Finite groups as algebraic groups in defining characteristic

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September 30, 2020

G finite group, p prime, P Sylow p-subgroup

p-local group theory: study of G via p-local subgroups $N_G(Q)$, $1 \neq Q \leq P$

Main method for the classification of finite simple groups ($p=2$)

Examples Brauer-Fowler 1955: G simple, $s \in G$ involution $\Rightarrow |G| \leq \left(2|C_G(s)|^2\right)!$

Brauer-Suzuki 1959: P quaternion group $\Rightarrow G$ not simple

Burnside 1897: P abelian, $x, y \in P$ conjugate in $G \Rightarrow$ conjugate in $N_G(P)$

Alperin 1967: Can tell if $x, y \in P$ are conjugate in G using p-local subgps.

p-local representation theory: study of rep. of G over $\mathbb{Z}_p$, $\mathbb{F}_p$ (or extensions)

using p-local subgps and their rep.

Alperin's conjecture (1986) $k=\overline{\mathbb{F}}_p$. $kG = (kG)' \oplus \overbrace{\text{semi-simple}}^{\text{maximal}}$ $B_0(kG)$: p^{al} block

$$\# \operatorname{Irr} B_0(kG) \overset{?}{=} \# \{(Q,V) \mid Q \text{ } p\text{-subgp of } G, V \text{ proj. simple } kN_G(Q)/Q\text{-module in } B_0(kN_G(Q))\text{-mod}\}/G$$

$\# \operatorname{Irr} B_0(kG)'$ determined locally?

Modelled on $G = \mathbb{G}(\mathbb{F}_{p^r})$ finite group of Lie type, p = defining char.

Broué's conjecture (1988). Assume P is abelian

$D^b((kG)'\text{-mod})$ is determined locally? $D^b(B_0(kG)) \overset{?}{\simeq} D^b(B_0(kN_G(P)))$

Modelled on $G = \mathbb{G}(\mathbb{F}_{\ell^r})$ finite group of Lie type, $p \neq$ defining char:

Equivalence from complex of chains of Deligne-Lusztig variety?

Broué $\Rightarrow$ Alperin. Problem: Extension of Broué's conjecture to non-abelian P?

Derived cat. is not locally determined in general.

Reformulation of Alperin's conj. (Knörr-Robinson)

$\mathcal{S}$: simplicial complex associated with poset of non-trivial subgps of G

n-simplices = chains $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$

$$\sum_{c \in \mathcal{S}/G} (-1)^{|c|} \# \mathrm{Irr}\, B_0(kG_c)' \overset{?}{=} 0$$

Can replace $\mathcal{S}$ by subcomplex with Q_i's elem. abelian, or $O_p N_G(Q_i) = Q_i$

Reconstruct something related to $D^b(B_0(kG)')$ by gluing $D^b(B_0(kG_c)')$'s ?

$H^*(G,k)$ is determined locally (P abelian: $H^*(G,k) = H^*(N_G(P),k) = H^*(P,k)^{N_G(P)}$)

Neighborhood of triv. rep determined locally / $(BG)^\wedge_p$ determined locally

Finite groups of Lie type, defining characteristic

$\mathbb{G}$ connected simple and simply connected alg. group over $\overline{\mathbb{F}}_p$. $\mathbb{T}$ max torus $\subset \mathbb{B}$ Borel

$G := \mathbb{G}(\mathbb{F}_q)$ finite group of Lie type, $q = p^{\nu}$. $\mathbb{U}(\mathbb{F}_q)$ = Sylow p-subgroup

$$\begin{tikzcd}
X^+ \arrow[r, "\sim"] & \mathrm{Irr}(\mathbb{G}) \arrow[d, "\mathrm{Res}"] & \\
\left\{ \lambda \in X^+ \,\middle|\, \begin{array}{l} \langle \lambda, \alpha \rangle < q \\ \forall \alpha \in S := \{\text{simple roots}\} \end{array} \right\} = X_q^+ \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Irr}(kG) \arrow[r, "\sim", "\text{Alperin}"'] & \{(Q,V) \mid Q \le G,\ V \text{ proj; simple } kN_G(Q)/Q\text{-mod}\}/G \\
\coprod_{I \subset S} \mathrm{Irr}(T/T \cap [L_I, L_I]) \arrow[r, "\sim"] & \coprod_{I \subset S} \{\text{Proj simple } k\,P_I/U_I\text{-modules}\} \arrow[ur, "\sim"] &
\end{tikzcd}$$

