Microlocal Sheaves on certain Affine Springer Fibers

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Plan:

1. Homogeneous affine Springer fibers of slope 1 as a Lagrangian cycle, and microlocal sheaves
2. Homogeneous affine Springer fibers of slope $\frac{1}{m}$ (not today)

1. Slope 1

Motivation

(1) $e \in \mathcal{N} \subset \mathfrak{g}$.

$$\mathcal{B}_e = \{ \mathfrak{b} \subset \mathfrak{g} \, ; \, e \in \mathfrak{b} \}.$$

$$\begin{tikzcd}
\mathcal{B}_e \arrow[r, hook, "\text{Lagrangian}"'] \arrow[d] & \tilde{S}_e \arrow[r, hook] \arrow[d] & T^*\mathcal{B} = \tilde{\mathcal{N}} \arrow[d] \\
\{e\} \arrow[r, hook] & S_e \arrow[r, hook] & \mathcal{N}
\end{tikzcd}$$

symplectic

S_e = Slodowy slice through e inside $\mathcal{N}$.

"Microlocal sheaves" supp on $\mathcal{B}_e$

$\longleftrightarrow$ repn of W-algebras

(2)

(2) Kazhdan-Lusztig map

$$\mathcal{N}/\sim \;\longrightarrow\; W/\sim$$

$$e \;\longmapsto\; \tilde{e} = e + x_1 t + x_2 t^2 + \cdots$$
$$\in \mathfrak{g}[[t]].$$

s.t. $\tilde{e}$ is reg. s.s. $\in \underline{\mathfrak{g}((t))}$

Cartan subgps in $G((t)) \longleftrightarrow W/\sim$.

$$\mathrm{Gal}\left(\overline{\mathbb{C}((t))}/\mathbb{C}((t))\right) \simeq \hat{\mathbb{Z}}$$

$$C_{G((t))}(\tilde{e}) \rightsquigarrow [w] \in W/\sim$$

splitting type of $\tilde{e}$

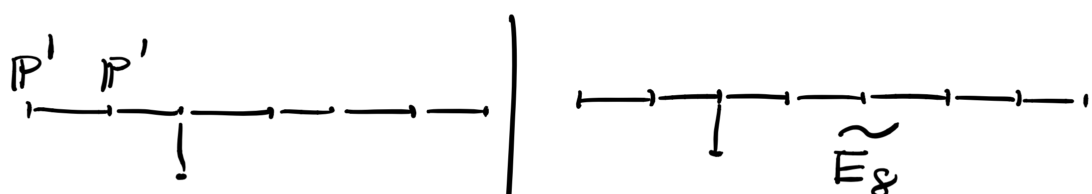
e.g. $P(x) \in \mathbb{C}((t))[x]$
char pol. of $\tilde{e}$.
factorizes $P_1\, P_2 \cdots P_\ell$
deg $n_1\; n_2 \cdots n_\ell$.
$[w]$. cycle type $n_1, n_2 \cdots n_\ell$.

Lusztig observed: $e \rightsquigarrow \tilde{e}$ (elliptic)

$$\mathcal{B}_e \qquad\qquad Fl_{\tilde{e}}$$

$$H^*(\mathcal{B}_e) \qquad\qquad H^{top}(\underline{Fl}_{\tilde{e}})$$

$e =$ subregular in E_8.



$\sum_i h^i(\mathcal{B}_e) = 1 + 8 = 9.$ | $h^2(Fl_{\tilde{e}}) = 9.$

Affine Springer Fibers:

Iwahori $= I \subset G[[t]]$ $\quad\quad Fl = G((t))/I$

$$\begin{array}{ccc} I & \subset & G[[t]] \\ \downarrow & & \downarrow t=0. \\ B & \subset & G \end{array}$$

$\gamma \in \mathfrak{g}((t)).$

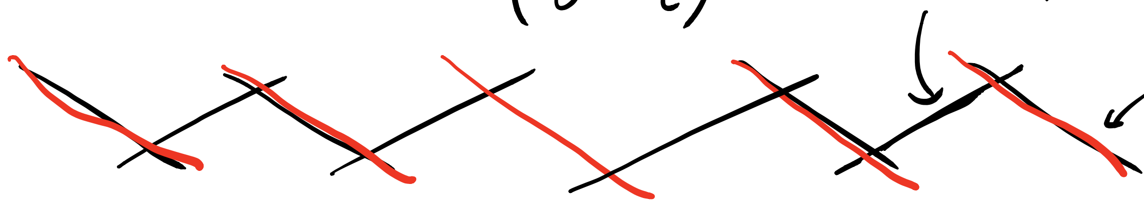
$$Fl_\gamma = \{ g \in G((t))/I \mid Ad(g^{-1})\gamma \in \mathrm{Lie}\, I \} \subset Fl.$$

