Remark on Frobenius weights.

Discussion equivalence $D_\lambda/F\ell) = D^\epsilon \text{Gr}^{\nu}(\text{St})$.

Here $K^0$ of both sides in $\mathbb{Z}[[\text{Witt}]]$. We are interested in $H_{\ell\ell}$.}

In Beilinson-Bernstein part of KL conjectures the problem is reduced to a general theorem $[\text{BBD}]:$ if $F, Y$ are pure wt $0$ (l-adic) Weil sheaves, then Frobenius acts on $\text{Ext}^n(F, Y)$ with positive weights. This is translated into a statement about $\text{Ext}$ between objects in category $\mathcal{O}$ using localization Theorem + Riemann-Hilbert + comparison of etale & classical settings.
The $\mathcal{F}_i$ etc are defined over $k$.

Thm. Assume $k=\overline{F}_p$ and we work with etale l-adic sheaves. The equivalences $D_\mathbb{I}(\mathbb{F}_i) \simeq D^b C^b_{\mathbb{I}}(\mathcal{St})$ intertwines Frobenius $F_{\mathbb{I}}$ with the functor $\mathfrak{f}^*$, where $\mathfrak{f} : (\mathfrak{f}_s, \mathfrak{f}_y, x) \mapsto (\mathfrak{f}_s, \mathfrak{f}_y, \mathfrak{f}^* x)$.

$[\mathcal{St} = \overline{\mathbb{N}} \times \overline{\mathbb{N}}]$  

This follows from the construction: $x$ comes from big monodromy acting on the central sheaves. So $F_{\mathfrak{f}}^* m = m^* F_{\mathfrak{f}}$ - a relation in $\text{Gal}(F_{\mathfrak{f}}(\mathbb{I}))$ implies the Thm.

Since $\mathfrak{f} \in D^b C^\infty_{\mathbb{I}}(\mathcal{St})$, its image in $D_\mathbb{I}(\mathbb{F}_i)$ acquires a Weil structure, for $\mathfrak{f}, \mathfrak{f}_j \in D^b C^\infty_{\mathbb{I}}(\mathcal{St})$,

$\text{Ext}^i(\mathfrak{f}, \mathfrak{f}_j) \otimes_{\mathfrak{f}} 1 d) = \text{Ext}^i(\mathfrak{f}(\mathfrak{f}), \mathfrak{f}(\mathfrak{f}_j))$ evaluated in the degree $d$ part.
The rest is about the question: what can be said about the image of perverse sheaves under the equivalences $D^b(\mathcal{F}) \cong D^b CH^*(\tilde{\mathcal{N}} \times \tilde{\mathcal{N}})$ etc.

A) $D^b(\mathcal{F})$ & nilpotent orbits

Recall the notion of a 2-sided cell in a Weyl group $W$ (either finite or affine).

These are subsets $\mathcal{C}$ in $W$, $W = \bigcup \mathcal{C}$.

The set of 2-sided cells is partially ordered.

Key property: $H_{\leq \mathcal{C}} := \langle C_w \mid w \in \mathcal{C} \leq \mathcal{C}' \rangle$ is a 2-sided ideal in the (affine) Hecke algebra $H$. 
This has a natural categorification,
The Hecke category:
\[ H = H_{\leq c} := \langle IC_w \mid w \leq c \leq c' \rangle \] - full triangulated subcategory.

Focus on \( W = W_{aff} \), monoidal 2-sided ideal

Lusztig constructed a bijection
\[ \{ 2 \text{- sided cells in } W_{aff} \} \leftrightarrow \text{nilpotent orbits in } o^g \]
\[ c \rightarrow O_c. \]

Let \( St_c \) be the preimage of \( O_c \) under the map \( St \rightarrow N^g \).

Let \( St_{\leq c} \) be the preimage of the closure of \( O_c \).

Prop. The equivalence
\[ H = D^b_{\mathcal{I}^c} (H) = D^b \text{Coh}^G (St) \]
restricts to \( H_{\leq c} = D^b \text{Coh}_{St_{\leq c}}^G (St) \).

and similarly for \( D_{\mathcal{I} \circ \mathcal{I}} \), \( D_{\mathcal{I}^c \circ \mathcal{I}^c} \) etc.

\[ \text{Coh}_2^G (X) - \text{coherent sheaves on } X \]
\[ \text{vanishing on } X \setminus 2 \]
B) Perv(FL) & perverse coherent sheaves on N.

