proj. as a } R\text{-mod} \end{array} \right\}$

$$\Lambda \to t^{-N} R[[t]]^n / \Lambda$$

is a well-defined ~~natural~~ equivalence of functors.

Lemma 2. $Gr^{(N), f} \overset{i}{\hookrightarrow} Gr(2nN)$ is a closed embedding

Lemma 1 is technical and Lemma 2 can be seen from multiplying by t gives a nilpotent operator on $\operatorname{Im}(i)$.

§1 Some Geometry

§1.1 Connected components

• Thm. The natural map $LG \to G$ induces

$\pi_0(Gr_G) \simeq \pi_0(LG) \simeq \pi_1(G)$

Proof for $G/\mathbb{C}$

In this case, $Gr_G \simeq \Omega K$: loop space of $K \subset G$ max. cpt subgroup of $G(\mathbb{C})$. Then $\pi_0(Gr_G) = \pi_0(\Omega K) = \pi_1(K) = \pi_1(G)$.

□

• Extra symmetries

Warning: no direct analogues in mixed char. setting.

• Def $\mathrm{Aut}(D) : \mathrm{Aff}_{\mathbb{k}} \to \mathrm{Grp}$.

$$R \mapsto \mathrm{Aut}_R(R[[t]]).$$

For any $\varphi \in \mathrm{Aut}(D)(R)$, φ is determined by the image of t

$\varphi(t) = a_0 + a_1 t + \cdots \in R[[t]]$.

Some calculation $\Rightarrow$ a_0 is nilp. and a_1 is inv.

Thus,

$$\mathrm{Aut}(D) \simeq \mathrm{Spf}\, \mathbb{k}[[a_0]] \times \mathrm{Spec}\, \mathbb{k}[a_1^{\pm 1}] \times \mathrm{Spec}\, \mathbb{k}[a_2, \cdots]$$

$\mathrm{Aut}(D)^+ := \mathrm{Spec}\, \mathbb{k}[a_1^{\pm 1}, a_2, \cdots]$ is a closed normal subgroup scheme of $\mathrm{Aut}(D)$, which classifies auto of $R[[t]]$, which reduces to Id mod t.

So $\mathrm{Aut}(D) = \mathrm{Aut}^+(D) \rtimes \hat{G}_a$.

$G_m^{rot} := \mathrm{Spec}\, \mathbb{k}[a_1^{\pm}]$: rotation torus

$$\mathrm{Aut}(D)^+ = \mathrm{Aut}^{++}(D) \rtimes G_m^{rot}$$

$$\simeq \mathrm{Spec}\, \mathbb{k}[a_2, \cdots] \quad \text{pro-unip. radical of } \mathrm{Aut}^+(D)$$

$\mathrm{Aut}(D) \times \mathrm{Gr}_G \to \mathrm{Gr}_G \qquad \epsilon, (\mathcal{E}, \beta) \mapsto (\epsilon^* \mathcal{E}, \epsilon^* \beta)$.

In particular, for $G = GL_n$,

$$(\epsilon, \Lambda \subseteq R((t))^n) \mapsto \Big(\Lambda \otimes_{R[[t]], \epsilon} R[[t]], \ \big(\Lambda \otimes_{R[[t]], \epsilon} R[[t]]\big) \otimes_{R[[t]]} \underbrace{R((t))}_{\simeq R((t))^n}\Big)$$

• Building Blocks

For any modification $\mathcal{E}_1|_{D_R^\times} \xrightarrow[\sim]{\beta} \mathcal{E}_2|_{D_R^\times}$, choose ϕ_1, ϕ_2

$$\begin{tikzcd}
\mathcal{E}_1|_{D_R^\times} \arrow[r, "\beta", "\sim"'] \arrow[d, "\phi_1", "\wr"'] & \mathcal{E}_2|_{D_R^\times} \arrow[d, "\wr", "\phi_2"'] \\
\mathcal{E}^{triv}|_{D_R^\times} \arrow[r, dashed, "\phi_2 \beta \phi_1^{-1}"] & \mathcal{E}^{triv}|_{D_R^\times}
\end{tikzcd}$$

$\Rightarrow \phi_2 \beta \phi_1^{-1} \in \mathrm{Aut}(\mathcal{E}^{triv}|_{D_R^\times}) = G(R((t)))$.

$\phi_1, \phi_2 \in \mathrm{Aut}(\mathcal{E}^{triv})$ or $\mathcal{E}_1 \xrightarrow{\phi_1} \mathcal{E}^{triv}$ and $\mathcal{E}_2 \xrightarrow{\phi_2} \mathcal{E}^{triv}$.

$$\mathrm{Aut}(\mathcal{E}^{triv}) = G(R[[t]]).$$

Cartan Decomp Thm $\Rightarrow$

$$G(\mathcal{O})\backslash G(F)/G(\mathcal{O}) \simeq X_\bullet^+(T)$$

$$\phi_2\beta\phi_1^{-1} \longmapsto \mathrm{Inv}(\beta) : \text{relative position}$$

• <u>Def</u> (Schubert Varieties and Schubert cells).

