Haff & NC Springer resolution.

I Remark on Frobenius weights.

Discussed equivalence $D_I(\mathcal{Fl}) \simeq D^b Coh^{G^\vee}(St)$ etc

- Here K^0 of both sides is $\mathbb{Z}[W_{aff}]$ | $H_{aff} = K\, Coh^{G^\vee \times \mathbb{G}_m}$

We are interested in H_{aff}.

- In Beilinson-Bernstein proof of KL conjectures the problem is reduced to a general Theorem [BBD]: if $\mathcal{F}, \mathcal{G}$ are pure wt 0 (ℓ-adic) Weil sheaves then Frobenius acts on $Ext^1(\mathcal{F}, \mathcal{G})$ with positive weights, this is translated into a statement about Ext between objects in category $\mathcal{O}$ using localization Theorem + Riemann-Hilbert + comparison of etale & classical settings.

$\mathcal{Fl}$ etc are defined over k.

Thm. Assume $k = \overline{\mathbb{F}_p}$ and we work with etale ℓ-adic sheaves. The equivalences

$$D_I(\mathcal{Fl}) \cong D^b Coh^{G^\vee}(St)$$ etc. intertwines Frobenius Fr_q with the functor q^*, where $q : (b_1, b_2, x) \mapsto (b_1, b_2, qx)$

$[St = \tilde{N} \times_{N} \tilde{N}]$

This follows from the construction: x comes from log monodromy acting on the central sheaves. So $Fr\, m = m^q\, Fr$ - a relation in $Gal(\mathbb{F}_q((t)))$ implies the Thm.

(assume it's in $Perv_I(\mathcal{Fl})$)

So if $\mathcal{F} \in D^b Coh^{G^\vee \times \mathbb{G}_m}(St)$, its image in $D_I(\mathcal{Fl})$ acquires a Weil structure, for $\mathcal{F}, \mathcal{G} \in D^b Coh^{G^\vee \times \mathbb{G}_m}(St)$

$$\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G})(d) \cong \mathrm{Ext}^\bullet(\Phi(\mathcal{F}), \Phi(\mathcal{G}))_{Fr\text{-wt } d}.$$

$\mathbb{G}_m$ degree d part

II The rest is about the question: what can be said about the image of perverse sheaves under the equivalences $D^b_I(\mathcal{F}\ell) \simeq D^b Coh^G(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}})$ etc.

A) $Perv(\mathcal{F}\ell)$ & nilpotent orbits

Recall the notion of a 2-sided cell [Lusztig] in a Weyl group W (either finite of affine)

These are subsets $\underline{c}$ in W, $W = \coprod \underline{c}$.
The set of 2-sided cells is partially ordered.

Key property: $H_{\leq \underline{c}} := \{ C_w \mid w \in \underline{c}' \leq \underline{c} \}$ is a 2-sided ideal in the (affine) Hecke algebra H.

This has a natural categorification.
The Hecke category:

$\mathcal{H} \supset \mathcal{H}_{\leq \underline{c}} := \langle IC_w \mid w \in \underline{c}' \leq \underline{c} \rangle$ - full triangulated subcategory - monoidal 2-sided ideal

Focus on $W = W_{aff}$.

Lusztig constructed a bijection

$$\{\text{2-sided cells in } W_{aff}\} \longleftrightarrow \text{nilpotent orbits in } \mathfrak{g}^\vee$$

$$\underline{c} \longmapsto O_{\underline{c}}.$$

Let $St_{\underline{c}}$ be the preimage of $O_{\underline{c}}$ under the map $St \longrightarrow \mathcal{N} \subset \mathfrak{g}^\vee$.

Let $St_{\leq \underline{c}}$ be the preimage of the closure of $O_{\underline{c}}$.

<u>Prop.</u> The equivalence

$$\mathcal{H} = D^b_I(\mathcal{Fl}) \simeq D^b Coh^{G^\vee}(St)$$

restricts to $\mathcal{H}_{\leq \underline{c}} \simeq D^b Coh^{G^\vee}_{St_{\leq \underline{c}}}(St)$.

and similarly for $D_{I^\circ I}$, $D_{I^\circ I^\circ}$ etc.

[$Coh_Z(X)$ - coherent sheaves on X vanishing on $X \setminus Z$].

B) Perv($\mathcal{F}\ell$) & perverse coherent (equivariant) sheaves on $\mathcal{N}$.

