Finite groups as algebraic groups in non-defining characteristic

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October 7, 2020

$\mathbb{G}$ connected reductive alg gp $/\overline{\mathbb{F}}_q$, F Frobenius endo. $G = \mathbb{G}(\mathbb{F}_q) = \mathbb{G}^F$

example: $\mathbb{G} = \mathbb{GL}_n$, $F((a_{ij})) = (a_{ij}^q)$, $G = \mathbb{GL}_n(\mathbb{F}_q)$

Jordan: $$\boxed{\mathrm{Cl}(G) \xrightarrow{\sim} \coprod_{(t)\in \mathrm{Cl}_{ss}(G)} \mathrm{Cl}_{unip}(C_G(t))}$$

- $$\begin{tikzcd} \mathcal{O} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\supseteq"] & K \arrow[dr, phantom, "\supseteq"] & \\ k \arrow[dr, phantom, "\supseteq"] & \mathbb{Z}_\ell \arrow[r, hook] \arrow[d, two heads] & \mathbb{Q}_\ell \\ & \mathbb{F}_\ell & \end{tikzcd}$$, $\ell \nmid q$. Lusztig's Jordan decomposition of characters:

$$\boxed{\mathrm{Irr}_K(G) \xrightarrow{\sim} \coprod_{(s)\in \mathrm{Cl}_{ss}(\mathbb{G}^*)^{F^*}} \mathrm{Unip}_K(C_{\mathbb{G}^*}(s)^{F^*})}$$

$\mathrm{Unip}_K(G^*) \xrightarrow{\sim} \mathrm{Unip}_K(G)$: rep. occurring in $H^*(X_w, K)$, X_w Deligne-Lusztig var.

Lusztig: combinatorial param. of $\mathrm{Unip}_K(G)$. Independent of q.

example: $G = GL_n(q)$, $\mathrm{Unip}_K(G) =$ rep. in $\mathrm{Ind}_B^G K \xrightarrow{\sim} \{\text{Partitions of } n\}$

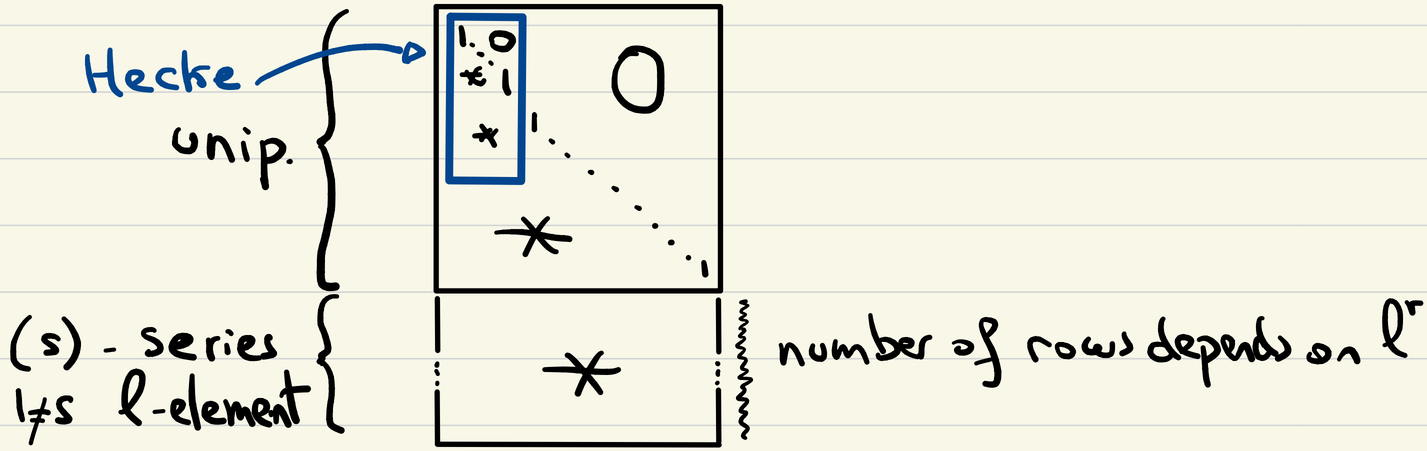
- Modular case $$\mathcal{O}G = \bigoplus_{(s)\in \mathrm{Cl}_{ss}(\mathbb{G}^*)^{F^*}_{\ell'}} b_{(s)}\mathcal{O}G \quad \text{(Broué-Michel)}$$

(partial) reduction to unipotent blocks = in $b_{(1)}\mathcal{O}G$ (Broué, Bonnafé-Dat-R.)