Particular elts: $\quad \gamma = t \cdot \psi.$

$\psi \in \mathfrak{g}^{rs}(\mathbb{C}).$

<u>Ex</u> $G = SL_2.$ $\quad \gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$

Fix Λ_0, varying Λ_1.

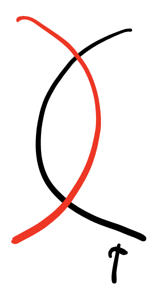


Fix Λ_1, vary Λ_0.

Chain of $\mathbb{P}^1$. $\quad \longrightarrow$

$\mathbb{Z}$

mod $\mathbb{Z}$



$\tilde{A}_1$ configuration.

$G = SL_n.$ $\quad Fl_\gamma$ parametrizes

$$\cdots \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \cdots$$

lattices in $\mathbb{C}((t))^n$

$(/\mathbb{C}[[t]])$.

s.t. • $t\Lambda_i = \Lambda_{i-n}$

• Λ_i/Λ_{i-1} is 1-dim'l $/\mathbb{C}$.

• $\gamma\Lambda_i \subset \Lambda_i$.

• Λ_0 has the same volume as $\mathbb{C}[[t]]^n$.

<u>Geometry of Fl_γ</u> when $\gamma = t\cdot\varphi$.

$\varphi \in \mathfrak{t}^{rs} \subset \mathfrak{g}^{rs}$. $C_G(\varphi) = T$.

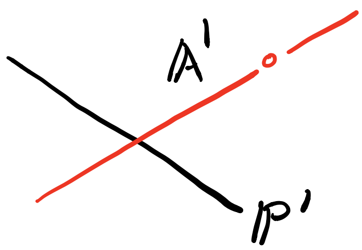
$\Rightarrow$ $X_*(T) \circlearrowright Fl_\gamma$. free action.

$\lambda \in X_*(T)$ acts by t^λ

Lusztig : $\exists$ <u>fund. domain</u> of $X_*(T)$-action on Fl_γ

$$\bigcup_w T^*_{X_w}\mathcal{B} = \text{"Steinberg var"} \subset \tilde{\mathcal{N}}$$

$$\begin{array}{ccc} \text{"Steinberg var"} & \subset & \tilde{\mathcal{N}} \\ \downarrow & & \downarrow \\ \mathfrak{n} & \subset & \mathcal{N} \end{array}$$



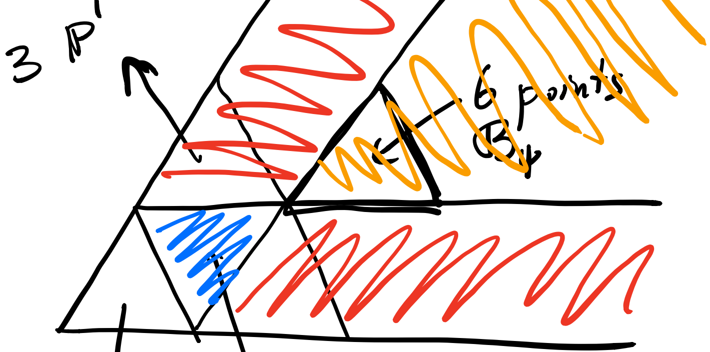
$\mathbb{P}^1 \cup T^*_\infty \mathbb{P}^1$

$\mathbb{G}_m$-action on <u>Fl_γ</u>.
(<u>loop rotation</u>)

Fixed points. $\longleftrightarrow$ Hessenberg varieties.

$G = SL_3$.

$G[[t]] \circlearrowright Fl$



orbits $\longleftrightarrow$ alcoves in dom. chamber.

$(\underline{\text{each orbit}}) \cap Fl_\gamma$

$\uparrow$

$\underline{\mathbb{G}_m\text{-fixed pts here}}$.

$$G/B \qquad \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3\} \simeq Bl_3(\mathbb{P}^2)$$

ψ.

