On two geometric Realisations of an affine Hecke algebra I

Notation: G reductive grp.

$\cup$

B Borel

$\cup$

T max'l torus

W_f Weyl group

W affine Weyl group

I Iwahori subgroup.

$I^\circ \subseteq I$ unipotent radical

$\mathcal{F}\ell = G(\mathcal{K})/I$ affine flag variety

$\widetilde{\mathcal{F}\ell} = G(\mathcal{K})/I^\circ \to \mathcal{F}\ell$ T-bundle

$G^\vee$ Langlands dual.

$\cup$

$B^\vee$

$\cup$

$T^\vee$

$\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$

$\tilde{\mathfrak{g}}^\vee$ = Grothendieck-Springer resolution.

$\tilde{N}^\vee$ = Springer resolution

$St = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee$

$\hat{\tilde{\mathfrak{g}}}^\vee$ completion at N

$\hat{St}$ " " "

The constructible categories

$D_{II} = D^b_I(\mathcal{F}\ell)$ cat of I-equivariant constructible sheaves on $\mathcal{F}\ell$.

$D_{I_0 I} = D^b_{I_0}(\mathcal{F}\ell)$ " " I_0 " " " " "

$D_{I_0 I_0}$ = I_0-equivariant sheaves on $\widetilde{\mathcal{F}\ell}$, T-monodromic of unipotent monodromy,

$\circlearrowleft$
$\mathcal{O}(t^\vee)$ is full subcat of $D^b_{I_0}(\widetilde{\mathcal{F}\ell})$ gen'd by $\pi^*\mathcal{F}$, $\pi : \widetilde{\mathcal{F}\ell} \to \mathcal{F}\ell$ $\mathcal{F} \in D_{I_0 I}$

$\hat{D}$ = completion of $D_{I_0 I_0}$ to contain free monodromic objects.

$\circlearrowleft$
$\mathcal{O}(\hat{t}^\vee)$ completion of $\mathcal{O}(t^\vee)$ at 0.

$D^I_{IW} = D^b_{IW}(\mathcal{F}\ell)$ Iwahori-Whittaker sheaves on $\mathcal{F}\ell$.

$D^{I_0}_{IW}$ = Iwahori-Whittaker sheaves on $\widetilde{\mathcal{F}\ell}$, T monodromic of unipotent monodromy.

$\hat{D}_{IW}$ = completion of $D^{I_0}_{IW}$

Basic notions of $\hat{D}$

The identity $1 \in \mathcal{F}l$ has preimage $T \subseteq \widetilde{\mathcal{F}l}$

We can construct monodromic sheaves on T by constructing representations of $\pi_1(T)$ we have a unipotent action of this on $\mathcal{O}(t^\vee)/m^n$ giving sheaves $\mathcal{F}_n$ on T where $\mathcal{O}(t^\vee)$ acts via $\mathcal{O}(t^\vee)/m^n$

$\hat{D}$ contains the pro-object $\tilde{\delta}$ limit of these $\mathcal{F}_n$

For $\mathcal{F}l_w \subseteq \mathcal{F}l$ the preimage on $\widetilde{\mathcal{F}l}$ is isomorphic to $\mathbb{A}^{\ell(w)} \times T$ so has a sheaf $j_{w!}(\mathbb{C}_{\mathbb{A}^{\ell(w)}} \boxtimes \tilde{\delta}) =: j_{w!} \in \hat{D}$ the standard objects.

Main results

Theorem 1

1) $\hat{D} \cong D^b Coh^{G^\vee}(\hat{St})$

2) $D_{I_0 I_0} \cong D^b Coh^{G^\vee}_{N^\vee}(St)$

3) $D_{I_0 I} \cong D^b Coh^{G^\vee}(\mathfrak{g}^\vee \times_{\mathfrak{g}^\vee} \tilde{N}^\vee)$

4) $D_{II} \cong D^b Coh^{G^\vee}(\tilde{N}^\vee \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{N}^\vee)$

follow by looking at the $\mathcal{O}(t^\vee)$ - action

Equivalences 1), 2) and 4) are as monoidal categories under convolution.
3) as module over 2) and 4).

