Broué's Abelian Defect Group Conjecture I

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All rings are assumed unital. If R is a ring then we take the term R-*module* to mean a finitely generated left R-module. We will denote by $R-\mathbf{mod}$ the category of all R-modules and by $\mathrm{Irr}(R)$ the isomorphism classes of all simple R-modules. If M is an R-module then we denote by $[M]$ its isomorphism class. The Grothendieck group $\mathcal{G}_0(R)$ of $R-\mathbf{mod}$ is a quotient of the free group generated by $\{[M] \mid M \in R-\mathbf{mod}\}$.

There are a number of standard references for the modular representation theory of finite groups. A recent comprehensive treatment is given in the two-volume text by Linckelmann [Lin18a; Lin18b]. An extremely comprehensive encyclopaedic text, also covering many other areas, is the multi-volume texts by Curtis–Reiner [CR06; CR81; CR87]. For an approach considering only kG-modules, see the book by Alperin [Alp86]. The texts by Isaacs [Isa06] and Navarro [Nav98] focus on (Brauer) characters. Finally the text by Nagao–Tsushima [NT89] is a relevant all rounder and the text by Lux–Pahlings [LP10] offers a more computational viewpoint.

1. **Blocks**

Let k be a field of characteristic $\ell \geqslant 0$. A k-algebra A will be assumed unital and finite dimensional as a k-module. If B is another k-algebra then M is said to be an (A, B)-bimodule if it is a left A-module and a right B-module such that $a(mb) = (am)b$ for all $a \in A$, $b \in B$, and $m \in M$. By convention, we assume that $cm = (c1_A)m = m(c1_B) = mc$ for all $c \in k$ and $m \in M$, where $1_A \in A$ and $1_B \in B$ are the units of the algebra.

Being an (A, B)-bimodule is then the same as being an $A \otimes_k B^{\mathrm{op}}$-module where B^{op} is the opposite algebra. Hence, usual module theoretic terms can be applied to bimodules.

Proposition 1.1. *For a k-algebra A the following are equivalent:*

(i) *A decomposition $A = A_1 \oplus \cdots \oplus A_r$ into indecomposable (A, A)-bimodules.*

(ii) *A decomposition $1 = e_1 + \cdots + e_r$ of the unit in A as a sum of primitive central idempotents $e_i \in Z(A)$ satisfying $e_ie_j = \delta_{i,j}e_i$ where $\delta_{i,j}$ is the Kronecker delta.*

We call $\mathrm{Bl}(A) = \{e_i \mid 1 \leqslant i \leqslant r\}$ the blocks of A and $A_i = Ae_i$ the corresponding block algebra.

If M is an A-module then $M = 1M = \sum_{e\in\mathrm{Bl}(A)} eM$ and this sum is direct. If S is a simple (or indecomposable) A-module then we must have $eS \neq \{0\}$ for a unique block and $S = eS$ for this block. If $\mathrm{Irr}(A \mid e) = \{[S] \in \mathrm{Irr}(A) \mid eS \neq \{0\}\}$ then we get a corresponding partition $\mathrm{Irr}(A) = \bigsqcup_{e\in\mathrm{Bl}(A)} \mathrm{Irr}(A \mid e)$. We have a similar partition of the isomorphism classes of indecomposable modules.

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Example 1.2. Assume G is a finite group and $k = \mathbb{C}$. To each simple kG-module S we have a corresponding character $\chi_S : kG \to k$ defined by taking the trace $\chi_S(g) = \mathrm{Tr}(g \mid S)$. The element $e_S = \frac{\chi_S(1)}{|G|} \sum_{g \in G} \chi_S(g^{-1})g \in kG$ is a block of kG with $\mathrm{Irr}(kG \mid e_S) = \{[S]\}$. Moreover, the block algebra $e_S kG \cong \mathrm{Mat}_{\chi(1)}(k)$ is a matrix algebra and this construction gives all blocks $\mathrm{Bl}(kG) = \{e_S \mid [S] \in \mathrm{Irr}(kG)\}$.

Example 1.3. If $\ell > 0$ and G is an ℓ-group then kG has a unique simple module, namely the trivial module. Hence, in this case there is only one block so kG is indecomposable as a (kG, kG)-bimodule. As we will see below this can also happen for groups that are not ℓ-groups.

The next examples are classical, see [Lin18a, §1.13] for instance.

Example 1.4. Assume $n \geqslant 1$ is an integer and D is a division ring with $Z(D)$ an extension field of k. Then D is a k-algebra and the k-algebra $A = \mathrm{Mat}_n(D) \cong D \otimes_k \mathrm{Mat}_n(k)$ is a simple, hence indecomposable, (A, A)-bimodule. If S is the set of column vectors of length n with entries in D then this is naturally a left A module and we have $\mathrm{Irr}(A) = \{[S]\}$ and $D \cong \mathrm{End}_A(S)^{\mathrm{op}}$.

Example 1.5. Assume A is a semisimple k-algebra and $\mathrm{Irr}(A) = \{[S_1], \dots, [S_r]\}$. Wedderburn's Theorem states that we have a decomposition of $A = A_1 \oplus \cdots \oplus A_r$ into blocks such that $A_i \cong \mathrm{Mat}_{n_i}(D_i)$ where $D_i = \mathrm{End}_A(S_i)^{\mathrm{op}}$ is a division ring. This implies $A \cong \bigoplus_{i=1}^r S_i^{n_i}$ as a left A-module.

Recall that a k-algebra A is said to be *split* if $\mathrm{End}_A(S) \cong k$ for any simple module S. By the above example a split semisimple algebra is necessarily isomorphic to $\bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(k)$. An algebra will split over a finite extension k' of k (recall our algebras are finite dimensional). Specifically, an extension field k' of k is said to be a *splitting field* for A if the k'-algebra $k' \otimes_k A$ is split.

Split semisimple algebras are the easiest examples of k-algebras. The following gives a criterion, internal to A, which characterises when A is semisimple over a splitting field, see [Lin18a, Prop. 1.16.19] and its proof.

Proposition 1.6. *If A is a k-algebra then the following are equivalent:*

(i) *A is projective as an (A, A)-bimodule,*

(ii) *there exists a finite field extension k'/k for which the k'-algebra $k' \otimes_k A$ is split semisimple.*

Remark 1.7. An algebra satisfying condition (i) of this proposition is usually said to be *separable*. The matrix algebra $\mathrm{Mat}_n(D)$, with D a division ring, is simple but may not be separable if the field extension $Z(D)/k$ is not separable.

2. Defect Groups

From this point forward G is a finite group and kG is the corresponding group algebra.

Let $H \leqslant G$ be a subgroup. We denote by $\mathrm{Ind}_H^G : kH-\mathbf{mod} \to kG-\mathbf{mod}$ and $\mathrm{Res}_H^G : kG-\mathbf{mod} \to kH-\mathbf{mod}$ the usual induction and restriction functors between group algebras so that $\mathrm{Ind}_H^G(M) = kG \otimes_{kH} M$.

Definition 2.1. An indecomposable module $M \in kG-\mathbf{mod}$ is said to be *relatively* H-*projective* if M is a direct summand of $\mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$.

Remark 2.2. Equivalently we have M is a direct summand of $\mathrm{Ind}_H^G(V)$ for some module $V \in kH-\mathbf{mod}$, see [Alp86, III.9, Prop. 1].

A subgroup $Q \leqslant G$ is said to be a *vertex* for the indecomposable module M if M is relatively Q-projective and Q is minimal with respect to this property. The following gives the fundamental properties of vertices, see [Alp86, III.9, Thm. 4] or [Lin18a, Thm. 5.1.2].

Proposition 2.3. *Assume* $M \in kG-\mathbf{mod}$ *is indecomposable.*

(i) *Any vertex of* M *is an* ℓ*-subgroup of* G.

(ii) *Any two vertices of* M *are conjugate in* G.

A vertex of M measures how far the indecomposable module M is from being projective. Indeed, the *Projective Indecomposable Modules (PIMs)* of kG are exactly the direct summands of kG viewed as a kG-module. Now if $H \leqslant G$ is the trivial subgroup then

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)) \cong (kG)^{\dim(M)}$$

for any kH-module M. Hence, it follows that M is relatively H-projective if and only if M is projective. Thus the indecomposable modules with vertex the trivial group are precisely the PIMs.