(same color, same Hessenberg var)

- Hitchin moduli

A.S.F. $\longleftrightarrow$ Hitchin fib.

Laumon–Ngô.

$$X = \mathbb{P}^1.$$

level str. at 0, ∞.

$$\mathrm{I}_0 \qquad G_\infty^2 = \ker\left(G[[t^{-1}]] \longrightarrow G[t^{-1}]/(t^{-1})^2 \right)$$

$$G_\infty^2 \subset G_\infty^1 \subset G[[t^{-1}]]$$

$$\leftarrow \mathfrak{g} \rightarrow \quad \leftarrow G \rightarrow$$

$$\mathrm{Bun}_G(\mathrm{I}_0, G_\infty^2) \circlearrowleft G_\infty^1/G_\infty^2 = \mathfrak{g}$$

$$\psi \in \mathfrak{g}^{rs} \simeq (\mathfrak{g}^*)^{rs}.$$

$$T^*\mathrm{Bun}_G(\mathrm{I}_0, G_\infty^2) /\!/_{\psi}\, \mathfrak{g} \quad \text{defined as:}$$

$$\mu : T^*(\text{—}) \longrightarrow \mathfrak{g}^*$$

$$\mu^{-1}(\psi) \longrightarrow \psi$$

$$\downarrow$$

$$\mu^{-1}(\psi)/\mathfrak{g} = T^*(-)/\!\!/_{\psi}\, \mathfrak{g}$$

$$T^*\mathrm{Bun}_G(I_0, G_\infty^2)/\!\!/_{\psi}\, \mathfrak{g} =: \mathcal{M}_\psi$$

parametrizes: $(\mathcal{E}, \tau_\infty, \mathcal{E}_{0,B}, \varphi)$.

$\mathcal{E} \downarrow$ G-bundle $\mathbb{P}^1$; τ_∞: triv. of $\mathcal{E}$ at ∞.; $\mathcal{E}_{0,B}$: $\boxed{\text{B-red. of } \mathcal{E} \text{ at } 0}$

$$\varphi \in H^0(\mathbb{P}^1, \mathrm{Ad}^*(\mathcal{E}) \otimes \underbrace{\omega_{\mathbb{P}^1}(2\cdot\underline{\infty} + 1\cdot\underline{0})}_{\mathcal{O}_{\mathbb{P}^1}(\underline{0}) \,\cong})$$

$\boxed{\mathbb{P}^1: \text{affine coord } t}$

s.t. near ∞. $\varphi = \psi \cdot dt + \underline{\text{higher}}$.

$\psi \downarrow$ (use τ_∞); $dt \to$ 2^{nd} order pole at ∞.

near 0. $\underline{\underline{\mathrm{Res}_0 \varphi}} \in \underline{\underline{\mathfrak{n}_{0,B}}}$.

<u>Ex</u>: $G = GL_n$.

$$\mathcal{M}_\psi = \{ \mathcal{E}: \text{rk } n. \text{ v.b.}, \ \tau_\infty. \text{ full flag at } 0, \ \varphi: \mathcal{E} \longrightarrow \mathcal{E}(\underline{0}) \simeq \mathcal{E}(1). + \cdots \}.$$

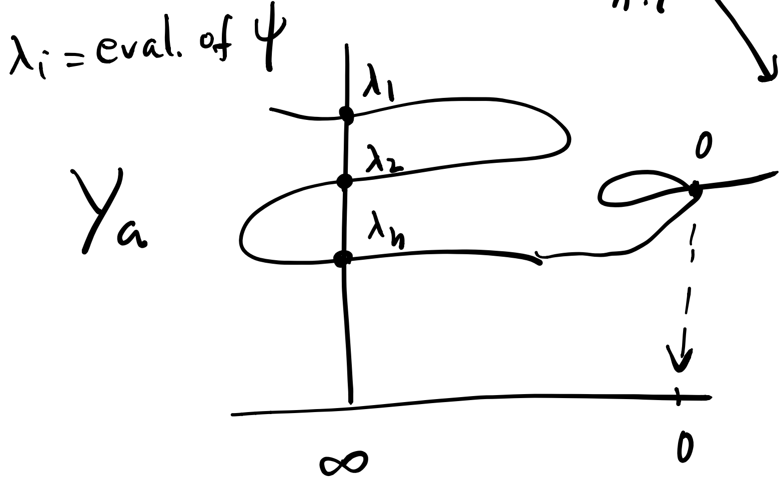
$$\downarrow f_\psi$$

$$A_\psi \subseteq \bigoplus_{i=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}(\underline{i}))_{\text{vanishing at } 0}.$$