Theorem 2

$$\hat{D}_{IW} \cong D^b Coh^{G^\vee}(\hat{\tilde{\mathfrak{g}}})$$

$$D^{I_0}_{IW} \cong D^b Coh^{G^\vee}_N(\tilde{\mathfrak{g}})$$

$$D^{I}_{IW} \cong D^b Coh^{G^\vee}(\tilde{N})$$ (equivalence from last week)

compatibly with the actions and the equivalences of Thm 1.

Motivation from decategorified statement

$$K^{G^\vee\times\mathbb{G}_m}(St) \cong H_{aff}$$

$$K^{G^\vee\times\mathbb{G}_m}(\tilde{\mathfrak{g}}) \cong H_{aff}\,\xi$$

$$\xi = \sum_{w\in W_f} (-1)^{\ell(w)} T_w$$

$$\pi_1^* \pi_{1*}[\mathcal{F}] = [\mathcal{F}]\,\xi$$

ring homomorphism

$$(K^{G^\vee\times\mathbb{G}_m}(\tilde{\mathfrak{g}}),\otimes) \xrightarrow{\delta_*} (K^{G^\vee\times\mathbb{G}_m}(St),*)$$

$$H_{aff}\,\xi\, H_{aff} \subseteq H_{aff}$$

$$\begin{tikzcd}
D^b Coh^{G^\vee}(\hat{St}) \arrow[r, phantom, "\cong"] \arrow[d, shift left, "\pi_{1*}"] & \hat{D} \arrow[d, shift left, "Av_{IW}"] \\
D^b Coh^{G^\vee}(\hat{\tilde{\mathfrak{g}}}) \arrow[r, phantom, "\cong"] \arrow[u, shift left, "\pi_1^*"] & D_{IW} \arrow[u, shift left, "\hat{Av}_{I_0}"]
\end{tikzcd}$$

$$\hat{\tilde{\Xi}} \in \hat{D} \cong D^b Coh^{G^\vee}(\hat{St}) \ni \mathcal{O}_{St}$$

$$Av_{I_0} Av_{IW}(\mathcal{F}) \cong \mathcal{F} * \hat{\Xi}$$

monoidal functor

$$\Phi_{diag} : D^b Coh^{G^\vee}(\hat{\tilde{\mathfrak{g}}}) \longrightarrow \hat{D}$$

$$\Phi_{perf} : D^b_{perf, G^\vee}(\hat{St}) \longrightarrow \hat{D}$$

$$\mathcal{O}(\lambda,\mu)\otimes V \longmapsto \hat{J}_\lambda * \hat{Z}_V * \hat{\Xi} * \hat{J}_\mu$$

<u>Corollary</u> We have commutative diagrams

$$\begin{tikzcd}
\hat{D} \arrow[r, shift left, "Av_{IW}"] \arrow[d, phantom, "\wr\wr"] & \hat{D}_{IW} \arrow[l, shift left, "Av_{I_0}"] \arrow[d, phantom, "\wr\wr"] \\
D^b Coh^{G^\vee}(\hat{St}) \arrow[r, shift left, "\pi_*"] & D^b Coh^{G^\vee}(\hat{\tilde{g}}) \arrow[l, shift left, "\pi^*"]
\end{tikzcd}$$

And similarly for the other settings.

<u>Pf</u>: this follows from compatibility with convolution and $Av_{IW}(\tilde{\delta}) \longleftrightarrow \mathcal{O}_{\hat{\tilde{g}}}$ in the equivalence, noting that $Av_{IW}(\mathcal{F})$ is given by convolution with $Av_{IW}(\tilde{\delta})$ and $\pi_*(\mathcal{F})$ " " " " " $\mathcal{O}_{\hat{\tilde{g}}}$.

Construction of functors from $D^b_{perf, \check{G}}(\tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee)$.