Coherent description of Perv(FL) has two aspects: global over N & local over an object O ∈ N.
Start with the global one.

Recall $D_{IW}(FL) = D^b_Coh_c(\tilde{\mathcal{N}})$.

Irreducible objects are $IC_w$, $w ∈ Fr W ∈ \mathcal{W}$.

$W = \{ w | L(w) ≤ L(z_w) \forall z ∈ W \}$

Let also $W^f = \{ w | L(w) ≤ L(u w z_w) \}$.

$\forall u, z ∈ W \forall n ≥ 0$

$D^{as_p}_{IW} = D_{IW}(FL) / \langle IC_w, w ∈ W^f \rangle$

$D_{IW}(FL) = D^{as_p}_{IW}$
\[ \text{Pew}^{\text{app}}_{\text{IW}} = \text{Pew}^{\text{app}}_{\text{IW}}(\pi^*) / \langle \text{IC}_{\text{IW}}, \text{IW} \rangle. \]

(categorification of \( e_\text{h}^* H_{\text{h}} e_\text{h}^* \), \( e_\text{h} \in H_{\text{h}} \), sign identical)

Thm a) \( D^{\text{app}}_{\text{IW}} = D^6 \text{Coh}_{\Sigma}^G(N) \).
b) \( \text{Pew}^{\text{app}}_{\text{IW}} = \text{PCoh}_{\Sigma}^G(N) \).

\( \text{PCoh}_{\Sigma}^G(N) \) are perverse coherent sheaves of middle perversity.

Comment on a) \( D^{\text{app}}_{\text{IW}} \) to \( D^{\text{app}}_{\text{IW}} \) is 
like \( D_{\text{IW}}(\pi^*) \) to \( D_{\text{IW}} \).

Proof of b) assuming a).
The perverse \( t \)-structure is defined on \( D^6 \text{Coh}(X) \) for a stack \( X \) and a perversity function \( p: |X| \to \mathbb{Z} \) which is monotone and comonotone; for \( y \in X \).

\[ 0 \leq p(y) - p(x) \leq \text{dim } X - \text{dim } y. \]
If the inequalities are strict then one can define the minimal (GM) extension and classify irreducible objects [Delegue (unpublished), Gabber, Kashiwara, Aunkin-B.

\[ \text{Ex} \quad X = \mathbb{N}/G^v, \quad \rho(x) = \text{codim}(x) \]

(use that all orbits have even dimension)

Another example: \( \mathbb{H}/G(1) \) [Rank in B. Finkelberg]

Related to cluster algebras and dual canonical basis: [Cautis-Williams, Finkelberg-Fujita]

Now to prove a) observe that \( \pi_* (L) \in \text{Pcoh}^F(N) \) for a line bundle \( L \) on \( \widetilde{\mathbb{N}} \) b/c it is semi-small.

Moreover, a bounded \( t \)-structure whose heart contains \( \psi_\ast(L) \) for every line bundle \( L \) equals \( \text{Pcoh}[\psi] \) [enough to take \( L = O_{\mathbb{N}} \)]

But \( \psi_\ast(L) \) is the image of \( \psi_* \in \text{Pev}^\mathbb{N}(H) \),

Wakimoto sheaves.
Rank. Perverse coherent sheaves of middle perversity appear also in geometric Langlands for $GL(1)$ [C. Schnell]. Is there a common context/generalization?
C) NC Springer resolution (local over $N$ aspect).