We now want to apply these ideas to the blocks of kG to give some measure of the complexity of a block. If $B \subseteq kG$ is a block then this is a $kG \otimes_k (kG)^{\mathrm{op}}$-module. The inversion map $G \to G$ extends to a k-algebra isomorphism $(kG)^{\mathrm{op}} \to kG$. Hence, we have an isomorphism of k-algebras $kG \otimes_k (kG)^{\mathrm{op}} \cong kG \otimes_k kG \cong k[G \times G]$. Thus B is an indecomposable $k[G \times G]$-module so has a vertex which can be described as follows, see [Alp86, IV.13, Thm. 4].

Proposition 2.4. *If* $e \in \mathrm{Bl}(kG)$ *is a block then* kGe*, as an indecomposable* $k[G \times G]$*-module, has a vertex of the form* $\Delta D := \{(g, g) \mid g \in D\}$ *for some* ℓ*-subgroup* $D \leqslant G$.

Remark 2.5. Note that two subgroups ΔD_1 and ΔD_2 are $G \times G$-conjugate if and only if D_1 and D_2 are G-conjugate. So the subgroup D in Proposition 2.4 is unique up to G-conjugacy.

Notation. If $n > 0$ is an integer then we denote by $\nu(n) = \nu_\ell(n)$ the largest integer $a \geqslant 0$ such that $n \in \ell^a\mathbb{Z}$.

Definition 2.6. An ℓ-subgroup $D \leqslant G$ is called a *defect group* of $e \in \mathrm{Bl}(kG)$ if ΔD is a vertex of the block algebra kGe viewed as a $k[G \times G]$-module. We call the integer $d(e) := \nu(|D|)$ the *defect* of the block.

One should view the defect groups as a measure of complexity of the block. For example, if kGe is split then $d(e) = 0$ if and only if kGe is semisimple by Proposition 1.6. At the other extreme there is a unique block $e_0 \in \mathrm{Bl}(kG)$, the *principal block* of kG, which contains the trivial module. This block has full defect $d(e_0) = \nu(|G|)$ so the defect groups of the principal block e_0 are the Sylow ℓ-subgroups of G. Therefore, we should consider the principal block to be one of the most complicated blocks of kG.

Notation. If $D \leqslant G$ is an ℓ-subgroup then we denote by $\mathrm{Bl}(kG \mid D) \subseteq \mathrm{Bl}(kG)$ those blocks whose defect group is G-conjugate to D.

Remark 2.7. It is not true that every ℓ-subgroup of G is necessarily the defect group of a block. Let $D \leqslant G$ be an ℓ-subgroup and $P \leqslant G$ a Sylow ℓ-subgroup containing D. Then two necessary conditions for D to be the defect group of a block are as follows:

(i) $D = P \cap {}^gP$ for some $g \in G$, see [Alp86, IV.13, Thm. 6],

(ii) $D = O_\ell(N_G(D))$ which is to say D is an *ℓ-radical subgroup* of G so that D is the largest normal ℓ-subgroup of $N_G(D)$, see [NT89, Chap. 5, Ex. 2.17].

Note neither of these statements are particularly easy to see.

3. Decomposition Maps

In general, directly computing the decomposition of the algebra kG into block algebras is very hard. However, one of the important features of the representation theory of finite groups is that we can pass, bi-directionally, between representations in characteristic zero and those of positive characteristic. To do this requires a little ring theory, we refer to [Jan73; Ser79] for more details.

We fix a Discrete Valuation Ring (DVR) $\mathcal{O}$ of characteristic zero. This is a local PID, which is not a field, so it has a unique maximal ideal $\mathfrak{l} \subsetneq \mathcal{O}$. The ideal $\mathfrak{l}$ defines a valuation on $\mathcal{O}$ called the $\mathfrak{l}$-adic valuation. If the quotient field K of $\mathcal{O}$ has characteristic zero and the residue field $k = \mathcal{O}/\mathfrak{l}$ has characteristic $\ell > 0$ then we call the triple $(K, \mathcal{O}, k)$ an *ℓ-modular system*. To visualise this we have the following commutative diagram of ring homomorphisms

$$\begin{array}{ccccc} \mathbb{Q} & \hookleftarrow & \mathbb{Z} & \twoheadrightarrow & \mathbb{F}_\ell \\ \downarrow & & \downarrow & & \downarrow \\ K & \hookleftarrow & \mathcal{O} & \twoheadrightarrow & k \end{array}$$

Example 3.1. If we fix a prime integer $\ell > 0$ then the localisation $\mathbb{Z}_{(\ell)}$ of $\mathbb{Z}$ at the prime ideal $(\ell) = \ell\mathbb{Z}$ is a DVR with maximal ideal $\mathfrak{l} = \ell\mathbb{Z}_{(\ell)}$. Its quotient field is $\mathbb{Q}$ and its residue field is the finite field $\mathbb{Z}_{(\ell)}/\mathfrak{l} \cong \mathbb{Z}/(\ell) = \mathbb{F}_\ell$. Hence we get the ℓ-modular system $(\mathbb{Q}, \mathbb{Z}_{(\ell)}, \mathbb{F}_\ell)$. We almost always want to work in the completion of the modular system defined with respect to the $\mathfrak{l}$-adic valuation. Completing in this case gives the ℓ-modular system $(\mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell)$ where $\mathbb{Z}_\ell = \hat{\mathbb{Z}}_{(\ell)}$ is the ring of ℓ-adic integers.

Example 3.2. We can generalise this example as follows. Let $K \supseteq \mathbb{Q}$ be an algebraic number field, such as a cyclotomic field $\mathbb{Q}(\zeta)$ with ζ a root of unity. The ring of algebraic integers $\mathfrak{O} \subseteq K$ is a Dedekind domain so the ideal $(\ell)\mathfrak{O}$ splits as a product $\mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_g^{e_g}$ of prime ideals. The $\mathfrak{l}_i$ are exactly the prime ideals containing (ℓ) and the localisation $\mathfrak{O}_{\mathfrak{l}_i}$ at $\mathfrak{l}_i$ is a DVR with maximal ideal $\mathfrak{l}_i\mathfrak{O}_{\mathfrak{l}_i}$ and residue field $k_i = \mathfrak{O}_{\mathfrak{l}_i}/\mathfrak{l}_i\mathfrak{O}_{\mathfrak{l}_i} \cong \mathfrak{O}/\mathfrak{l}_i$. We have k_i is a finite field of characteristic ℓ and the triple $(K, \mathfrak{O}_{\mathfrak{l}_i}, k_i)$ is an ℓ-modular system.

Remark 3.3. Let $f_i = [k_i : \mathbb{F}_\ell]$. In the setting of Example 3.2 the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ permutes transitively the ideals occurring in the decomposition of $(\ell)\mathfrak{O}$ and if $e := e_1 = \cdots = e_g$ and $f := f_1 = \cdots = f_g$ then $efg = [K : \mathbb{Q}]$, see [Jan73, Thm. 6.8]. Note, even though all choices are Galois conjugate our choice of prime ideal can still cause issues computationally.

The ring homomorphisms between K, $\mathcal{O}$, and k, extend naturally to ring homomorphisms between the corresponding group algebras

$$KG \longleftarrow \mathcal{O}G \longrightarrow\!\!\!\!\!\rightarrow kG$$

We write the projection map as $\bar{\ }: \mathcal{O}G \to kG$. Thus if V is an $\mathcal{O}G$-module we get a KG-module $V^K := K \otimes_{\mathcal{O}} M$ and a kG-module $\overline{V} := k \otimes_{\mathcal{O}} M$.

As an $\mathcal{O}$-module the $\mathcal{O}G$-module V may have torsion, which is lost on passage to K. Usually we want to avoid this so we typically assume that V is an *$\mathcal{O}G$-lattice*, which means V is free as an $\mathcal{O}$-module of finite rank. Equivalently, we could say that V is projective, or torsion free, as an $\mathcal{O}$-module because $\mathcal{O}$ is a PID, see [Lin18a, Thm. 4.2.4].