$$\left(\begin{array}{c} \exists V \\ \Downarrow \\ O_p N_G(Q) = Q \\ \Downarrow \\ N_G(Q) \text{ parabolic} \end{array} \right)$$

$$\# \mathrm{Irr}(kG) = q^{\mathrm{rank}(G)} = \sum_{I \subset S} (q-1)^{|S \setminus I|}$$

Group of $\mathbb{F}_q$-rank 1

More general setting: $G \circlearrowleft \sigma$, σ^d Frobenius giving $\mathbb{F}_q$-structure

$A_1(q) = SL_2(q)$, ${}^2A_2(q) = SU_3(q)$, ${}^2B_2(2^{2f+1})$ Suzuki group, ${}^2G_2(3^{2f+1})$ Ree group

Sylow p-subgroups are Trivial Intersection: $P \cap gPg^{-1} = 1$ if $g \notin N_G(P)$.

$$\Rightarrow (kG)'\text{-mod}/_{\text{proj}} \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} (kB)'\text{-mod}/_{\text{proj}} \quad \text{by Mackey's formula}$$

$$\begin{array}{ccc} \| & & \| \\ D^b((kG)')/_{\text{perfect}} & \overset{\sim}{\rightleftarrows} & D^b((kB)')/_{\text{perfect}} \\ \uparrow & & \uparrow \\ D^b((kG)') & \overset{?}{\underset{\sim}{\dashrightarrow}} & D^b((kB)') \end{array}$$

- Presumably no for ${}^2A_2, {}^2B_2, {}^2G_2$ (Known for ${}^2B_2$; for ${}^2G_2$: $Z((kG)') \not\cong Z((kB)')$)
  same ${}^2A_2$ for some q's.
- Yes for $SL_2(q)$, Okuyama 2000.

For $q=p$, equiv $\quad L(2m) \mapsto \begin{cases} \mathrm{Res}_B^G L(2m) & 0 \leq m < (p-2)/4 \\ \Omega^{-1}(\mathrm{Res}_B^G L(2m))[-1] & \text{oth.} \end{cases}$

$\Omega^{-1}(M) := \text{Inj.hull}(M)/_M$

Rational representations (Arkhipov-Bezrukavnikov-Ginzburg)

$p >$ Coxeter number
R : root lattice

$$
\begin{tikzcd}
D^b(B) \arrow[r, "R\mathrm{Ind}_B^G"] & D^b(G) \\
\langle R_\mu \rangle_{\mu \in pR} \arrow[u, hook] \arrow[r, "\sim"] \arrow[d, "\sim"] & D^b(B_0(G)) \arrow[u, hook] \\
D^b_G(\tilde{N}\text{-dg}) & \\
D^b(\mathrm{Perv}_{\substack{\text{Iwahori}\\ \text{constr.}}}(Gr_{G^\vee})) \arrow[u, "\sim"] &
\end{tikzcd}
$$

quantum case

perverse : filtration by $\angle(\lambda)$, $\lambda < \lambda_0$,
shift $\angle(\lambda)$ by $[-\ell(\omega)]$, $\omega \cdot \lambda \in pX$, $\omega \in W$

SL_2 : $\quad R_{pm} \longmapsto \begin{cases} \nabla(pm) & m \geq 0 \\ \Delta(-pm-2)[-1] & m < 0 \end{cases}$

mod-p local Langlands

Rep of $GL_2(\mathbb{Q}_p)$ over k $\xrightarrow{\text{Res}}$ Rep of $\mathbb{B}(\mathbb{Q}_p)$ over k

(smooth, finite length, comp. fact. have a central char.)

Berger: (injective on K_0)

$\updownarrow$ Colmez

(φ, Γ)-modules

$\downarrow$ Fontaine

$\{\text{blocks}\} \xrightarrow[\text{Breuil, Paskunas}]{\sim}$ semi-simple 2-dim rep of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over k

Other groups: too many mod p irr. rep.