Notice that the categories $D_{II}, \hat{D}_{II}$ come with a (a priori weak) homomorphism from the (extended) affine braid group $\overline{B}$ to $\hat{D}_{II}, \hat{D}_{II}.$

$$\overline{B} = \langle \tilde{w} | w \in W_{aff} \rangle / \tilde{w}_1 \tilde{w}_2 = \tilde{w}_1 \tilde{w}_2$$

$$\forall \ e(w, w_1) = e(w) + e(w_2)$$

$$\tilde{w} \mapsto T^{-1}_{\tilde{w}} = j^* w^n \text{ (costandard object)}$$

Theorem: There are (a priori weak) homomorphisms

$$\overline{B} \rightarrow \mathcal{D}^6 \text{Coh}^\wedge_\mathfrak{g} / \mathcal{G}^\wedge \times \mathcal{G}^\wedge$$

compatible with the above.
Explicit description:

$$\mathbb{B}aff = \langle \tilde{w}, \theta_\Delta \mid w \in W \subset \text{Waff}, \rangle$$

$$\Lambda = X^*_+ (\tilde{T}^\vee) = X^*_+ (\tilde{T})$$

$$\tilde{w} \mapsto \Theta_{\tilde{w}}, \quad Z_w = \Gamma_w \subset \tilde{\mathcal{O}}_{\tilde{T}} \times \tilde{\mathcal{O}}_{\tilde{T}}$$

$$\Gamma_w$$ is the graph of $$w$$ acting on $$\tilde{\mathcal{O}}_{\tilde{T} \text{ reg}}$$.

$$\theta_\Delta \mapsto \theta_\Delta (\Lambda)$$ (line bundles on the diagonal)


Key point: the same $$\mathbb{B}aff$$ action arises in other contexts.

$$\text{D}^b \text{ Coh}^\sim (\tilde{\mathcal{N}})$$ is identified with a derived category of representations (principal block), $$\mathbb{B}aff$$ acts by reflection functors.
Theorem. a) There exists a unique $t$-structure on $\mathcal{D}^b \text{Coh}(\widetilde{\mathfrak{g}}), \mathcal{D}^b \text{Coh}(\mathfrak{g})$, s.t.

i) (normalization) $R\mathcal{F}$ is $t$-exact

ii) (braid positivity) The generators $\mathcal{S}_a$ (a runs over vertices of affine Dynkin graph) act by right exact functors.

b) Some (hence any) projective generator for the heart is a vector bundle.

If $E$ is a projective generator, $A = \text{End}(E)^{\text{op}}$, then $\mathcal{F} \mapsto R\text{Hom}(E, \mathcal{F})$ is an equivalence $\mathcal{D}^b \text{Coh}(\widetilde{\mathfrak{g}}) \cong \mathcal{D}^b(A\text{-mod})$.

$A$ is the noncommutative Springer resolution - a NC resolution of $\mathfrak{g}$ (form by van der Bergh, bondal-orel).

$A' = \text{End}(E^{\text{op}})^{\text{op}}, \quad A = A' \otimes_{\mathbb{Z}[T]} \mathbb{K}.$

$E_{\mathcal{A}}, \ G = \text{SL}(2), \ E = \mathcal{O} \oplus \mathcal{O}(1), \ A\text{-mod} = \text{Coh}(\mathbb{A}^1/\mathcal{A}^1).$
\[ A \simeq \left( \begin{array}{cc} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{array} \right) \in \text{Mat}_2(k[x,y]). \]

Equivariant version.

\[ D^b \text{Col}_h^G(N) \simeq D^b(A-\text{mod}^G_v) \]

equivariant \( A \)-modules.

Thus

\[ D^b \text{Iw}(\mathcal{F} \ell) \simeq D^b(\text{Col}^G_v(N)) = D^b_! A-\text{mod}^G_v \]

\[ \text{Rep}_1 \text{Iw}(\mathcal{F} \ell) \xrightarrow{\sim} D(PA-\text{mod}^G_v) \]

\[ \text{perverse coherent} \]

\[ \text{equivariant } A \text{-modules of middle perversity.} \]
D) NC Springer and RT.

Thin Fix $e \in N$ and consider $A_e = \mathcal{A} \otimes \mathcal{K}_e$.

a) [B - Mirković]

If $\text{char } (k) = p > 1$, then $A_e$-mod = $\mathcal{O}_g$-mod, modules with a fixed central char.

b) [B - Qian Lih], based on Freulein - Gaitsgory.

If $\text{char } k = 0$, $A_e \cong (\hat{g}, I)$-modules at the critical level with a fixed central character (open with residue).

c) (Conjecture by B - Mirković, A Backelin - Kremmizer, Tanisaki)

If $\text{char } k = 0$, $A_e \cong \mathcal{U}_q$-mod, modules over the Kac - De Concini quantum group at a root of 1, with a fixed central char.