Lemma 3.4. *If M is a KG-module then there exists an $\mathcal{O}G$-lattice V such that the product map $V^K \to M$ is an isomorphism of KG-modules.*

Proof. If $\{v_1, \dots, v_n\} \subseteq M$ is a K-basis of M then take $V = \mathcal{O}Gv_1 + \cdots + \mathcal{O}Gv_n \subseteq M$. ∎

This now gives a method for going from characteristic zero to characteristic $\ell > 0$. If M is a KG-module we pick an $\mathcal{O}G$-lattice as in Lemma 3.4 and then consider the reduction $\overline{V}$ modulo $\mathfrak{l}$. There is an issue here that the choice of $\mathcal{O}G$-lattice V is far from unique. Moreover, if V_1, V_2 are two $\mathcal{O}G$-lattices such that $V_1^K \cong V_2^K$ then it may happen that $\overline{V}_1 \not\cong \overline{V}_2$. However this issue is removed by considering only composition series, see [CR81, Prop. 16.16].

Proposition 3.5. *We have a well-defined group homomorphism* $\mathrm{dec}_{\mathcal{O}G} : \mathcal{G}_0(KG) \to \mathcal{G}_0(kG)$ *defined by setting* $\mathrm{dec}_{\mathcal{O}G}([V^K]) = [\overline{V}]$ *for any $\mathcal{O}G$-lattice V.*

If $[S] \in \mathrm{Irr}(KG)$ is the isomorphism class of a simple KG-module then for each isomorphism class $[T] \in \mathrm{Irr}(kG)$ there exists a non-negative integer $d_{[S][T]} \geqslant 0$ such that

$$\mathrm{dec}_{\mathcal{O}G}([S]) = \sum_{[T] \in \mathrm{Irr}(kG)} d_{[S][T]}[T].$$

These integers are encoded in a matrix $D = (d_{[S][T]})$ called the *decomposition matrix*. This matrix is extremely challenging to calculate in general. If this matrix is known then one can calculate the dimensions of all simple kG-modules, which remains a difficult open problem. We end by noting that the number of rows and columns of this matrix is understood, see [CR06, Thm. 83.5, Cor. 83.7].

Theorem 3.6. *Assume* KG *is split semisimple. Then* kG *is also split semisimple and* $|\mathrm{Irr}(kG)| \leqslant |\mathrm{Irr}(KG)|$. *More precisely we have*

(i) $|\mathrm{Irr}(KG)|$ *is the number of conjugacy classes in* G,

(ii) $|\mathrm{Irr}(kG)|$ *is the number of* ℓ*-regular conjugacy classes in* G.

(Recall an element $g \in G$ *with order* $o(g)$ *is said to be* ℓ*-regular if* $\nu(o(g)) = 0$.*)*

4. Blocks in Characteristic Zero

Let $(K, \mathcal{O}, k)$ be an ℓ-modular system for the finite group G. For block theory one needs to place some additional assumptions on this modular system. Typically we require that $\mathcal{O}$ is complete with respect to its $\mathfrak{l}$-adic valuation. However, one can also work with the assumption that the K-algebra KG is split semisimple. This will be the case if K contains a primitive $|G|$th root of unity. Following Curtis–Reiner [CR87, Def. 56.3] we say that an ℓ-modular system is *admissible* for G if either of these conditions hold. The following is [CR87, Prop. 56.7].

Proposition 4.1 (Idempotent Lifting). *Assume* $(K, \mathcal{O}, k)$ *is an* ℓ*-modular system that is admissible for* G. *Then to each block* $b \in \mathrm{Bl}(kG)$ *there exists a unique central primitive idempotent* $b^* \in Z(\mathcal{O}G)$ *such that* $\overline{b^*} = b$.

> From this point forward we assume that $(K, \mathcal{O}, k)$ is an ℓ-modular system which is large enough such that for any subgroup $H \leqslant G$ we consider KH is split semisimple.

As before the blocks $\mathrm{Bl}(\mathcal{O}G) = \{b^* \mid b \in \mathrm{Bl}(kG)\}$ of $\mathcal{O}G$ are the central primitive idempotents. As for the k-algebra kG we have decompositions

$$\mathcal{O}G = \bigoplus_{b \in \mathrm{Bl}(kG)} \mathcal{O}Gb^* \qquad \rightsquigarrow \qquad KG = \bigoplus_{b \in \mathrm{Bl}(kG)} KGb^*$$

as a direct sum of block algebras. In turn this gives a partition of the isomorphism classes of simple KG-modules $\mathrm{Irr}(KG) = \bigsqcup_{b \in \mathrm{Bl}(kG)} \mathrm{Irr}(KG \mid b)$ into ℓ-blocks. We now wish to show how we can calculate this partition.

Recall that if M is a KG-module then we have a corresponding character $\chi_M : KG \to K$ defined by taking the trace $\chi_M(g) = \mathrm{Tr}(g \mid M)$ at $g \in KG$. If $\mathrm{Class}_K(G) = \{f : G \to K \mid f(x^{-1}gx)$ for all $x, g \in G\}$ then we have an injective group homomorphism $\mathcal{G}_0(KG) \to \mathrm{Class}_K(G)$ satisfying $[M] \mapsto \chi_M$ for any $M \in KG-$**mod**. In fact, extending scalars, this defines an isomorphism $K \otimes_{\mathbb{Z}} \mathcal{G}_0(KG) \to \mathrm{Class}_K(G)$ of K-vector spaces. We will make this identification implicitly moving forward.

Central Characters

Suppose $\chi \in \mathrm{Irr}(KG)$ is an irreducible character afforded by the simple module $S \in KG-\mathbf{mod}$. As we assume KG is split we have $\mathrm{End}_{KG}(S) \cong K$. Hence we have a corresponding K-algebra homomorphism $\omega_\chi : Z(KG) \to K$ defined such that $zs = \omega_\chi(z)s$ for all $z \in Z(KG)$ and $s \in S$. In terms of χ we have

$$\omega_\chi(z) = \frac{\chi(z)}{\chi(1)}.$$

We call ω_χ the *central character* of χ. As one might expect the central character plays a role in determining the blocks.

For any subset $X \subseteq G$ we denote by $\widehat{X} = \sum_{x \in X} x \in KG$ with $\widehat{\varnothing} = 0$. If $g \in G$ we let $g^G = \{x^{-1}gx \mid x \in G\}$ be the corresponding conjugacy classes. If $\mathrm{Cl}(G)$ is the set of all conjugacy classes of G then the centre $Z(KG)$ of the group algebra has a K-basis given by the class sums $\{\widehat{C} \mid C \in \mathrm{Cl}(G)\}$. If $C = g^G \in \mathrm{Cl}(G)$ then

$$\omega_\chi(\widehat{C}) = \frac{|C|\chi(g)}{\chi(1)} \in \mathcal{O}.$$

The following shows that the partition $\mathrm{Irr}_K(G) = \bigsqcup_{b \in \mathrm{Bl}(kG)} \mathrm{Irr}_K(G \mid b)$ can be determined from the central characters, see [CR87, Cor. 56.24].

Theorem 4.2. *Two characters $\chi, \psi \in \mathrm{Irr}(KG)$ belong to the same kG-block if and only if*

$$\omega_\chi(\widehat{C}) \equiv \omega_\psi(\widehat{C}) \pmod{\mathfrak{l}}$$

for all $C \in \mathrm{Cl}(G)$.

Blocks and Decomposition Matrices

The blocks can also be seen in the decomposition matrix. We define a relation on $\mathrm{Irr}(KG)$ by setting $[S_1] \sim [S_2]$ if there exists $[T] \in \mathrm{Irr}(kG)$ for which $d_{[S_1][T]} \neq 0$ and $d_{[S_2][T]} \neq 0$. The sets $\mathrm{Irr}(KG \mid e)$, with $e \in \mathrm{Bl}(kG)$, are then the equivalence classes for the *transitive closure* of the relation $\sim$.

Projectives

If $V \in \mathcal{O}G-\mathbf{mod}$ is a projective $\mathcal{O}G$-module then the modular reduction $\overline{V} \in kG-\mathbf{mod}$ is a projective kG-module. Moreover, if V is indecomposable then so is $\overline{V}$. The map $V \mapsto \overline{V}$ defines a bijection between isomorphism classes of projective modules sending indecomposable modules to indecomposable modules.

Now suppose $V \in \mathcal{O}G-\mathbf{mod}$ is a *PIM*, as in Section 2, and let $T = \overline{V}/\mathrm{Rad}(\overline{V})$ be the head of $\overline{V} \in kG-\mathbf{mod}$, which is a simple kG-module. If $\Phi_{[T]}$ is the character of the KG-module V^K then, as a consequence of Brauer reciprocity, we have

$$\Phi_{[T]} = \sum_{[S] \in \mathrm{Irr}(KG)} d_{[S][T]}[S],$$

see [CR87, 56.26]. We will refer to $\Phi_{[T]}$ as the character of the PIM. Once the decomposition matrix is known then all characters of PIMs can be determined.