$\forall$ $\mu \in X_\bullet^+(T)$,

$Gr_{\le\mu} = \{(\mathcal{E},\beta) \in Gr_G \mid \mathrm{Inv}(\beta) \le \mu\}$

$Gr_\mu = \{(\mathcal{E},\beta) \in Gr_G \mid \mathrm{Inv}(\beta) = \mu\}$.

• <u>Lemma</u>. ① $(Gr_G)_{\mathrm{Red}} = \mathrm{colim}_\mu\, Gr_{\le\mu}$

② Gr_μ is the orbit of $t^\mu := \mathrm{Im}(\mu : \mathbb{G}_m \to T \hookrightarrow G)$ under the L^+G-action, and it is a smooth, quasi-proj. var. of dim $(2\rho,\mu)$.

③ $Gr_{\le\mu}$ is the Zariski closure of Gr_μ, and is thus a proj. variety.

<u>Proof</u> Omitted for time reason.

• Example

• $G = GL_2$.

① $Gr_{(0,0)} = pt$

② $Gr_{(1,0)} \simeq L^+GL_2 / L^+GL_2 \cap \begin{pmatrix} t & \\ & 1\end{pmatrix} L^+GL_2 \begin{pmatrix} t^{-1} & \\ & 1\end{pmatrix}$

$$\simeq L^+GL_2 / \begin{pmatrix} \mathcal{O}^\times & t\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times \end{pmatrix} \quad \leftarrow L^+G\text{-orbit point of view.}$$

$$\simeq \mathbb{P}^1$$

Or, $Gr_{(1,0)} = \{\Lambda \subset R((t))^2 \mid R[[t]]^2 \supsetneq \Lambda \supsetneq t\cdot R[[t]]^2$ s.t. $R[[t]]^2/\Lambda$ has length 1. i.e. 1-dim subspace inside $R^2\}$

$$\simeq \mathbb{P}^1$$

③ $Gr_{(1,1)} = pt$, $(1,1)$ central

④ $Gr_{\le(2,0)} = \{\Lambda \subset \mathbb{k}[[t]]^2 \mid \mathrm{length}\,(\mathbb{k}[[t]]^2/\Lambda) = 2\}$.

$\mathbb{k}[[t]]^2/\Lambda$ as a quotient space of $\mathbb{k}[[t]]^2/t^2\mathbb{k}[[t]]^2 \simeq \mathbb{k}^4$.

$\Rightarrow Gr_{\le(2,0)} \hookrightarrow Gr(2,4)$

Consider

$\widetilde{Gr}_{\le(2,0)} = \{ \Lambda_1 \subset \Lambda_2 \subset \Lambda_0 \mid \dim(\Lambda_0/\Lambda_2) = \dim(\Lambda_2/\Lambda_1) = 1 \}$.

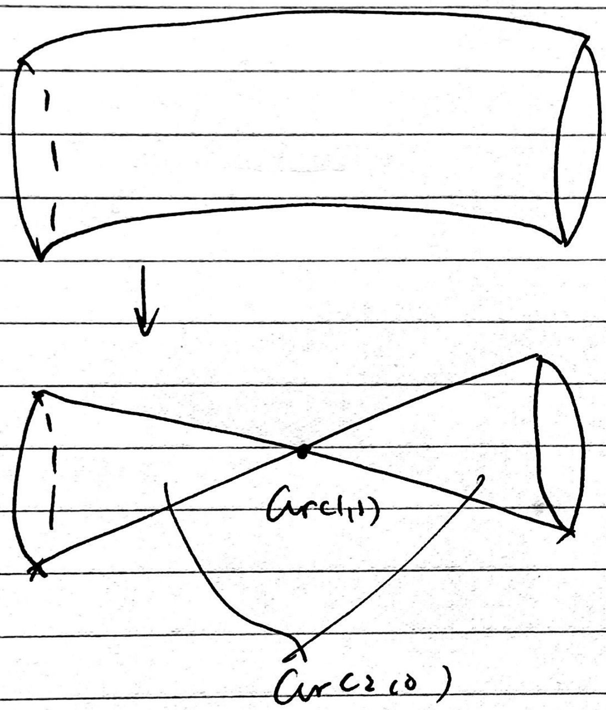
This is the Demazure resolution of $Gr_{\le(2,0)}$

$\widetilde{Gr}_{\le(2,0)} \to Gr_{\le(2,0)} \quad \Lambda_1 \subset \Lambda_2 \subset \Lambda_0 \mapsto \Lambda_1 \subset \Lambda_0$

if $\Lambda_1 \neq t\Lambda_0 \quad \exists!\ \Lambda_2$ s.t. $\Lambda_1 \subset \Lambda_2 \subset \Lambda_0$

if $\Lambda_1 = t\Lambda_0 \quad \exists\ \mathbb{P}^1$ - choices of Λ_2.