$$\begin{array}{ccc} \downarrow & & \downarrow{\scriptstyle ev_\infty} \\ [\psi] & \in & \mathbb{A}^n = \mathrm{Sym}^n(\mathbb{A}^1). \end{array}$$

$$\begin{tikzcd} a \in A_\psi \arrow[r, squiggly] & Y_a \arrow[dr, "n:1"'] & \subset \mathbb{P}^2 \arrow[d, dashed] \\ & & \mathbb{P}^1 \end{tikzcd}$$

degree n (Spectral curve)



$$f^{-1}(a) \simeq \overline{\mathrm{Pic}}(Y_a)_\infty$$

∞ = trivialization above ∞.

T change trivial'n.

$X_*(T)$ modifying line bundle above ∞.

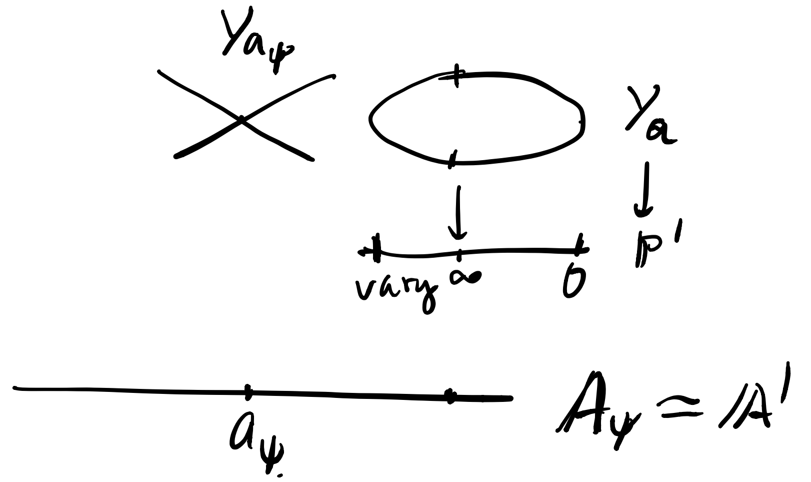
<u>Fact</u>:

- M_ψ is symplectic. variety.
- $f: M_\psi \longrightarrow A_\psi$ comp. integrable system.
- $a_\psi \in A_\psi$ (given by char. pol. of $\psi \underline{\underline{dt}}$)

  $f^{-1}(a_\psi) \cong \underline{Fl_{t\psi}}$.

generic fibers of f: T-bundles over $\mathrm{Pic}(\underline{Y_a})$.

SL_2: Y_a rat'l.



"Microlocal sheaves" on $\mathcal{M}_\psi$ supp on $Fl_{t\psi}$.

- char p. / D-module setting.

$$Sh\left(\mathrm{Bun}_G(I_0, G_\infty^2)\right)_{(\mathfrak{g},\psi)}.$$

$$\mathfrak{g} \xrightarrow{\psi} \mathbb{A}^1.$$

$\psi^* AS$ is a character sheaf on $\mathfrak{g}$.

- Gaitsgory's Kirillov model.

$$\mathbb{G}_a \rtimes \mathbb{G}_m \circlearrowright X$$

$$\mathrm{Kir}(X) = Sh_{\mathbb{G}_m}(X) / Sh_{\mathbb{G}_a \rtimes \mathbb{G}_m}(X).$$

$$\mathrm{Kir}(X) \simeq Sh_{(\mathbb{G}_a,\psi)}(X).$$

$$\left(\text{Fourier dual to} \quad Sh(\mathbb{G}_m) \simeq Sh(\mathbb{A}^1)/Sh_{\underline{\{0\}}}(\mathbb{A}^1) \right)$$