Remark 4.3. If $V \in \mathcal{O}G-\mathbf{mod}$ is a projective module then V must be a direct sum of PIMs so we have $\chi_{V^K} = \sum_{[T]\in\mathrm{Irr}(kG)} m_{[T]}\Phi_{[T]}$ with $m_{[T]} \geqslant 0$ an integer.

5. An Example: The group $\mathfrak{A}_5$

g	$()$	$(12)(34)$	(123)	(12345)	(12354)
$\lvert g^G \rvert$	1	15	20	12	12
χ_1	1	1	1	1	1
χ_4	4	.	1	-1	-1
χ_5	5	1	-1	.	.
$\chi_{3,+}$	3	-1	.	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\chi_{3,-}$	3	-1	.	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Table 5.1: Character Table of $\mathfrak{A}_5$

Let $G = \mathfrak{A}_5$ be the alternating group of order $60 = 2^2 \cdot 3 \cdot 5$ which we consider as a subgroup of the symmetric group $\mathfrak{S}_5$. The ordinary character table of G is given in Table 5.1. The columns represent the conjugacy classes and are labelled by a representing element of the class. The rows give the irreducible characters. The two characters of degree 3 are distinguished by their value at the 5-cycle (12345).

g	$()$	$(12)(34)$	(123)	(12345)	(12354)
ω_1	1	15	20	12	12
ω_4	1	.	5	-3	-3
ω_5	1	3	-4	.	.
$\omega_{3,+}$	1	-5	.	$2(1+\sqrt{5})$	$2(1-\sqrt{5})$
$\omega_{3,-}$	1	-5	.	$2(1-\sqrt{5})$	$2(1+\sqrt{5})$

Table 5.2: Central Character Table of $\mathfrak{A}_5$

The quadratic field $\mathbb{Q}(\sqrt{5})$ is a splitting field for $G = \mathfrak{A}_5$. For each prime integer $\ell \mid |G|$ we pick a prime ideal $\mathfrak{l} \subseteq \mathfrak{O}$ containing $\ell > 0$ and work with the admissible modular system $(K, \mathfrak{O}_\mathfrak{l}, k)$ where $k = \mathfrak{O}_\mathfrak{l}/\mathfrak{l}\mathfrak{O}_\mathfrak{l}$ which we identify with $\mathfrak{O}/\mathfrak{l}$. We give the blocks, defect groups, and decomposition matrices in each case. We also describe the normaliser $N_G(P)$ of a Sylow ℓ-subgroup $P \leqslant G$ and its blocks.

Quadratic Fields

To do these calculations we need to understand how to choose $\mathfrak{l}$. For this we recall a little elementary algebraic number theory. Let $p > 0$ be an *odd* prime and let $\varepsilon(p) = (-1)^{(p-1)/2}$. Then $L = \mathbb{Q}(\sqrt{\varepsilon(p)p})$ is a quadratic extension of $\mathbb{Q}$. If $\mathfrak{O}_L \subseteq L$ is the ring of algebraic integers then $\mathfrak{O}_L = \mathbb{Z}[\frac{1+\sqrt{\varepsilon(p)p}}{2}]$, see [Jan73, I, Thm. 9.2]. There are three possibilities for the ideal $(\ell)\mathfrak{O}_L$, see Remark 3.3 and [Jan73, I, Lem. 11.5].

- If $\ell = p$ then $(\ell)\mathfrak{O}_L = (\pi\mathfrak{O}_L)^2$ where $\pi = \sqrt{\varepsilon(p)p}$ so ℓ is *fully ramified* in L and $\mathfrak{O}_L/\pi\mathfrak{O}_L \cong \mathbb{F}_\ell$.
- If $\ell \neq p$ and ℓ is not a quadratic residue modulo p then the ideal $(\ell)\mathfrak{O}_L$ is prime and $\mathfrak{O}_L/(\ell)\mathfrak{O}_L \cong \mathbb{F}_{\ell^2}$.
- If $\ell \neq p$ and ℓ is a quadratic residue modulo p then the ideal $(\ell)\mathfrak{O}_L = \mathfrak{l}_1\mathfrak{l}_2$ splits as a product of two distinct prime ideals and $\mathfrak{O}_L/\mathfrak{l}_i \cong \mathbb{F}_\ell$.

Exercise 5.1. Let $g(X) = X^2 - X + \frac{1-\varepsilon(p)p}{4} \in \mathbb{Z}[X]$, which is the minimal polynomial of $\frac{1+\sqrt{\varepsilon(p)p}}{2}$ over $\mathbb{Q}$. Let $\bar{g}(X) \in \mathbb{F}_\ell[X]$ be the modular reduction of this polynomial. Show that $\mathfrak{O}_L/(\ell)\mathfrak{O}_L \cong \mathbb{F}_\ell[X]/(\bar{g}(X))$. When $p = 5$ use this, together with Remark 3.3, to show that the ideal $(\ell)\mathfrak{O}_L$ is prime when $\ell \in \{2, 3\}$.

The prime 3

We have that 3 is not a quadratic residue modulo 5 so we take $\mathfrak{l} = 3\mathfrak{O}$. Reducing the rows of the central character table in Table 5.2 modulo $3\mathfrak{O}$ we see that there are three blocks of G: the principal block $\{\chi_1, \chi_4, \chi_5\}$ with defect group P, and the two singleton blocks $\{\chi_{3,+}\}$ and $\{\chi_{3,-}\}$ with trivial defect. We have $|\mathrm{Irr}(kG)| = 4$ and the decomposition matrix in this case is

$$\begin{array}{c} \chi_1 \\ \chi_4 \\ \chi_5 \\ \chi_{3,+} \\ \chi_{3,-} \end{array} \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ 1 & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

Hence the dimensions of the simple kG-modules are: 1, 4, 3, and 3.

g	$()$	$(12)(34)$	(123)
$\lvert g^{N_G(P)}\rvert$	1	3	2
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{ref}	2	.	-1

Table 5.3: Character Table of $N_G(P)$ with $\ell = 3$.

We have $P = \langle(123)\rangle$ is a Sylow 3-subgroup of G and its normaliser $N_G(P) = P \rtimes \langle(12)(45)\rangle$ is isomorphic to $\mathfrak{S}_3$. The character table of $N_G(P)$ is as in Table 5.3. There is only one block, the principal block, and $|\mathrm{Irr}(kN_G(P))| = 2$.

The prime 5

Here we are in the fully ramified case so we take $\mathfrak{l} = \pi\mathfrak{O}$ where $\pi = \sqrt{5}$. There are two blocks of G: the principal block $\{\chi_1, \chi_4, \chi_{3,+}, \chi_{3,-}\}$ with defect P, and the singleton

$\{\chi_5\}$ with trivial defect. We have $|\mathrm{Irr}(kG)| = 3$ and the decomposition matrix in this case is

$$\begin{array}{c} \chi_1 \\ \chi_{3,+} \\ \chi_{3,-} \\ \chi_4 \\ \chi_5 \end{array} \left[\begin{array}{cc|c} 1 & . & . \\ . & 1 & . \\ . & 1 & . \\ 1 & 1 & . \\ \hline . & . & 1 \end{array}\right]$$

Hence the dimensions of the simple kG-modules are: 1, 3, and 5.

g $\lvert g^{N_G(P)}\rvert$	$()$ 1	$(12)(34)$ 5	(12345) 2	(12354) 2
χ_{triv}	1	1	1	1
χ_{sgn}	1	-1	1	1
$\chi_{\mathrm{ref},+}$	2	.	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
$\chi_{\mathrm{ref},-}$	2	.	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

Table 5.4: Character Table of $N_G(P)$ with $\ell = 5$.

We have $P = \langle(12345)\rangle$ is a Sylow 5-subgroup of G and its normaliser $N_G(P) = P \rtimes \langle(25)(34)\rangle$ is isomorphic to a dihedral group D_{10} of order 10. The character table of $N_G(P)$ is given in Table 5.4. There is only one block, the principal block, and $|\mathrm{Irr}(kN_G(P))| = 2$.