Coherent description of Perv ($\mathcal{F}\ell$) has two aspects: global over $\mathcal{N}$ & local over an orbit $\mathbb{O} \subset \mathcal{N}$.

Start with the global one.

Recall $\quad D_{IW}(\mathcal{F}\ell) \simeq D^b Coh^{G^\vee}(\tilde{\mathcal{N}})$.

Irreducible objects are IC_w, $\quad w \in {}^f W \subset W_{aff}$

$${}^f W = \{ w \mid \ell(w) \le \ell(zw) \ \forall\ z \in W \}$$

Let also ${}^f W^f = \{ w \mid \ell(w) \le \ell(uwz) \}$. $\ \forall\ u, z \in W$

$$\updownarrow$$

$$\Lambda^+$$

$$D_{IW}^{asp} = D_{IW}(\mathcal{F}\ell) / \langle IC_w,\ w \notin {}^f W^f \rangle$$

$D_{IW}(\mathcal{F}\ell) \simeq$
$\simeq D^b(\mathrm{Perv}_{IW}(\mathcal{F}\ell))$

$\cup$

$$\mathrm{Perv}^{asp}_{IW} = \mathrm{Perv}_{IW}(\mathcal{Fl}) / \langle IC_w \mid w \notin {}^f W^f \rangle.$$

(categorification of $e_- \mathcal{H}_{aff} e_-$, $e_- \in H_f$)
e_-: sign idempotent

Thm a) $D^{asp}_{IW} \simeq D^b \mathrm{Coh}^{G^\vee}(\mathcal{N})$.

b) $\mathrm{Perv}^{asp}_{IW} \simeq P\mathrm{Coh}^{G^\vee}(\mathcal{N})$.

$P\mathrm{Coh}^{G^\vee}(\mathcal{N})$ are perverse coherent sheaves of middle perversity.

Comment on a): D^{asp}_{IW} to D_{IW} is like $D_{IW}(\mathcal{Fl})$ to D_{II}.

$$\begin{tikzcd} D_{IW}(\mathcal{Fl}) \simeq D^b\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \arrow[d] \arrow[d, shift left=20ex, "R\pi_*"] \\ D^{asp}_{IW}(\mathcal{Fl}) \simeq D^b\mathrm{Coh}^{G^\vee}(\mathcal{N}) \end{tikzcd}$$

$\pi: \tilde{\mathcal{N}} \to \mathcal{N}$

Proof of b) assuming a).

The perverse t-structure is defined on $D^b\mathrm{Coh}(X)$ for a stack X and a perversity function $p: |X| \to \mathbb{Z}$ which is monotone and comonotone: for $y \in \overline{x}$.

$$0 \le p(y) - p(x) \le \dim \overline{x} - \dim \overline{y}.$$

$\mathcal{O}_x \xrightarrow{f} \mathcal{O}_x$ - cone is not locally free on X

dual perversity $\Leftrightarrow$ $p'(x) = -\dim(X) - p$ is monotone.

If the inequalities are strict then one can define the minimal (GM) extension and classify irreducible objects [Deligne (unpublished), Gabber, Kashiwara, Arinkin-B.].

Ex $X = \mathcal{N}/G^\vee$, $p(x) = \frac{\operatorname{codim}(x)}{2}$

(use that all orbits have even dimension)

another example: $Gr/G(O)$ [Rmk in B-Finkelberg-Mirković]
Related to cluster algebras and dual canonical bases; Cautis-Williams; Finkelberg-Fujita

Now to prove a) observe that $\pi_*(L) \in PCoh^{G^\vee}(\mathcal{N})$ for a line bundle L on $\tilde{\mathcal{N}}$ b/c π is semismall.

Moreover, a bounded t-structure whose heart contains $\pi_*(L)$ for every line bundle L equals $PCoh$ [enough to take $L = O(\lambda)$, $\pm\lambda \in \Lambda^+$, these are (co) standard objects]

But $\pi_*(L)$ is the image of $J_\lambda \in Perv_I(Fl)$.

Wakimoto sheaves.

Rmk. Perverse coherent sheaves of middle perversity appear also in geom. Langlands for $GL(1)$ [C. Schnell].
[Popa–Schnell, Rothstein, Laumon]

Is there a common context/generalization?