Dec. matrix $b_{(1)}\mathcal{O}G$

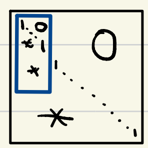
(Brunat-Dudas-Taylor, after Geck, Dipper, James, Grüber, Hiß)

$(p, \ell$ good, $\ell \nmid |Z(\mathbf{G})^F|)$



$\Rightarrow$ $\mathrm{Irr}_K(b_{(1)}kG) \xrightarrow{\sim} \mathrm{Unip}_K(G)$: depends only on the type of G

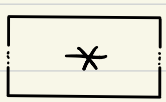
$d \geq 1$ min, $\ell \mid q^d - 1$. $r := \nu_\ell(q^d-1)$.

Conjecture | If $\ell \gg 0$, then  depends only on d and on the type of G

$G = GL_n(q)$: true, dec. mat of a q-Schur algebra over $\mathbb{C}$, $q = \sqrt[d]{1}$ (Dipper-James)

Problem : char. 0 version? Combinatorics for dec. numbers?

Conjectural solutions via Broué's conjecture.

Instead of removing , let $r = \nu_\ell(q^d-1) \to \infty$

Lang map $\mathcal{L}: \mathbb{G} \to \mathbb{G}, g \mapsto g^{-1}F(g)$ étale Galois covering of group G

$\mathbb{P} \subset \mathbb{G}$ parabolic, $\mathbb{U} = R_u(\mathbb{P})$, $\mathbb{L}$ Levi complement. Assume $\mathbb{L}$ is F-stable

<u>Deligne-Lusztig varieties</u>:

$$\begin{array}{ccc} G \circlearrowright \mathcal{L}^{-1}(F(\mathbb{U})) & \twoheadrightarrow & Y_{\mathbb{U}} = \mathcal{L}^{-1}(F(\mathbb{U}))/\mathbb{U} \cap F(\mathbb{U}) \\ \circlearrowleft \ \mathbb{L}^F = L & & \downarrow \\ & & G \circlearrowright X_{\mathbb{U}} = Y_{\mathbb{U}}/L \end{array}$$

$\mathbb{P}$ F-stable $\Rightarrow Y_{\mathbb{U}} = G/U \twoheadrightarrow X_{\mathbb{U}} = G/P$

$H^*(X_{\mathbb{U}}, \bar{\mathbb{Q}}_\ell)$? (Independent of ℓ)

• Assume $\ell \nmid |W|$, S: Sylow ℓ-subgp of G. $\mathbb{L} = C_{\mathbb{G}}(S)$: F-stable Levi subgp

<u>Conj</u> (Broué) | $\exists \mathbb{P}$ parab. with $\mathbb{P} = \mathbb{U} \rtimes \mathbb{L}$ such that the action of $G \times L$ on $R\Gamma(Y_{\mathbb{U}}, k)$ extends to $G \times N_G(S)$ and $R\Gamma(Y_{\mathbb{U}}, k) \otimes_{kN_G(S)} - : D^b(B_0(kN_G(S))) \xrightarrow{\sim} D^b(B_0(kG))$

<u>Example</u> (Lusztig) $G = GL_n(q)$, $n < \ell \mid q^n - 1$, $L = T = \mathbb{F}_{q^n}^\times$, $N_G(S) = \mathbb{F}_{q^n}^\times \rtimes \underset{\text{acts as } F \text{ on } T}{\mathbb{Z}/n}$

$X_{\mathbb{U}} \simeq \mathbb{P}^{n-1} \setminus \mathbb{P}^{n-1}(\mathbb{F}_q)$. $H^i(X_{\mathbb{U}}, \bar{\mathbb{Q}}_\ell) = \mathrm{irr}_{(n-i,1,\dots,1)}$. Broué's conj ok (Bonnafé-R.)