Exercise 5.2. Assume $G = \mathfrak{A}_5$ and $\ell = 2$. Calculate the partition of $\mathrm{Irr}(KG)$ into ℓ-blocks, a Sylow ℓ-subgroup of G, its normaliser $N_G(P)$, and the partition of $\mathrm{Irr}(KN_G(P))$ into ℓ-blocks.

g $\lvert g^G\rvert$	1a 1	2a 1584	3a 720	6a 720	4a 165	8a 440	8b 1320	5a 990	11a 990	11b 990
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	2	1	-1	2	.	.	.	-1	-1
χ_3	10	-2	1	1	.	$-\sqrt{-2}$	$\sqrt{-2}$	.	-1	-1
χ_4	10	-2	1	1	.	$\sqrt{-2}$	$-\sqrt{-2}$	.	-1	-1
χ_5	11	3	2	.	-1	-1	-1	1	.	.
χ_6	16	.	-2	.	.	.	.	1	$\frac{-1-\sqrt{-11}}{2}$	$\frac{-1+\sqrt{-11}}{2}$
χ_7	16	.	-2	.	.	.	.	1	$\frac{-1+\sqrt{-11}}{2}$	$\frac{-1-\sqrt{-11}}{2}$
χ_8	44	4	-1	1	.	.	.	-1	.	.
χ_9	45	-3	.	.	1	-1	-1	.	1	1
χ_{10}	55	-1	1	-1	-1	1	1	.	.	.

Table 5.5: Character Table of M_{11}.

Exercise 5.3. The ordinary character of the finite simple group $G = M_{11}$ with order $2^4 \cdot 3^2 \cdot 5 \cdot 11$ is given in Table 5.5. The labelling of the conjugacy classes and the table itself are taken from the GAP character table library [Gap]. It is known that the field $K = \mathbb{Q}(\sqrt{-2}, \sqrt{-11}) \cong \mathbb{Q}(\sqrt{-2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-11})$ is a splitting field for G. Using this compute the ℓ-blocks of G. By [Jan73, I, Thm. 9.5] the ring of algebraic integers $\mathfrak{O} \subseteq K$ is $\mathbb{Z}[\sqrt{-2}, \frac{-1+\sqrt{-11}}{2}]$.

6. Brauer's First Main Theorem

Assume $H \leqslant G$ is a subgroup. The subgroup H acts by left and right multiplication on G and these actions preserve the partition $G = H \sqcup (G - H)$ where $G - H = \{g \in G \mid g \notin H\}$. If $k[G-H] := \bigoplus_{x \in G-H} kx$ then we have two decompositions of kG

$$kH \oplus k[G-H] = kG = \bigoplus_{e \in \mathrm{Bl}(kG)} kGe. \tag{6.1}$$

By restriction each block algebra of kG is a (kH, kH)-bimodule so these are direct sum decompositions of (kH, kH)-bimodules.

Every block algebra of kH occurs as a summand of the left hand side. Hence, for each block $b \in \mathrm{Bl}(kH)$ there is at least one block $e \in \mathrm{Bl}(kG)$ such that $be \neq 0$, i.e., the projection of b onto kGe is nonzero. In general, we could have $be \neq 0$ for several kG-blocks $e \in \mathrm{Bl}(kG)$. Moreover, if $e \in \mathrm{Bl}(kG)$ then the projection of e onto the summand kH may be zero. However when $H = N_G(D)$ is the normaliser of an ℓ-subgroup $D \leqslant G$ then the following result of Brauer gives the best possible relationship for those blocks of kH with defect group D, see [Alp86, IV.14, Thm. 2] for a proof in the language we use here.

Theorem 6.2 (Brauer's First Main Theorem). *If $D \leqslant G$ is an ℓ-subgroup then there is a bijection $\mathrm{Bl}(kN_G(D) \mid D) \to \mathrm{Bl}(kG \mid D)$ denoted by $b \mapsto b^G$ such that $b \in kGb^G$.*

The block b^G occurring in Theorem 6.2 is called the *Brauer correspondent* of b or we say b and b^G are *Brauer corresponding* blocks. Note that, despite the suggestive notation, the Brauer correspondence is not given by induction but is closely related to it. It is at the very heart of modular representation theory of finite groups to determine to what extent Brauer corresponding blocks determine each other.

Remark 6.3. Recall that not every ℓ-subgroup of G is necessarily a defect group so $\mathrm{Bl}(kG \mid D)$ may be empty. Of course, $\mathrm{Bl}(kN_G(D) \mid D)$ may also be empty. If D is a Sylow subgroup of $N_G(D)$ then $\mathrm{Bl}(kN_G(D) \mid D) \neq \varnothing$ because it must contain the principal block. Hence, in this case D must be the defect group of a block of kG.

In fact, we can be more precise here. Assume $b \in \mathrm{Bl}(kN_G(D) \mid D)$ is a block. Then by another result of Brauer, known as Brauer's Third Main Theorem [NT89, Chap. 5, Thm. 6.1], we have that if $b^G \in \mathrm{Bl}(kG)$ is the principal block then $b \in \mathrm{Bl}(kN_G(D))$ is the principal block.

Computing the Brauer Correspondence

One remarkable property of the Brauer correspondence is that it can be determined from the ordinary character table of G. This is not the case for the Green correspondence [Alp86, III.11, Thm. 1], of which the Brauer correspondence is a special case. To state the relevant result we associate to each block $e \in \mathrm{Bl}(kG)$ a group homomorphism $\mathrm{Proj}_e : \mathcal{G}_0(KG) \to \mathcal{G}_0(KG)$ by setting

$$\mathrm{Proj}_e\left(\sum_{\chi \in \mathrm{Irr}(KG)} m_\chi \chi\right) = \sum_{\chi \in \mathrm{Irr}(KG|e)} m_\chi \chi.$$

With this we have the following characterisation of the Brauer correspondence at the level of characters, see [NT89, Chap. 5, Cor. 3.2].

Theorem 6.4. *Assume* $D \leqslant G$ *is an* ℓ*-subgroup and* $b \in \mathrm{Bl}(kN_G(D) \mid D)$ *and* $e \in \mathrm{Bl}(kG \mid D)$ *are blocks. If* $\xi \in \mathrm{Irr}(KN_G(D) \mid b)$ *and* $\psi = \mathrm{Ind}_{N_G(D)}^G(\xi)$ *is the induced character then the following hold:*

(i) *If* $e = b^G$ *then* $\nu(\mathrm{Proj}_e(\psi)(1)) = \nu(\psi(1))$,

(ii) *If* $e \neq b^G$ *then* $\nu(\mathrm{Proj}_e(\psi)(1)) > \nu(\psi(1))$.

7. Alperin's Conjecture

In the case where D is abelian Broué's Conjecture predicts a very strong structural correspondence between a block and its Brauer correspondent. We'll take this up in the next few sections. Before doing this we mention a conjecture of Alperin, generalising a conjecture of McKay, that gives some numerical evidence to suggest that Brauer corresponding blocks are very closely related.

Assume $e \in \mathrm{Bl}(kG)$ is a block and $\chi \in \mathrm{Irr}(KG \mid e)$. Then there exists an integer $\mathrm{ht}(\chi) \in \mathbb{Z}$ such that the ℓ-part of the character degree $\chi(1)$ is described by the formula

$$\nu(\chi(1)) = \nu(|G|) - d(e) + \mathrm{ht}(\chi).$$

Here $d(e)$ is the defect of the block defined as in Definition 2.6. Note that $\chi(1) \mid |G|$ so we have $0 \leqslant \nu(\chi(1)) \leqslant \nu(|G|)$. We call the integer $\mathrm{ht}(\chi)$ the *height* of χ. For the following see [Lin18b, Prop. 6.5.12].

Proposition 7.1. *For any character* $\chi \in \mathrm{Irr}(KG)$ *we have* $\mathrm{ht}(\chi) \geqslant 0$.

Remark 7.2. It is known that each block contains a height zero character. For example, if e is the principal block then the trivial character has height 0.

In general when we consider equivalences between blocks, like Morita equivalences, the degrees of irreducible characters will not be preserved. However, the height is a relative measure that can be interpreted somewhat independently of the block and group to which the character belongs. Letting

$$\mathrm{Irr}_0(KG \mid e) = \{\chi \in \mathrm{Irr}(KG \mid e) \mid \mathrm{ht}(\chi) = 0\}$$

be the height zero characters in a block we have the following conjecture of Alperin, see [Alp76].