C) NC Springer resolution (local over $\mathcal{N}$ aspect).

Notice that the categories D_{II}, $\widehat{D_{I_0 I_0}}$ come with a (a'priori weak) homomorphism from the (extended) affine braid group: to $D^{\times}_{II}$, $D^{\times}_{I^{\circ} I^{\circ}}$.

$$B_{aff} = \langle \tilde{w} \mid w \in W_{aff} \rangle / \widetilde{w_1 w_2} = \tilde{w}_1 \tilde{w}_2$$
$$\text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$$

$$\tilde{w} \longmapsto T_{w^{-1}}^{-1} = j_{w*} \quad \text{(costandard object)}$$

<u>Theorem</u> There are (a'priori weak) homomorphisms

$$\begin{array}{ccc} B_{aff} & \Longrightarrow & D^b Coh^{G^\vee}(\tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee)^{\wedge} \\ \downarrow & & \\ D^b Coh^{G^\vee}(St) & & \end{array}$$

compatible with the above

Explicit description:

$$B_{aff} = \left\langle \tilde{w}, \theta_\lambda \;\middle|\; \begin{array}{l} w \in W \subset W_{aff}, \\ \lambda \in \Lambda = X^*(T^\vee) = X_*(T) \end{array} \right\rangle$$

$\tilde{w} \longmapsto \mathcal{O}_{Z_w}$, $\quad Z_w = \overline{\Gamma_w} \subset \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ $\quad \boxed{\begin{array}{ccc} \tilde{\mathfrak{g}} & \supset & \tilde{\mathfrak{g}}^{reg} \circlearrowleft W \\ \downarrow & & \downarrow \\ \mathfrak{g} & \supset & \mathfrak{g}^{reg} \end{array}}$

Γ_w is the graph of w acting on $\tilde{\mathfrak{g}}^{reg}$.

$\theta_\lambda \longmapsto \mathcal{O}_\Delta(\lambda)$ (line bdle on the diagonal).

[B, Riche – uses char p localization].

Key point: the same B_{aff} action arises in other contexts.

$D^b Coh^{G^\vee}(\tilde{\mathcal{N}})$ is identified with a derived category of representations (principal block), B_{aff} acts by reflection functors.

<u>Theorem</u>. a) There exists a unique t-structure on $D^b Coh(\tilde{\mathcal{N}})$, $D^b Coh(\tilde{\mathfrak{g}})$ s.t.

i) (normalization) $R\Gamma$ is t-exact

ii) (braid positivity) The generators $\tilde{S}_\alpha$ (α runs over vertices of <u>affine</u> Dynkin graph) act by right exact functors.

b) Some (hence any) projective generator for the heart is a vector bundle.

If $\mathcal{E}$ is a projective generator, $A = End(\mathcal{E})^{op}$, then $\mathcal{F} \longmapsto R Hom(\mathcal{E}, \mathcal{F})$ is an equivalence $D^b Coh(\tilde{\mathcal{N}}) \simeq D^b(A\text{-mod})$.

fin. gen-d modules.

A is the <u>noncommutative Springer resolution</u>. - a NC resolution of $\mathcal{N}$ (term by van den Bergh Bondal-Orlov).

$A' = End(\mathcal{E}_{\tilde{\mathfrak{g}}})^{op}$, $A = A' \otimes_{k[\mathfrak{t}]} k$.

Ex. $G = SL(2)$, $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$.

$A\text{-mod} \simeq Coh^{\mathbb{Z}/2}(\mathbb{A}^2)$.

$$A \simeq \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \subset Mat_2(k[x,y]).$$

Equivariant version.

$$D^b Coh^{G^\vee}(\tilde{\mathcal{N}}) \simeq D^b(A\text{-mod}^{G^\vee})$$

equivariant A-modules.

<u>Thm</u>

$$\begin{array}{ccc} D^b_{IW}(\mathcal{F}\ell) & \simeq D^b(Coh^{G^\vee}(\tilde{\mathcal{N}})) \simeq & D^b(A\text{-mod}^{G^\vee}) \\ \cup & & \cup \\ Perv_{IW}(\mathcal{F}\ell) & \xrightarrow{\sim} & D(PA\text{-mod}^{G^\vee}) \end{array}$$

perverse coherent equivariant A-modules of middle perversity.

D) NC Springer and RT.

