Parabolic version of 2 realizations
& applications to modular representations
(jt. w. Bezrukavnikov)

Part 1

Part 2

Goal of the project:

Bezrukavnikov-Mirkovic proved Lusztig's conjecture identifying
- the basis of simple $\mathfrak{g}$-modules w. given p-character, $p \gg 0$
- & the canonical basis in K-theory of Springer fiber.

However, in general, the latter is too implicit to get Kazhdan-Lusztig type character formulas for the simples.

It turns out, that modifying the category of $\mathfrak{g}$-modules "slightly" (look at modules that are equivariant w.r.t. a reductive subgroup $\tilde{T}_0 \subset G$ w. $\tilde{T}_0^{\circ}$ torus) we can still get KL type character formulas for the simples (in the equivariant category).

When the p-character is distinguished (to be recalled later) the character formula is a K-theoretic consequence of an equivalence that is deduced from a parabolic version of the two realizations theorem.

1) Parabolic version of 2 realizations

1.1) Reminder on 2 realizations.

G connected reductive group $\rightsquigarrow$ Langlands dual $G^\vee$

$St := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightsquigarrow D^b(Coh^G St)$

$G^\vee \rightsquigarrow G^\vee((t)) \supset I \supset I^\circ$ (Iwahori & pro-unip't radical) $\rightsquigarrow \mathcal{F}\ell = G^\vee((t))/I \rightsquigarrow$ constructible category $D^b_{I^\circ}(\mathcal{F}\ell)$.

Thm (Bezrukavnikov) Have equivalence of triang'd cat's
$$\tau: D^b_{I^\circ}(\mathcal{F}\ell) \xrightarrow{\sim} D^b(Coh^G St).$$

Important: this is a bimodule equivalence w.r.t.
- "completed" version of affine Hecke category, $D^b(Coh^G St^\wedge)$ in the coherent version acting on the left &
- "specialized" version $D^b_I(\mathcal{F}\ell) \xrightarrow[\tau]{\sim} D^b(Coh^G \tilde{\mathcal{N}} \times^R_{\mathfrak{g}} \tilde{\mathcal{N}})$ acting on the right.

Also important: τ is nicely compatible w. t-structures:
- for perverse t-structure on $D^b_{I^\circ}(\mathcal{F}\ell)$
- & perverse bimodule (over NC Springer) t-structure on $D^b(Coh^G St)$.

— will elaborate on this later.

1.2) Statement of parabolic version

Notation: $P \subset G$ parabolic $\rightsquigarrow \tilde{\mathcal{N}}_P = T^*(G/P) \rightsquigarrow$
$St_P = \tilde{\mathfrak{g}} \times^R_{\mathfrak{g}} \tilde{\mathcal{N}}_P \rightsquigarrow D^b(Coh^G St_P)$.

$P \subset G \rightsquigarrow P^\vee \subset G^\vee \rightsquigarrow$ parahoric $\mathcal{J} \subset G^\vee((t)) \rightsquigarrow$
$\mathcal{F}\ell_P = G^\vee((t))/\mathcal{J} \rightsquigarrow D^b_{I^\circ}(\mathcal{F}\ell_P)$

Thm (R.B.-I.L.) $\exists$ equivalence of triangulated categories:

$$\mathcal{T}_P: D^b_{I^\circ}(\mathcal{F}\ell_P) \xrightarrow{\sim} D^b(Coh^G St_P)$$

that has properties 1-4 below.

Property 1: $\mathcal{T}_P$ is compatible with $\mathcal{T}$: the following diagram is commutative:

$$\begin{tikzcd}
D^b_{I^\circ}(\mathcal{F}\ell_P) \arrow[r, "\mathcal{T}_P"'] \arrow[d, "\varphi_2"] & D^b(Coh^G St_P) \arrow[d, "\varphi_1"] \\
D^b_{I^\circ}(\mathcal{F}\ell) \arrow[r, "\mathcal{T}"'] & D^b(Coh^G St)
\end{tikzcd}$$

where φ_1, φ_2 are as follows:

- φ_2: $\eta: \mathcal{F}\ell \twoheadrightarrow \mathcal{F}\ell_P$ (w. fiber $P^\vee/B^\vee$), $\varphi_2 := \eta^*[\dim P^\vee/B^\vee]$

- φ_1: Set $Z := G \times^B \mathfrak{p}^\perp \rightsquigarrow \iota: Z \hookrightarrow \tilde{\mathcal{N}}, \varpi: Z \twoheadrightarrow \tilde{\mathcal{N}}_P$

$$\rightsquigarrow \quad \begin{tikzcd} \tilde{\mathfrak{g}} \times^R_{\mathfrak{g}} \mathcal{Z} \arrow[r, hook, "\iota"'] \arrow[d, two heads, "\varpi"] & St \\ St_p \end{tikzcd}$$

Then $\varphi_1 := \iota_* \varpi^*(\cdot)$ (line bundle twist)

Property 2: τ_p is equivariant w.r.t. the action of (the completed version of) affine Hecke category.

Property 3: Description of $\tau_p(\underline{\mathbb{C}}_{pt})$:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d, two heads] & \tilde{\mathfrak{g}} \\ \tilde{\mathcal{N}}_p \end{tikzcd} \rightsquigarrow \mathcal{Z}_{diag} \hookrightarrow St_p$$

Then $\tau_p(\underline{\mathbb{C}}_{pt}) = \mathcal{O}_{\mathcal{Z}_{diag}}$

Property 4: t-exactness w.r.t. perverse t-structures

On the coherent side this t-structure is as follows:

Tilting generator $\mathcal{E}$ on $\tilde{\mathfrak{g}} \rightsquigarrow \mathcal{A} = End(\mathcal{E})$

$\mathcal{E}_p := \varpi_* \iota^* \mathcal{E}$ is a tilting generator on $\tilde{\mathcal{N}}_p$ (R.B.-Mirković)

$\rightsquigarrow \mathcal{A}_p := End(\mathcal{E}_p)$.

$$\left(\text{recall} \quad \begin{tikzcd} \mathcal{Z} \arrow[r, hook, "\iota"] \arrow[d, two heads, "\varpi"] & \tilde{\mathfrak{g}} \\ \tilde{\mathcal{N}}_p \end{tikzcd}\right)$$

Both $\mathcal{A}, \mathcal{A}_p$ are $\mathbb{C}[\mathfrak{g}^*]$-algebras

$\rightsquigarrow \mathcal{A} \otimes_{\mathbb{C}[\mathfrak{g}^*]} \mathcal{A}_p^{opp}$; G acts on these algebras

$\rightsquigarrow \mathcal{A} \otimes_{\mathbb{C}[\mathfrak{g}^*]} \mathcal{A}_p^{opp}\text{-mod}^G$

$$R\Gamma(\mathcal{E}\otimes\mathcal{E}_p^*\otimes\bullet): D^b(Coh^G St_p) \xrightarrow{\sim} D^b(\mathcal{A}\otimes_{\mathbb{C}[\mathfrak{g}^*]}\mathcal{A}_p^{opp}\text{-}mod^G).$$

Have perverse t-structure on r.h.s. (see Roman's talk).

Remark: the reason we care about 2&3 is that they allow to get info on images of (co)standards under τ_p.

1.3) Steps of proof

Step 1: φ_i are faithful but not full. We show that $\tau: D^b_{I^\circ}(\mathcal{F}\ell) \xrightarrow{\sim} D^b(Coh^G St)$ intertwines the "full images" of $\varphi_2: D^b_{I^\circ}(\mathcal{F}\ell_p) \longrightarrow D^b_{I^\circ}(\mathcal{F}\ell)$ & $\varphi_1: D^b(Coh^G St_p) \longrightarrow D^b(Coh^G St)$.