Conjecture 7.3 (Alperin–McKay Conjecture). *Assume* $D \leqslant G$ *is an* ℓ*-subgroup of* G*. For any block* $b \in \mathrm{Bl}(kN_G(D) \mid D)$ *with defect group* D *we have*

$$|\mathrm{Irr}_0(KN_G(D) \mid b)| = |\mathrm{Irr}_0(KG \mid b^G)|.$$

Let us denote by $\mathrm{Irr}_{\ell'}(KG) = \{\chi \in \mathrm{Irr}(KG) \mid \nu(\chi(1)) = 0\}$ the set of irreducible characters with degree coprime to ℓ. If $P \leqslant G$ is a Sylow ℓ-subgroup then it is clear that we have

$$\mathrm{Irr}_{\ell'}(KG) = \bigsqcup_{b \in \mathrm{Bl}(kG \mid P)} \mathrm{Irr}_0(KG \mid b)$$

are the height zero characters in the blocks of full defect. Applying this remark in $N_G(P)$, together with the Brauer correspondence, we recover the following earlier conjecture of McKay.

Conjecture 7.4 (McKay's Conjecture). *If* $P \leqslant G$ *is a Sylow* ℓ*-subgroup then* $|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N_G(P))|$.

This conjecture is a major focus of the representation theory of finite groups and has recently been solved in the case $\ell = 2$ using the classification of finite simple groups, see [MS16]. It has a number of fascinating generalisations involving Galois automorphisms, local Schur indices, etc., which we will not discuss here.

8. Equivalences Between Blocks: The Morita Case

The strongest possible equivalence relation between rings is given by isomorphism. A weaker equivalence, still preserving representation theoretic data, is given by a Morita equivalence. By Morita's Theorem if A and B are k-algebras then any equivalence $A-\mathbf{mod} \to B-\mathbf{mod}$ is of the form $M \otimes_B -$ where M is an (A, B)-bimodule which is projective as a left A-module and right B-module, see [Lin18a, Thm. 2.8.2].

Morita Equivalences Between Blocks

The first basic question we can ask in the direction of Brauer corresponding blocks is as follows. If $D \leqslant G$ is an ℓ-subgroup and $b \in \mathrm{Bl}(kN_G(D) \mid D)$ is a block then are the block algebras $kN_G(D)b$ and kGb^G Morita equivalent? However, we're interested in preserving both characteristic zero and positive characteristic information simultaneously. So a better question would be the following.

Question 8.1. If $b \in \mathrm{Bl}(\mathcal{O}N_G(D) \mid D)$ is a block then when are the block algebras $\mathcal{O}N_G(D)b$ and $\mathcal{O}Gb^G$ Morita equivalent?

To answer this question we need to know when an $(\mathcal{O}Gb^G, \mathcal{O}N_G(D)b)$-bimodule V gives a Morita equivalence between $\mathcal{O}N_G(D)b$ and $\mathcal{O}Gb^G$. The following result of Broué, see [Bro90a, Thm. 0.2], shows how to reduce this to a computation with characters in characteristic zero, assuming we know our modules are projective.

Theorem 8.2 (Broué). *Assume $e \in \mathrm{Bl}(\mathcal{O}G)$ and $b \in \mathrm{Bl}(\mathcal{O}H)$ are blocks of two finite groups G and H. Let V be an $(\mathcal{O}Ge, \mathcal{O}Hb)$-bimodule that is projective as a left $\mathcal{O}Ge$-module and right $\mathcal{O}Hb$-module. Then the following are equivalent:*

(i) *The functor $V \otimes_{\mathcal{O}Hb} - : \mathcal{O}Hb-\mathbf{mod} \to \mathcal{O}Ge-\mathbf{mod}$ is a Morita equivalence.*

(ii) *The group homomorphism $\mathcal{G}_0(KHb) \to \mathcal{G}_0(KGe)$, given by $[S] \mapsto [V^K \otimes_{KHb} S]$, restricts to a bijection $\mathrm{Irr}(KH \mid b) \to \mathrm{Irr}(KG \mid e)$.*

Remark 8.3. For a generalisation of this to symmetric algebras see [Bro90a, Thm. 2.4].

To do the calculation in (ii) we need to understand what happens when we tensor with a bimodule. So assume G and H are finite groups and M is a (KG, KH)-bimodule. If χ_M is the character of M, viewed as a $(KG \otimes KH)$-module, then we have

$$\chi_M(g,h) = \sum_{[S]\in\mathrm{Irr}(KH)} \chi_{M\otimes_{KH}S}(g)\chi_S(h) \tag{8.4}$$

for any $g \in G$ and $h \in H$. Moreover, for any $g \in G$ we have

$$\chi_{M\otimes_{KH}S}(g) = |H|^{-1}\sum_{h\in H} \chi_M(g,h^{-1})\chi_S(h). \tag{8.5}$$

By the orthogonality relations these decompositions are equivalent. Hence, understanding M as a (KG, KH)-bimodule is equivalent to understanding the corresponding homomorphism $\mathcal{G}_0(KHb) \to \mathcal{G}_0(KGe)$.

Exercise 8.6. Taking the group algebra $M = KG$ as a (KG, KH)-bimodule, under left and right multiplication, obtain from the formula above the usual formula for an induced character.

The Green Correspondence in the TI Case

A subgroup $P \leqslant G$ is said to be *TI* or have the *trivial intersection property* if $P \cap {}^gP$ is either $\{1\}$ or P for each $g \in G$, where ${}^gP = gPg^{-1}$ is the conjugate. Of course, any cyclic subgroup is TI. In the next section we will need the following special case of the Green correspondence, see [Lin18a, Cor. 5.2.4] (for a proof over k see [Alp86, III.10, Thm. 1]). In both cases the result is a consequence of the Mackey formula for induction and restriction.

Theorem 8.7. *Assume $P \leqslant G$ is a Sylow ℓ-subgroup that is TI and let $H \leqslant G$ be a subgroup containing $N_G(P)$. Suppose $U \in \mathcal{O}G-\mathbf{mod}$ and $V \in \mathcal{O}H-\mathbf{mod}$ are* non-projective *indecomposable modules. Then the following hold:*

(i) $\mathrm{Res}_H^G(U) = \phi(U) \oplus Q$ *with* $Q \in \mathcal{O}H-\mathbf{mod}$ *a projective module and* $\phi(U) \in \mathcal{O}H-\mathbf{mod}$ *a non-projective indecomposable module,*

(ii) $\mathrm{Ind}_H^G(V) = \psi(V) \oplus R$ *with* $R \in \mathcal{O}G-\mathbf{mod}$ *a projective module and* $\psi(V) \in \mathcal{O}G-\mathbf{mod}$ *a non-projective indecomposable module.*

Moreover, we have $\psi(\phi(U)) \cong U$ *and* $\phi(\psi(V)) \cong V$.

Remark 8.8. In thinking of generalisations of this result one should replace the condition "non-projective" with "has a non-trivial vertex".

An Example: The Principal Block of $G = \mathfrak{A}_5$ *when* $\ell = 3$

We assume now that $G = \mathfrak{A}_5$ and $\ell = 3$. Let $P = \langle(123)\rangle$ and set $H = N_G(P) = P \rtimes \langle(12)(45)\rangle \cong \mathfrak{S}_3$. The characters of the PIMs for G and H are given by

$$\Phi_1 = \chi_1 + \chi_5 \qquad \Phi_{\mathrm{triv}} = \chi_{\mathrm{triv}} + \chi_{\mathrm{ref}}$$
$$\Phi_4 = \chi_4 + \chi_5 \qquad \Phi_{\mathrm{sgn}} = \chi_{\mathrm{sgn}} + \chi_{\mathrm{ref}}.$$

Let $e \in \mathrm{Bl}(\mathcal{O}G)$ be the principal block. Then the character of $e\mathcal{O}G$ is $\Phi_1 + 4\Phi_4$. There is only one block of $\mathcal{O}H$ and its character is $\Phi_{\mathrm{triv}} + \Phi_{\mathrm{sgn}}$. We wish to show that $e\mathcal{O}G$ is Morita equivalent to $\mathcal{O}H$.