Microlocalizing Kirillov category:

$$\boxed{\mu Sh\left(T^*X /\!\!/_1 \mathbb{G}_a\right)}$$

$$\mu_{\mathbb{G}_a} : T^*X \longrightarrow \mathbb{C}.$$

$$\boxed{\mu_{\mathbb{G}_a}^{-1}(\mathbb{C}^\times)} \subset T^*X.$$

Kashiwara-Schapira $\Omega \mapsto Sh(X; \underline{\Omega})$

sheafify to get $\mu Sh(\Omega)$. $\qquad \Omega \underset{\text{open}}{\subset} T^*X.$

$$\left(\mu_{\mathbb{G}_a}^{-1}(\mathbb{C}^\times) \cap \mu_{\mathbb{G}_m}^{-1}(0)\right) / \mathbb{G}_m$$

$$\cong$$

$$\underline{T^*X /\!\!/_1 \mathbb{G}_a}$$

$$\mu : \mathrm{Kir}(X) \longrightarrow \mu Sh\left(T^*X /\!\!/_1 \mathbb{G}_a\right)$$

Apply to $X = Bun_G(I_0, G_\infty^2) \circlearrowleft \mathfrak{g} \rtimes \mathbb{G}_m \xrightarrow{\psi} \mathbb{G}_a.$

$$\rightsquigarrow \mathrm{Kir}_\psi(Bun_G(I_0, G_\infty^2)) \xrightarrow{\mu} \mu Sh(\mathcal{M}_\psi).$$

when AS makes sense

$$\mathrm{Sh}\Big((G[t^{-1}]_1,\psi)\backslash \underbrace{G((t))/I}_{Fl}\Big) \circlearrowleft \mathcal{H}$$

$$\cup$$

$$\mathcal{D}_\psi \overset{\text{def}}{=\!=} \text{generated by pullback from } Gr \text{ under } \mathcal{H}\text{-convolution.}$$

Thm. $$\mathcal{D}_\psi \xrightarrow{\ \mu\ } \mu\mathrm{Sh}_{Fl_{t\psi}}(\mathcal{M}_\psi)$$

is fully faithful.

Thm.

$$\begin{array}{ccc} K_0(\mathcal{H}_\psi) & \overset{\sim}{\longrightarrow} & \mathbb{Z}[X_*(T)] \\ \circlearrowright\ \infty & & \circlearrowright \\ K_0(\mathcal{D}_\psi) & \xrightarrow{\ cc\ } & H_{top}(Fl_{t\psi}). \\ \circlearrowleft\ 0 & & \circlearrowleft \text{ affine Springer} \\ K_0(\mathcal{H}) & = & \mathbb{Z}[\widetilde{W}] \end{array}$$

Conj: $$\mathcal{D}_\psi \simeq D^b\mathrm{Coh}^{T^\vee}_{B^\vee}(\widetilde{\mathcal{N}}^\vee)$$

$$\mathrm{Perv}\big((G[t^{-1}]_1,\psi)\backslash Gr\big) \simeq \mathrm{Rep}(T^\vee).$$

$$\boxed{G[t^{-1}]_1\cdot t^\lambda} \qquad \lambda\in X_*(T).$$

Conj: $$\mathcal{D}_{\psi,\mathbf{P}} \simeq D^b\,\mathrm{Coh}^{C_e}_{\mathcal{B}_e}(S^\vee_{\underline{e}})$$

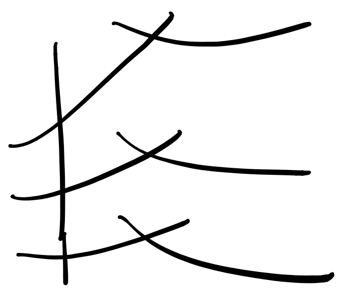
$$\Big\downarrow$$

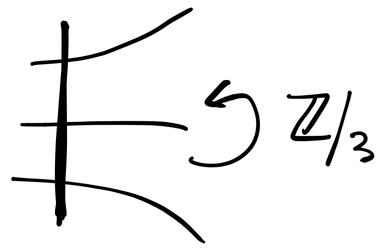
$$H_{top}(Fl_\psi) \longleftrightarrow K^{\underline{C_e}}(\mathcal{B}_e)$$

$$\updownarrow$$

$$H_{top}(Fl_{\psi^\vee}).$$

G_2 subregular $\qquad\qquad K^{\mathbb{Z}/3}(\mathcal{B}_e)$





7 $\qquad\qquad 3+3+1=7$