Applications: in a) "Lusztig's Hope"

Proof of conjectures on character of irreducible $\mathcal{O}_g$-modules for (indefinitely) large $p$, answer in terms of the canonical basis in $V_\lambda (\mathfrak{g})$.

Potential applications in b,c) - character formulas for these modules.

\[ K(\text{A}_e\text{-mod}) \cong K(\mathcal{B}_e) \cong \text{H}^0(\mathcal{B}_e) \] - the standard rem. of Hall
Morale of the story: $A_e$-mod is the $I$-invariant part of the local Langlands category. Langlands categories beyond $I$-invariants deserve further study, Freiedel-Gaitsgory say, Thm 6) should generalize.

Another generalization: replace $N$ by a symplectic singularity (workshop talk).

Further applications:
* study of asymptotic Hecke algebra category $J$.
* Local $L$-functions of tamely ramified reps Galois representations (Braverman, Finkelberg, Kazhdan, in progress).

$E$ - vector bundle on $\tilde{\mathcal{X}}$ or $\tilde{\mathcal{Y}}$.

1) $E^*_B = F^*_\mathcal{X}(\mathcal{O}_B)$

$s$ even $p > 1$

$F^*_\mathcal{X} \mathcal{O}_p^s = \mathcal{O} \oplus \mathcal{O}(-1)^{p-1}$
The story works for \( \mathfrak{g}/\mathfrak{p} \), for \( P^n \), get \( 0 \oplus O(1) \oplus \cdots \oplus O(r-1) \).

For \( B = SU(3) \), \( E \) has 3 summands of rank 1

\[
\begin{align*}
3 & \rightarrow \begin{cases} \text{rank } 1 \quad & \text{one of these is not pulled back for } B, \\
1 & \end{cases}
\end{align*}
\]

Introduce \( E_d \) - geometric wall crossing: \( D'' \mathcal{O}_H(0) \)

\( d \)-vertex of affine Dynkin.

If \( d = 1 \), then \( E_d = \prod \text{finite } \eta_1 \cdots \eta_k \Rightarrow \eta_j \rightarrow \overline{\eta}_j \).

\[
E = \bigoplus \frac{\mathbb{C}^*}{\mathbb{C}^*} \frac{\mathbb{C}^n}{\mathbb{C}^n} (0)
\]

\[
\text{Perv}_{E_0}(\mathcal{F}) \ni E_w \ast \lambda(\nu) \Rightarrow \text{tilting}.
\]

\[
\text{Tw}_{\nu}, \text{Tw for } w \text{-max in } W \text{ in } W \text{ in a summand}
\]

Recall the asymptotic Hecke algebra \( \mathcal{J} \).

\[
\mathcal{J} = \bigoplus \mathcal{J}_e.
\]

\[
\text{Con. } \mathcal{J}_e = K \left( \mathcal{A}_e \otimes \mathcal{A}_e^{\geq 0} \text{ mod } \mathcal{I}_e \right), \ e \in O_e.
\]

"Clear" we get based ideals \( e \leq e' \).

Comparing bijections \( \mathcal{I} \leq \mathcal{J} \rightarrow \mathcal{N}/\mathcal{G} \).

Cuspidal uses the hom 

\[
\text{Hass} \rightarrow \mathcal{J} \left[ \mathfrak{g}, \mathfrak{g}^{-}, \mathfrak{g}; \mathfrak{g}^{+} \right]
\]

Start with a rep \( \mathfrak{g} \rightarrow \mathcal{J}_e \) (other \( \mathcal{G} \))

pull-back - take central char.

get \( \mathcal{Z} \in \mathcal{G}^{'}(\mathcal{O}[\mathfrak{g}, \mathfrak{g}^{-}, \mathfrak{g}; \mathfrak{g}^{+}]) \).

\[
\mathcal{Z} = z \circ \chi \left( \text{deg}_{\nu}(\mathfrak{g}, \mathfrak{g}^{-}) \right), \ \chi : \text{SL}(2) \rightarrow \mathcal{G}^{'}.
\]
\[
\frac{d\theta}{\theta} = e
\]

This connects to our construction using Calabi-Yau weight.

weight = monadric field.