We consider the functor $e\mathcal{O}G \otimes_{\mathcal{O}H} - : \mathcal{O}H-\mathbf{mod} \to \mathcal{O}Ge-\mathbf{mod}$ which is simply inducing from H to G and then cutting by the block. Computing the induction from H to G we see that the character of eKG as a (KGe, KH)-bimodule is given by

$$\begin{aligned}\chi_{eKG} &= (\chi_1 + \chi_4 + \chi_5) \otimes \chi_{\mathrm{triv}} + \chi_4 \otimes \chi_{\mathrm{sgn}} + (\chi_4 + 2\chi_5) \otimes \chi_{\mathrm{ref}}\\ &= (\chi_1 + \Phi_4) \otimes \chi_{\mathrm{triv}} + \chi_4 \otimes \chi_{\mathrm{sgn}} + (\chi_5 + \Phi_4) \otimes \chi_{\mathrm{ref}}\\ &= (\chi_1 \otimes \chi_{\mathrm{triv}} + \chi_4 \otimes \chi_{\mathrm{sgn}} + \chi_5 \otimes \chi_{\mathrm{ref}}) + \Phi_4 \otimes \Phi_{\mathrm{triv}}.\end{aligned}$$

This calculation hints that the functor is potentially very close to giving a Morita equivalence. Whilst this character calculation is yet to be borne out by a calculation with the underlying modules, it suggests that the bimodule $e\mathcal{O}G$ gives a Morita equivalence up to removing a projective error term. More precisely, one can show that $e\mathcal{O}G$ induces a stable equivalence of Morita type between $\mathcal{O}H$ and $\mathcal{O}G$, see [Lin18a, §2.17] and [Lin18b, Prop. 9.8.3] for more details on this.

For our purposes we will be able to establish the Morita equivalence using just Theorem 8.7. By the definition of blocks, the bimodule $e\mathcal{O}G$ is indecomposable as an $\mathcal{O}[G \times G]$-module. Moreover, it must be non-projective otherwise $\overline{\mathcal{O}Ge} \cong kGe$ would be a projective $k[G \times G]$-module so split semisimple as a k-algebra by Proposition 1.6. This is clearly not the case.

Clearly $P \times P \leqslant G \times G$ is a Sylow ℓ-subgroup that is TI and $G \times H$ is a subgroup containing $N_{G\times G}(P \times P) = H \times H$. Hence, applying Theorem 8.7 in $G \times G$ we see that $\mathrm{Res}_{G\times H}^{G\times G}(e\mathcal{O}G) = V \oplus R$ with $V \in \mathcal{O}[G \times H]-\mathbf{mod}$ a non-projective indecomposable module and $R \in \mathcal{O}[G \times H]-\mathbf{mod}$ a projective module. Clearly we have an equality of

characters
$$\chi_{eKG} = \chi_{V^K} + \chi_{R^K}.$$

The PIMs of $\mathcal{O}[G \times H]$ are of the form $Q_G \otimes_{\mathcal{O}} Q_H$ with Q_G a PIM of $\mathcal{O}G$ and Q_H a PIM of $\mathcal{O}H$. It follows easily from the calculation of χ_{eKG} above and Remark 4.3 that $\chi_{R^K} = \Phi_4 \otimes \Phi_{\mathrm{triv}}$ so we must have

$$\chi_{V^K} = \chi_1 \otimes \chi_{\mathrm{triv}} + \chi_4 \otimes \chi_{\mathrm{sgn}} + \chi_5 \otimes \chi_{\mathrm{ref}}.$$

Hence the bimodule V satisfies the condition in (ii) of Theorem 8.2.

Being a block algebra $e\mathcal{O}G$ is a summand of $\mathcal{O}G$ so V is a summand of $\mathcal{O}G$. As $\mathcal{O}G = \bigoplus_{gH \in G\backslash H} g\mathcal{O}H$ is free both as a left $\mathcal{O}G$-module and a right $\mathcal{O}H$-module it follows that V is projective as a left $\mathcal{O}G$-module and a right $\mathcal{O}H$-module. Hence all the assumptions of Theorem 8.2 are satisfied so the functor $V \otimes_{\mathcal{O}H} -$ gives a Morita equivalence between $\mathcal{O}H$ and $\mathcal{O}Ge$.

Exercise 8.9. Express the character of M^K first as a left KG-module and then as a right KH-module as a sum of PIMs.

Remark 8.10. Explicitly constructing the underlying bimodule V can be quite challenging. Here we have $\dim(V^K) = 15$. The subgroup $U = \langle (12)(34), (14)(23) \rangle \leqslant G$ is a Sylow 2-subgroup normalised by P. The induced module $V = \mathrm{Ind}_U^G(\mathcal{O}) \cong \mathcal{O}[G/U]$ is naturally an $(\mathcal{O}G, \mathcal{O}P)$-bimodule that is projective as a left $\mathcal{O}G$-module and as a right $\mathcal{O}P$-module because $|U|$ is invertible in k. Moreover, $\dim(V^K) = 15$. This module can be extended to an $(\mathcal{O}G, \mathcal{O}H)$-bimodule, giving the summand of $\mathcal{O}Ge$ used above, but this takes work.

An Example: The Principal Block of $G = \mathfrak{A}_5$ *when* $\ell = 3$

We continue to assume that $G = \mathfrak{A}_5$ but now $\ell = 5$. Let $P = \langle (12345) \rangle$ and set $H = N_G(P) = P \rtimes \langle (25)(34) \rangle \cong D_{10}$. This time the characters of the PIMs for G and H are given by

$$\begin{aligned} \Phi_1 &= \chi_1 + \chi_4 & \Phi_{\mathrm{triv}} &= \chi_{\mathrm{triv}} + \chi_{\mathrm{ref},+} + \chi_{\mathrm{ref},-} \\ \Phi_3 &= \chi_{3,+} + \chi_{3,-} + \chi_4 & \Phi_{\mathrm{sgn}} &= \chi_{\mathrm{sgn}} + \chi_{\mathrm{ref},+} + \chi_{\mathrm{ref},-}. \end{aligned}$$

Let $e \in \mathrm{Bl}(\mathcal{O}G)$ be the principal block. Then the character of $e\mathcal{O}G$ is $\Phi_1 + 3\Phi_3$. Again, there is only one block of $\mathcal{O}H$ and its character is $\Phi_{\mathrm{triv}} + \Phi_{\mathrm{sgn}}$. We wish to investigate whether $e\mathcal{O}G$ is Morita equivalent to $\mathcal{O}H$.

Again we consider the functor $e\mathcal{O}G \otimes_{\mathcal{O}H} - : \mathcal{O}H\text{-}\mathbf{mod} \to e\mathcal{O}G\text{-}\mathbf{mod}$. The character of eKG as a (KGe, KH)-bimodule is given by

$$\begin{aligned} \chi_{eKG} &= \chi_1 \otimes \chi_{\mathrm{triv}} + (\chi_{3,+} + \chi_{3,-}) \otimes \chi_{\mathrm{sgn}} + (\chi_{3,+} + \chi_4) \otimes \chi_{\mathrm{ref},+} + (\chi_{3,-} + \chi_4) \otimes \chi_{\mathrm{ref},-} \\ &= (\chi_1 \otimes \chi_{\mathrm{triv}} - \chi_4 \otimes \chi_{\mathrm{sgn}} - \chi_{3,-} \otimes \chi_{\mathrm{ref},+} - \chi_{3,+} \otimes \chi_{\mathrm{ref},-}) + \Phi_3 \otimes \Phi_{\mathrm{sgn}}. \end{aligned}$$

Again, this seems close to giving a Morita equivalence. However, in this case we can only split off a projective summand by introducing signs and constructing a virtual character. So this won't come from a factorisation of $e\mathcal{O}G$ in terms of modules.

In fact in this instance the block algebras $e\mathcal{O}G$ and $\mathcal{O}H$ are *not* Morita equivalent. To see this we compute the Brauer trees of these blocks. This is a combinatorial structure which encodes the decomposition of the PIMs into ordinary irreducible characters, see [HL89] for more details and examples of Brauer trees. Indeed, each PIM is the sum of two adjacent vertices on the tree.

The darkly shaded node on the tree is called the *exceptional vertex*.

Theorem 8.11 (Brauer). *Assume* G *and* H *are two finite groups with blocks* $e \in \mathrm{Bl}(kG)$ *and* $b \in \mathrm{Bl}(kH)$. *If the defect groups of* e *and* b *are both cyclic of the same order then the blocks* ekG *and* bkH *are Morita equivalent if and only if they have the same planar embedded Brauer tree.*

9. Equivalences Between Blocks: The Derived Case

The Morita equivalences considered in the previous section are too narrow to capture all the relationships between blocks that we want to consider. Instead, Broué proposed the following more general relationship between a block and its Brauer correspondent.