$\widetilde{Gr}_{\le(2,0)}$



$Gr_{\le(2,0)}$

• $G = PGL_2$

$Gr_{(0,0)} = pt$, $Gr_{(0,-1)} = \mathbb{P}^1$, $Gr_{(0,-2)}$: line bundle / $\mathbb{P}^1$

$\mathcal{O}_{\mathbb{P}^1}(-1)$

• $G = SL_2$, $PGL_2 \twoheadrightarrow SL_2 \Rightarrow Gr_{SL_2} \hookrightarrow Gr_{PGL_2}$ embedding into the even component of Gr_{PGL_2}.

• Opposite Schubert Varieties & Transversal Slices.

Warning ① Opposite S.V. are not "real" varieties

② No direct analogue

• Def $L^-G(R) := G(R[t^{-1}])$, $L^{<0}G = \ker(L^-G \to G,\ g \mapsto g \bmod t^{-1})$.

Fix $T \subset B \subset G \Rightarrow \quad G(F) = \coprod_{\mu \in X_*(T)^+} G(k[t^{-1}])\, t^{\mu}\, G(\mathcal{O})$ (Birkhoff decomp

$\Rightarrow L^{-}G(\mathbb{K})\backslash G(F)/G(\mathcal{O}) \simeq X_\bullet(T)^+$

• **Def** (Gr^μ, $Gr^{\geq\mu}$)

- $Gr^\mu = L^-G \cdot t^\mu \subset Gr_G$
- $Gr^{\geq\mu} = \coprod_{\lambda\geq\mu} Gr^\lambda$

• **Prop** ① Gr^μ is locally closed subscheme of Gr_G

② $Gr_\mu \cap Gr^\lambda \neq \phi \iff \mu \geq \lambda$ $\quad Gr_\mu \cap Gr^\mu = G \cdot t^\mu \simeq G/P_\mu$.

③ $Gr^{\geq\mu}$ is Zariski closed, and contains Gr^μ as an open dense subset.

④ $\operatorname{codim} Gr^{\geq\mu} = (2\rho, \mu) - \dim G/P_\mu$.

• **Prop** $L^{<0}G \cdot t^\lambda$ is a transversal slice of Gr_λ in Gr_G i.e. $\forall \lambda \leq \mu$, $S^\lambda_{\leq\mu} := L^{<0}G \cdot t^\lambda \cap Gr_{\leq\mu}$. Then $S^\lambda_{\leq\mu} \cap Gr_\nu$ is transversal $\forall \lambda \leq \nu \leq \mu$.

§2. The Satake Category

Recall that $\forall \mu \in X_\bullet^+(T)$, $L^+G \curvearrowright Gr_{\leq\mu}$ through a finite type quotient $L^n G$ $\Rightarrow$ makes sense to define

$$P_{L^+G}(Gr_G, \Lambda) \qquad \Lambda = \begin{cases} \mathbb{C} & \text{if } \mathbb{K}=\mathbb{C} \\ \overline{\mathbb{Q}}_\ell & \text{if } \mathbb{K}=\overline{\mathbb{F}}_p. \end{cases}$$

• **Def** (Satake Category)

$\mathrm{Sat}_{G,\Lambda} = \operatorname{colim}_\mu P_{L^+G}(Gr_{\leq\mu}, \Lambda)$, where $\Lambda = \mathbb{C}$ if $\mathbb{K}=\mathbb{C}$ and $\Lambda = \overline{\mathbb{Q}}_\ell$ if $\mathbb{K}=\overline{\mathbb{F}}_q$, where $\ell \neq \operatorname{char} \mathbb{F}_q$.

In fact, we can endow $\mathrm{Sat}_{G,\Lambda}$ w/ a monoidal structure as by Lusztig's convolution construction

Def. $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G := [LG \times^{L^+G} \mathrm{Gr}_G]$ $\quad L^+G \curvearrowright LG \times \mathrm{Gr}_G$ anti-diagonally

or equiv.

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G(R) := \{(\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \mid \mathcal{E}_2 \xrightarrow{\beta_2} \mathcal{E}_1 \xrightarrow{\beta_1} \mathcal{E}^0\}.$$

There is a proper and stratified semi-small morphism

$m: \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \to \mathrm{Gr}_G$

$([g_1], [g_2]) \mapsto ([g_1 g_2])$

$(\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \longmapsto (\mathcal{E}_2, \beta_2 \circ \beta_1)$

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$

$\mathcal{F} \boxtimes \mathcal{G}$

$\exists! \ \mathcal{F} \tilde{\boxtimes} \mathcal{G} \in P_{L^+G^2}(\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G)$ s.t. $\quad q^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \simeq p^*(\mathcal{F} \boxtimes \mathcal{G})$

$\mathcal{F} * \mathcal{G} := Rm_!(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$.