Chuang-R: Broué's "abstract" abelian defect conjecture holds for $GL_n(q)$. Uses 2-rep. of $\widehat{\mathfrak{sl}}_p$. 2-rep theory was introduced to solve that question, generalizing the problem and reducing it to simple 2-rep. of $\mathfrak{sl}_2$

Conj (Chuang-R). The Deligne-Lusztig equivalence conjectured by Broué is perverse

$\mathcal{A} = A\text{-mod}$, $\mathcal{A}' = A'\text{-mod}$, A, A' fin. dim k-alg. $p : \mathrm{Irr}(A) \to \mathbb{Z}$, $p' : \mathrm{Irr}(A') \to \mathbb{Z}$

$\mathcal{A}_i = \langle p^{-1}(\mathbb{Z}_{\leq i}) \rangle \subset \mathcal{A}$, $\mathcal{A}'_i = \langle p'^{-1}(\mathbb{Z}_{\leq i}) \rangle$ Serre subcat.

Def: $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is perverse for p, p'

if

$$\begin{tikzcd}
D^b(\mathcal{A}) \arrow[r, "F", "\sim"'] & D^b(\mathcal{A}') \\
D^b_{\mathcal{A}_i}(\mathcal{A}) \arrow[u, phantom, "\cup"] \arrow[r, dashed] \arrow[d] & D^b_{\mathcal{A}'_i}(\mathcal{A}') \arrow[u, phantom, "\cup"] \arrow[d] \\
D^b_{\mathcal{A}_i}(\mathcal{A}) / D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \arrow[r, dashed] & D^b_{\mathcal{A}'_i}(\mathcal{A}') / D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}') \\
\mathcal{A}_i / \mathcal{A}_{i-1} \arrow[u, "\mathrm{can}"] \arrow[r, dashed, "\sim"'] & \mathcal{A}'_i / \mathcal{A}'_{i-1} \arrow[u, "{\mathrm{can}[-i]}"']
\end{tikzcd}$$

- $\mathcal{A}$ and p determine $\mathcal{A}'$ and p'
- Can. bij. $\mathrm{Irr}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{A}')$
- $A' = B_0(kN_G(S)) \Rightarrow$ dec mat $\mathcal{A}$ is unitriang.
- Above: $p : \mathrm{Unip}_{\mathbb{F}_\ell}(G) \xrightarrow{\sim} \mathrm{Unip}_{\overline{\mathbb{Q}}_\ell}(G) \xrightarrow{\gamma_d} \mathbb{Z}$

  $[H^{\gamma_d(V)}(X_w, \overline{\mathbb{Q}}_\ell) : V] \neq 0$
- Craven: conjecture for γ_d, depends only on the generic degree

Assume $G = GL_n(q)$, $\ell > n$, $\ell \mid q^d - 1$, $1 \leq d \leq n$. $r := v_\ell(q-1)$. $n = dm + m'$, $0 \leq m' < d$.

$B_0(kN_G(S)) \simeq k\left(\mathbb{Z}/\ell^r\right)^m \rtimes \left(\left(\mathbb{Z}/d\right)^m \rtimes \mathfrak{S}_m\right)$

$k\,\mathbb{Z}/\ell^r \simeq k[x]/x^{\ell^r} = H^*\left(\left(k[h][x,y]/y^2, d(y) = hx^{\ell^r}\right)_{|h=1}\right)$

$\rightsquigarrow k\,\mathbb{Z}/\ell^r$ deformation of $k[x] \otimes \Lambda(y)$

$B_0(kN_G(S))$ deformation of $A_d = \left(S(V) \otimes \Lambda(V)\right) \rtimes \left(\left(\mathbb{Z}/d\right)^m \rtimes \mathfrak{S}_m\right)$

$$\begin{array}{ccccc} \mathbb{Z} & \xleftarrow{\gamma_d} & \mathrm{Irr}(B_0 kG) & \xrightarrow{\sim} & \{\lambda \vdash n \mid d\text{-core}(\lambda) = (m')\} \\ & \nwarrow^{\gamma'_d} & & & \uparrow\wr \\ & & \mathrm{Irr}\left(\left(\mathbb{Z}/d\right)^m \rtimes \mathfrak{S}_m\right) & \xrightarrow{\sim} & d\text{-multipart of } m \end{array}$$