Conjecture 9.1 (Broué's Abelian Defect Group Conjecture). *Assume* $D \leqslant G$ *is an abelian* ℓ*-subgroup of* G. *Then for any block* $b \in \mathrm{Bl}(\mathcal{O}N_G(D) \mid D)$ *there is a derived equivalence*

$$\mathscr{D}^b(\mathcal{O}N_G(D)b\text{--}\mathbf{mod}) \simeq \mathscr{D}^b(\mathcal{O}Gb^G\text{--}\mathbf{mod})$$

between the bounded derived categories of the module categories of b *and its Brauer correspondent.*

If A and A' are two $\mathcal{O}$-free $\mathcal{O}$-algebras then a *Rickard equivalence* between A and A' is an equivalence of triangulated categories $\mathscr{C} \overset{\mathrm{L}}{\otimes}_{A'} - : \mathscr{D}^b(A'\text{--}\mathbf{mod}) \to \mathscr{D}^b(A\text{--}\mathbf{mod})$ where $\mathscr{C}$ is a complex of (A, A')-bimodules that is perfect as a complex of left A-modules and also perfect as a complex of right A'-modules. In other words, when viewed from the left and right these should be complexes of projective modules.

Remark 9.2. If $\mathscr{C} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}Hb} -$ is a Rickard equivalence between two block algebras $\mathcal{O}Hb$ and $\mathcal{O}Ge$ of finite groups then we naturally obtain Rickard equivalences $\mathscr{C}^K \overset{\mathrm{L}}{\otimes}_{KHb} -$ and $\overline{\mathscr{C}} \overset{\mathrm{L}}{\otimes}_{kHb} -$.

An Example: The Principal Block of $G = \mathfrak{A}_5$ *when* $\ell = 5$

Let $P = \langle(12345)\rangle$, $H = N_G(P)$, and $R_1, R_3 \in \mathcal{O}G\text{--}\mathbf{mod}$ be the two PIMs affording the characters Φ_1 and Φ_3 respectively. Similarly, we let $Q_{\mathrm{triv}}, Q_{\mathrm{sgn}} \in \mathcal{O}H\text{--}\mathbf{mod}$ be the PIMs affording the characters Φ_{triv} and Φ_{sgn} respectively. Let $e\mathcal{O}G$ be the principal block and let $P = \langle (12345) \rangle$. If $M \in \mathcal{O}G-\mathbf{mod}$ then we denote by $M^* = \mathrm{Hom}_{\mathcal{O}G}(M, \mathcal{O})$ the dual module. In [Rou95, Lem. 2] Rouquier shows that we have a projective cover

$$\psi : (R_1 \otimes Q^*_{\mathrm{triv}}) \oplus (R_3 \otimes Q^*_{\mathrm{sgn}}) \to e\mathcal{O}G$$

of $(\mathcal{O}G, \mathcal{O}H)$-bimodules.

Proposition 9.3 (Rouquier [Rou95]). *Assume* $G = \mathfrak{A}_5$ *and* $\ell = 5$. *Let* $Q = R_3 \otimes Q^*_{\mathrm{sgn}}$. *Then the 2-term complex*

$$0 \longrightarrow Q \xrightarrow{\psi} e\mathcal{O}G \longrightarrow 0,$$

with $e\mathcal{O}G$ *in degree* 0*, gives a Rickard equivalence between* $\mathcal{O}Ge$ *and* $\mathcal{O}H$.

Remark 9.4. The results in [Rou95] actually apply to all blocks with cyclic defect group and yield a positive answer to Broué's conjecture in this case, see [Lin18b, Thm. 11.12.1].

Let us briefly explain how this relates to the character calculation we did in the previous section. Given a bounded complex $\mathscr{C}$

$$\cdots \longrightarrow M_{-2} \longrightarrow M_{-1} \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots$$

of (KG, KH)-bimodules we define a corresponding virtual character

$$\chi_{\mathscr{C}} = \sum_{i \in \mathbb{Z}} (-1)^i \chi_{M_i} \in \mathcal{G}_0(KG \otimes_K KH).$$

Viewing a module $S \in KH-\mathbf{mod}$ as a complex in degree 0 we can form the tensor product $\mathscr{C} \overset{L}{\otimes}_{KH} S$. Replacing M by $\mathscr{C}$, and tensor products of modules with those of complexes, the formulas in (8.4) and (8.5) still hold. However these are now expressions between virtual characters not genuine characters.

Now let $\mathscr{C}$ be the complex given in Proposition 9.3 then we saw in the previous section that

$$\chi_{\mathscr{C}^K} = -\Phi_3 \otimes \Phi_{\mathrm{sgn}} + \chi_{eKG} = \chi_1 \otimes \chi_{\mathrm{triv}} - \chi_4 \otimes \chi_{\mathrm{sgn}} - \chi_{3,-} \otimes \chi_{\mathrm{ref},+} - \chi_{3,+} \otimes \chi_{\mathrm{ref},-}.$$

Hence tensoring with $\mathscr{C}^K$ gives a signed bijection $\mathrm{Irr}(KH) \to \mathrm{Irr}(KG \mid e)$. This is an example of a *perfect isometry*, which is a certain signed bijection between irreducible characters introduced by Broué in [Bro90b].

It was shown by Broué that a Rickard equivalence between block algebras induces a perfect isometry and that such a signed bijection between irreducible characters is height preserving, see [Lin18b, Cor. 9.2.5, Cor. 9.3.3]. Hence, in the case of abelian defect groups Broué's conjecture gives a structural explanation for Alperin's conjecture described in Section 7.

References

[Alp76] J. L. Alperin, *The main problem of block theory*, in: *Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975)*, 1976, 341–356.

[Alp86] ______, *Local representation theory*, vol. 11, Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, 1986.

[Bro90a] M. Broué, *Isométries de caractères et équivalences de Morita ou dérivées*, Inst. Hautes Études Sci. Publ. Math. (1990) no. 71, 45–63.

[Bro90b] ______, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque (1990) no. 181-182, 61–92.

[CR81] C. W. Curtis and I. Reiner, *Methods of representation theory. Vol. I*, With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication, New York: John Wiley & Sons Inc., 1981.

[CR87] ______, *Methods of representation theory. Vol. II*, Pure and Applied Mathematics (New York), With applications to finite groups and orders, A Wiley-Interscience Publication, New York: John Wiley & Sons Inc., 1987.

[CR06] ______, *Representation theory of finite groups and associative algebras*, Reprint of the 1962 original, AMS Chelsea Publishing, Providence, RI, 2006.

[Gap] *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, The GAP Group, 2014.

[HL89] G. Hiss and K. Lux, *Brauer trees of sporadic groups*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1989.

[Isa06] I. M. Isaacs, *Character theory of finite groups*, Corrected reprint of the 1976 original [Academic Press, New York], AMS Chelsea Publishing, Providence, RI, 2006.

[Jan73] G. J. Janusz, *Algebraic number fields*, Pure and Applied Mathematics, Vol. 55, Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973.

[Lin18a] M. Linckelmann, *The block theory of finite group algebras. Vol. I*, vol. 91, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2018.

[Lin18b] ______, *The block theory of finite group algebras. Vol. II*, vol. 92, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2018.

[LP10] K. Lux and H. Pahlings, *Representations of groups*, vol. 124, Cambridge Studies in Advanced Mathematics, A computational approach, Cambridge University Press, Cambridge, 2010.

[MS16] G. Malle and B. Späth, *Characters of odd degree*, Ann. of Math. (2) **184** (2016) no. 3, 869–908.

[NT89] H. Nagao and Y. Tsushima, *Representations of finite groups*, Translated from the Japanese, Academic Press, Inc., Boston, MA, 1989.

[Nav98] G. Navarro, *Characters and blocks of finite groups*, vol. 250, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1998.

[Rou95] R. Rouquier, *From stable equivalences to Rickard equivalences for blocks with cyclic defect*, in: *Groups '93 Galway/St. Andrews, Vol. 2*, vol. 212, London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1995, 512–523.

[Ser79] J.-P. Serre, *Local fields*, vol. 67, Graduate Texts in Mathematics, Translated from the French by Marvin Jay Greenberg, Springer-Verlag, New York-Berlin, 1979.