This is based on three observations

up to Karoubian completion

i) The full image of φ_2 is $\{\mathcal{F} * \underline{\mathbb{C}}_{P^\vee/B^\vee}\}$

ii) The full image of φ_1 is $\{\mathcal{G} * \mathcal{O}_{\mathcal{Z}\times_{\tilde{\mathcal{N}}_p}\mathcal{Z}}(\text{line bundle twist})\}$

iii) The equivalence $\tau: D^b_I(\mathcal{F}\ell) \xrightarrow{\sim} D^b(Coh^G \tilde{\mathcal{N}}\times^R_{\mathfrak{g}}\tilde{\mathcal{N}})$ sends $\underline{\mathbb{C}}_{P^\vee/B^\vee}$ to $\mathcal{O}_{\mathcal{Z}\times_{\tilde{\mathcal{N}}_p}\mathcal{Z}}(\text{twist})[\text{shift}]$.

Step 2: While φ_1, φ_2 are not full, they are t-exact and restrict to full embeddings between the hearts. The triangulated categories $D^b_{I^\circ}(\mathcal{F}\ell_p), D^b(Coh^G St_p)$

are derived categories of their hearts. This, together w. Step 1, allows to define t-exact

$$\tau_p : D^b_{I^\circ}(\mathcal{F}\ell_p) \xrightarrow{\sim} D^b(Coh^G St_p)$$

satisfying property 1 (& 4). Then one checks 2 & 3.

1.4) Quotient category: this is what we need for applic'ns to modular representation theory.

Categories $D^b_{I^\circ}(\mathcal{F}\ell_p)$ and $D^b(Coh^G St_p)$ come w. filtrations by 2-sided cells $\xleftrightarrow{\sim}$ nilp. orbits

Let's look at the top quotient of $D^b(Coh^G St_p)$

$\widetilde{\mathbb{O}}_p'$:= open G-orbit in $\widetilde{\mathcal{N}}_p$, $\mathbb{O}_p$:= image of $\mathbb{O}_p'$ in $\mathfrak{g}^*$, Richardson orbit for P: generic $e \in \mathfrak{p}^\perp$ lies in $\mathbb{O}_p$ & $\mathbb{O}_p' \simeq G/Z_p(e)$ &

$$\tilde{\mathfrak{g}} \times^R_{\mathfrak{g}} \mathbb{O}_p'/G \simeq \mathcal{B}_e/Z_p(e), \quad \mathcal{B}_e := \tilde{\mathfrak{g}} \times^R_{\mathfrak{g}} \{e\}$$

— Spinger fiber.

So the top quotient of $D^b(Coh^G St_p)$ is $D^b(Coh^{Z_p(e)} \mathcal{B}_e)$. It inherits the t-structure whose heart is identified with $\mathcal{A}_e$-mod$^{Z_p(e)}$, $\mathcal{A}_e$:= fiber of $\mathcal{A}$ over $e \in \mathfrak{g}^*$.

Conclusion: Have exact functor

$$\begin{tikzcd} Perv_{I^\circ}(\mathcal{F}\ell_p) \arrow[rr] \arrow[dr, two heads] & & \mathcal{A}_e\text{-mod}^{Z_p(e)} \\ & \text{top cell quotient} \arrow[ur, "\sim"] & \end{tikzcd}$$

Remark: still have control over the images of (co)standards in $\mathcal{A}_e\text{-mod}^{Z_P(e)}$ – thx to properties 2 & 3 above.

2) Applications to modular representations.

2.1) Setting

$\mathbb{F}$ algebraically closed field of char $p \gg 0$.

Can assume e from above is defined over $\mathbb{Z}$ $\rightsquigarrow$ reduce mod p $\rightsquigarrow$ $\mathcal{U}^0_{e,\mathbb{F}}$:= principal block of the p-central reduction of $\mathcal{U}(\mathfrak{g}_{\mathbb{F}})$ at $e \in \mathfrak{g}_{\mathbb{F}}^{(1),*}$.

Fact (Bezrukavnikov-Mirković) $\mathcal{A} = End(\mathcal{E})$ is defined over finite localization of $\mathbb{Z}$, and its fiber $\mathcal{A}_{e,\mathbb{F}}$ is Morita equivalent to $\mathcal{U}^0_{e,\mathbb{F}}$.

For this talk we'll be interested in the case when e is distinguished $\Leftrightarrow$ not contained in a proper Levi. Equivalently, the maximal reductive subgroup of $Z_G(e)$ is finite. Denote it by Γ, it coincides with $Z_G(e)/Z_G(e)^\circ$.