Perversity data $\gamma'_d \rightsquigarrow$ abelian cat $\mathcal{A}$, $D^b(\mathcal{A}) \xrightarrow{\sim} D^b(A_d)$

<u>Conj</u> (Craven-R) | (square part) Dec mat $\mathcal{A}$ = (square part) dec mat $B_0(kG)$

Works for any G, bijections use d-Harish Chandra theory (Broué-Malle-Michel)

$\mathcal{A}$ can be defined in char. 0.

$$D^b\big((S(V)\otimes\Lambda(V))\rtimes((\mathbb{Z}/d)^m\rtimes\mathfrak{S}_m)\big)\xrightarrow[\text{Koszul}]{\sim} D^b_{dg}\big((S(V)\otimes S(V^*))\rtimes((\mathbb{Z}/d)^m\rtimes\mathfrak{S}_m)\big)\xrightarrow[\text{McKay}]{\sim} D^b_{dg}(\mathrm{Hilb}^m X_d)$$

$X_d \to \mathbb{A}^2/(\mathbb{Z}/d)$ minimal resolution

t-struct. on $D^b_{dg}(\mathrm{Hilb}^m X_d)$: heart $\mathcal{H}_d$ $\begin{cases} \text{quantizes (conjecturally) to } D^b(B_0\, kG) \\ \text{defined in char. } 0 \end{cases}$

$$\begin{tikzcd}
\text{rational Chered. alg} & \mathcal{O}_{\mathfrak{S}_n} \arrow[d, leftrightarrow, "\text{level rank duality}"', "\sim \text{der}"] \arrow[rr, bend left, dashed, leftarrow, ""] \arrow[rrrr, bend left=30] & & \mathcal{H}_d \arrow[ll, dashed] \arrow[rr, dashed] \arrow[d, leftrightarrow, "\text{der}\sim"'] & & B_0(kG)\text{-mod} \arrow[d, leftrightarrow, "\text{der}\sim"', "\text{Broué}"] \\
\text{(Shan-Varagnolo-Vasserot-R)} & \mathcal{O}_{(\mathbb{Z}/d)^m\rtimes\mathfrak{S}_m} \arrow[rrrr, bend right=30] & & \mathrm{Hilb}^m X_d\text{-coh} \arrow[ll, dashed] \arrow[rr, dashed] & & B_0(kN_G(S))\text{-mod}
\end{tikzcd}$$

Dudas-R (in progress): bigradings $\rightsquigarrow$ conjectural 2-var. dec. numbers
starting point: Macdonald polynomials

Example: $\left[\Delta(1^9):L(32^21^2)\right]=30=1+2v+4v^2-uv+6v^2-2uv^2+8v^4-4uv^3+9v^5-5uv^4+9v^6-6uv^5+8v^7-5uv^6+6v^8-4uv^7+4v^9-2uv^8+2v^{10}-uv+v^{11}\Big|_{\substack{u=1\\v=1}}$
$U_g(q)$, $d=1$

$$\mathrm{Hilb}^n\mathbb{A}^2\text{-coh}\overset{?}{\simeq}\left[\frac{\mathbb{G}}{\mathbb{G}}\right]\text{-sh}_{\mathrm{unip}}\;;\;\left[\frac{\mathbb{G}(\mathbb{C})}{\mathbb{G}(\mathbb{C})}\right]^{\wedge}_{\ell}\sim\mathcal{L}(B\mathbb{G}(\mathbb{C}))^{\wedge}_{\ell}=\left(B\mathbb{G}(\mathbb{C})^{\wedge}_{\ell}\right)^{h\mathrm{id}}=\lim_{v_\ell(q-1)\to\infty}\left(B\mathbb{G}(\mathbb{F}_q)\right)^{\wedge}_{\ell}$$