Prop $\mathcal{F} * \mathcal{G} \in \mathrm{Sat}_{G,\Lambda}$.

Prop ① The Satake cat. $\mathrm{Sat}_{G,\Lambda}$ is semi-simple w/ simple objects $\mathcal{I}C_\mu$, $\mu \in X_*^+(T)$

② Objects in $\mathrm{Sat}_{G,\Lambda}$ are automatically $\mathrm{Aut}^+(D)$-equiv.

~~Prop~~ Thm The hypercohomology functor

$H^*: \mathrm{Sat}_{G,\Lambda} \longrightarrow \mathrm{Vect}_\Lambda$ lifts to an equiv. of symmetric monoidal cats

$H^*: \mathrm{Sat}_{G,\Lambda} \simeq \mathrm{Rep}_\Lambda(\check{G}_\Lambda)$, where $\check{G}_\Lambda$ is the Langlands dual group of G.

Prop. The geom. Sat. equiv. admits the following props

① $H^*(\mathcal{I}C_\mu) \simeq V_\mu$: highest weight repn. of $G^\vee$ asso. w/ μ.

② $F_\nu := H_c^*(S_\nu, -) : \mathrm{Sat}_{G,\Lambda} \to \mathrm{Vect}_\Lambda$, F_ν: weight functors.

③ Forgetful $\circ H^* \simeq \bigoplus_{\nu \in X_*(T)} F_\nu : \mathrm{Sat}_{G,\Lambda} \to \mathrm{Vect}_\Lambda$.

④ $\forall \mathcal{F} \in \mathrm{Sat}_G$, $F_\nu(\mathcal{F}) = \nu$- weight space of $H^*(\mathcal{F})$.

⑤ $H^*({}^p j_* \underline{\Lambda}_{Gr_\mu}[\langle 2\rho, \mu\rangle]) \simeq S(\mu)$ $(:= \mathrm{Ind}_{B^\vee}^{G^\vee} \Bbbk_\mu)$: Schur module

⑥ $H^*({}^p j_! \underline{\Lambda}_{Gr_\mu}[\langle 2\rho, \mu\rangle]) \simeq W(\mu) := (S(-w_0\mu)^*)$.

§3 Semi-∞ orbits and weight functors

• Semi - ∞ orbits

Chose $T \subset B \subset G$

$\rho^\vee = \frac{1}{2}\sum$ posit. coroots

$\mathbb{G}_m \xrightarrow{2\rho^\vee} T \to G \subset L^+G \rightsquigarrow \mathbb{G}_m \curvearrowright Gr_G$ w/ invariant pts.

$\{t^\lambda \mid \lambda \in X_*(T)\}$.

Def $S_\lambda = \{x \in Gr_G \mid \lim_{t\to 0} 2\rho^\vee(t)\cdot x = t^\lambda\}$.

$T_\lambda = \{x \in Gr_G \mid \lim_{t\to\infty} 2\rho^\vee(t)\cdot x = t^\lambda\}$

$\forall \lambda \in X_*(T)$.

Lemma ① $S_\lambda = LU\cdot t^\lambda$, $T_\lambda = LU^-\cdot t^\lambda$

② $S_{\le\lambda} := \bigcup_{\lambda'\le\lambda} S_{\lambda'}$ is closed, and $S_\lambda \subset S_{\le\lambda}$ is open & dense

$\Rightarrow S_\lambda$ is locally closed and in Gr_G and $S_{\le\lambda} = \overline{S}_\lambda$.

③ Similar results in ② hold for T_λ.

• **Thm**. $S_\nu \cap Gr_\mu \neq \phi \Leftrightarrow t^\nu \in Gr_{\le\mu}$

If $S_\nu \cap Gr_{\le\mu}$, then it is of pure dim $\rho(\mu+\nu)$.

$\mathrm{Irr}(S_\nu \cap Gr_{\le\mu})$ are called Mirković - Vilonen cycles.

• **Def** $F_\nu := H_c^*(S_\nu, -) : \mathrm{Sat}_{G,\Lambda} \to \mathrm{Vect}_\Lambda$

Prop ① $F_\nu \simeq H^{*}_{\bar{F}_\nu}(-)$.

② F_ν ~~is~~ vanishes outside deg $(2\mathcal{J}, \nu)$.

③ $S(\mu)_\nu = W(\mu)_\nu$ and have a canonical basis $\mathrm{Irr}(Gr_{\mu} \cap S_\nu)$ which is called the Mirkovic - Vilonen basis