Example: $\mathfrak{g} = \mathfrak{sp}_{2n}$. Nilpotent orbits are classified by their Jordan type. Distinguished $\Leftrightarrow$ no repeated parts ($(2,4,6)$ distinguished, $(2,5,5)$ isn't). Have

$\Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^{\#\text{ parts}}$ (for distinguished element).

Classical fact: $\exists$ parabolic P s.t. e is Richardson for $\mathfrak{p}$ (Pe is open in $\mathfrak{p}^{\perp}$) & $Z_P(e) = Z_G(e)$.

This gives us a description of $\mathcal{A}_e$-mod$^{Z_G(e)}$ as the quotient of $Perv_{I^0}(\mathcal{F}l_P)$.

Observations:

1) $K_0(\mathcal{A}_e\text{-mod}^{Z_G(e)}) \simeq K_0(\mathcal{A}_e\text{-mod}^{\Gamma}) \simeq K_0(\mathcal{A}_{e,\mathbb{F}}\text{-mod}^{\Gamma})$, isomorphisms of <u>based</u> abelian groups, b/c $Z_G(e) = \Gamma \ltimes$ unip't.
2) $\mathcal{A}_{e,\mathbb{F}}\text{-mod}^{\Gamma} \xrightarrow{\sim} \mathcal{U}^0_{e,\mathbb{F}}\text{-mod}^{\Gamma}$.

2.2) K_0-classes of simples.

So as a based module, $K_0(\mathcal{U}^0_{e,\mathbb{F}}\text{-mod}^{\Gamma})$ is the top cell quotient of $K_0(Perv_{I^0}(\mathcal{F}l_P)) = \mathbb{Z}W^a \otimes_{\mathbb{Z}W_P} sgn$, where $W^a = W \ltimes \mathfrak{X}(T)$ is the affine Weyl group & $W_P \subset W^a$ finite parabolic corresponding to P.

To describe the classes of simples we need to describe the images (in $K_0(\mathcal{U}^0_{e,\mathbb{F}}\text{-mod}^{\Gamma})$) of the standards (in $K_0(Perv_{I^0}(\mathcal{F}l_P))$). These classes go to classes of certain induced modules.

Set $W^{a,P} := \{x \in W^{a,P} \mid x \text{ is longest in } xW_P\}$. This is the labeling set for standards in $Perv_{I^0}(\mathcal{F}l_P)$. We have action $W^a \curvearrowright \mathfrak{X}(T)$: $w.\mu = w\cdot\mu$, $\theta.\mu = \mu + p\theta$, $w \in W$, $\theta \in \mathfrak{X}(T) \subset W^a$, $\mu \in \mathfrak{X}(T)$.

Let $L \subset P$ be Levi subgroup; $\mu_x := x^{-1}\cdot(-2\rho)$.
$x \in W^{a,P} \Leftrightarrow \mu_x$ is dominant for $L \rightsquigarrow$
Weyl module $W_{L,\mathbb{F}}(\mu_x)$. Can assume $\Gamma \subset L$, then we can view $W_{L,\mathbb{F}}(\mu_x)$ as an object of $\mathcal{U}_{0,\mathbb{F}}(\mathfrak{l})\text{-mod}^\Gamma$.

Further, $\mathcal{U}_{0,\mathbb{F}}(\mathfrak{p}) \hookrightarrow \mathcal{U}_{e,\mathbb{F}}(\mathfrak{g})$ b/c $e \in \mathfrak{p}^\perp$ & $\mathcal{U}_{e,\mathbb{F}}$ is a free rank $p^{\dim \mathfrak{g}/\mathfrak{p}}$-module over $\mathcal{U}_{0,\mathbb{F}}(\mathfrak{p})$.

Proposition: (R.B.-I.L.) The class of standard labelled by x goes to that of $\mathcal{U}_{e,\mathbb{F}}(\mathfrak{g}) \otimes_{\mathcal{U}_{0,\mathbb{F}}(\mathfrak{p})} W_{L,\mathbb{F}}(\mu_x)$.