Thm Fix $e \in \mathcal{N}$ and consider $A_e = A \otimes_{k[\mathcal{N}]} k_e$.

a) [B - Mirković]
If $\mathrm{char}(k) = p > h$ then $A_e\text{-mod} \simeq \mathfrak{g}\text{-mod}_e$, modules with a fixed central char.

b) [B - Qian Lin], based on Frenkel - Gaitsgory.
If $\mathrm{char}\, k = 0$, $A_e \simeq (\widehat{\mathfrak{g}^\vee}, I)$-modules at the critical level with a fixed central character (oper with residue e).

c) (Conjecture by B-Mirković, cf. Backelin-Kremnizer, Tanisaki)
If $\mathrm{char}\, k = 0$, $A_e \simeq \mathcal{U}_q^{KDC}\text{-mod}_e$ - modules over the Kac-De Concini quantum group at a root of 1 with a fixed central char.

Applications: in a) "Lusztig's hope"
Proof of conjectures on characters of irreducible $\mathfrak{g}$-modules for (indefinitely) large p, answer is in terms of the canonical basis in $H_*(B_e)$.
Potential applications in b, c) - character f-las for these modules.

$K(A_e\text{-mod}) \simeq K(B_e) \simeq H_*(B_e)$ - the standard repr-n of H_{aff}

springer fiber

Moral of the story: A_e-mod is the I-invariant part of the local Langlands category.

Langlands categories beyond I-invariants deserve further study, Frenkel-Gaitsgory say Thm 6) should generalize.

Another generalization: replace $\mathcal{N}$ by a symplectic singularity (workshop talk).

Further applications:

- study of asymptotic Hecke algebra & category J.

Local L-functions of tamely ramified rep-s

Galois representations. (w. Braverman, Finkelberg, Kazhdan, in progress).

$\mathcal{E}$ - vector bundle on $\tilde{\mathcal{N}}$ or $\tilde{\mathfrak{g}}$.

1) $\mathcal{E}^*|_{\mathcal{B}} \simeq Fr_*(\mathcal{O}_{\mathcal{B}})$

in char $p > h$

$Fr_* \mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O} \oplus \mathcal{O}(-1)^{p-1}$

The story works for G/P, for $\mathbb{P}^n$, get $\mathcal{O}\oplus\mathcal{O}(1)\oplus\cdots\oplus\mathcal{O}(p-1)$.

For ~~Bef~~ $\tilde{\mathcal{N}}_{SL(3)}$, $\mathcal{E}$ has 3 summands of rank 1

3 — " — of rank 2

one of these is not pulled back for B.

Introduce Ξ_α - geometric wall crossing : $D^b Coh(\tilde{\mathfrak{g}}) \circlearrowleft$

α - vertex of affine Dynkin.

If α - " - of finite - " - : $\Xi_\alpha = \pi_\alpha^* \pi_{\alpha *}$

$\pi_\alpha : \tilde{\mathfrak{g}} \to \tilde{\tilde{\mathfrak{g}}}_\alpha$.

$\mathcal{E} = \oplus\, \Xi_{\alpha_1} \cdots \Xi_{\alpha_n} (\mathcal{O})$

$Perv_{I_0}(\mathcal{F}l) \ni \Xi_{w_0} * Z(V)$ - tilting.

T_{w_0}

T_w for w - max in WwW is a summand

Recall the asymptotic Hecke algebra J.

$J = \bigoplus_{\underline{c}} J_{\underline{c}}$.

Cor. $J_{\underline{c}} = K\left(A_e \otimes A_e^{op} \text{-}mod_{ss.}^{Z(e)}\right)$, $e \in O_{\underline{c}}$.

"Clear" we get based ideals $\leftrightarrow$ $\underline{c}$.

Comparing bijections $\{\underline{c}\} \leftrightarrow \mathcal{N}/G^\vee$.

Lusztig uses the hom-m $H_{aff} \to J[z,z^{-1}]$

Start with a ~~pat~~ repr-n of $J_{\underline{c}}$ (over $\mathbb{C}$!)

pull-back, take central char.

get $z \in G^\vee(\mathbb{C}[z,z^{-1}])$.

$z = z_0 \cdot \gamma(diag(z,z^{-1}))$, $\gamma : SL(2) \to G^\vee$.

$$d\gamma\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e.$$

This connects to our construction using Gabber's Thm

weight = monodromic filtr-n.