<u>Recall from last time</u>: To construct a monoidal functor $D^b Coh^{G^\vee} \tilde{\mathfrak{g}}^\vee \to \mathcal{C}$

we need: 1) $F : Rep(G^\vee \times T^\vee) \to \mathcal{C}$. monoidal

2) $\mathcal{O}(t^\vee) \to End(F)$. $f_{v_1 \otimes v_2} = f_{v_1} \otimes id = id \otimes f_{v_2}$ $\quad v_1, v_2 \in Rep(G \times T)$, $f \in \mathcal{O}(t)$

3) $N_v : F(V) \to F(V)$ $\quad V \in Rep(G^\vee)$ tensor endomorphisms.

4) $B_\lambda : F(V_\lambda) \to F(\lambda)$

Satisfying:

$$\begin{tikzcd} F(V_\lambda \otimes V_\mu) \arrow[r] \arrow[d, "B_\lambda \otimes B_\mu"'] & F(V_{\lambda+\mu}) \arrow[d, "B_{\lambda+\mu}"] \\ F(\lambda) \otimes F(\mu) \arrow[r, "\sim"] & F(\lambda+\mu) \end{tikzcd} \qquad \begin{tikzcd} F(V_\lambda) \arrow[r, "B_\lambda"] \arrow[d, "N_{V_\lambda}"'] & F(\lambda) \arrow[d, "\lambda \in t \subseteq \mathcal{O}(t^\vee)"] \\ F(V_\lambda) \arrow[r, "B_\lambda"'] & F(\lambda) \end{tikzcd}$$

$K_\lambda = \left(\cdots \otimes \Lambda^2 V_\lambda \to V_\lambda \otimes \mathcal{O} \to \mathcal{O}(\lambda) \right)$ are sent to acyclic complexes

Construction of functors from $D^b_{perf,G^\vee}(\tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee)$.

Just as in D_{II}, we can construct Gaitsgory central sheaves $\hat{Z}_v$ and Wakimoto sheaves $\hat{J}_\lambda$ on $\hat{D}$, equipped with

$N_v : \hat{Z}_v \to \hat{Z}_v$ $\qquad B_\lambda : \hat{Z}_{v_\lambda} \to \hat{J}_\lambda$, $\quad \hat{Z}_v$ filtered by $\hat{J}_\lambda$

To get a monoidal functor $\Phi_{diag} : D^b Coh^{G^\vee}(\hat{\tilde{\mathfrak{g}}}^\vee) \to \hat{D}$

Acting by left + right convolution get an action of $D^b Coh^{G^\vee \times G^\vee}(\hat{\tilde{\mathfrak{g}}}^\vee \times \hat{\tilde{\mathfrak{g}}}^\vee)$ on $\hat{D}$, but note that right and left convolution by central sheaves are canonically isomorphic so this induces an action of $D^b_{perf,G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee})$

$$\begin{tikzcd}
G^\vee \backslash \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (G^\vee \backslash \tilde{\mathfrak{g}}^\vee)^2 \arrow[d] \\
G^\vee \backslash \mathfrak{g}^\vee \arrow[r, "\Delta"] & (G^\vee \backslash \mathfrak{g}^\vee)^2
\end{tikzcd}$$

The object $\hat{\Xi}$

We want to define $\underline{\Phi}_{perf}: D^b_{perf,G^\vee}(\hat{St}) \to \hat{D}$ by acting on a particular object.

Note that $\pi: St \to \tilde{\mathfrak{g}}^\vee$ $\qquad \pi^* \pi_* \mathcal{F} \cong \mathcal{O}_{St} * \mathcal{F}$

So for compatibility we want $\tilde{\underline{\Phi}}_{perf}(\mathcal{O}_{St}) = \tilde{\underline{\Phi}}_{perf}(\pi^* \pi_* \mathcal{O}_\Delta) = Av_{I_0} Av_{I_W}(\tilde{\delta}) = \hat{\Xi}$

Note that

$$Av_{I_0} A_{I_W}(\mathcal{F}) \cong \hat{\Xi} * \mathcal{F} \qquad \mathcal{F} \in \hat{D}$$

and

$$\pi^* \pi_*(\mathcal{G}) \cong \mathcal{O}_{\hat{St}} * \mathcal{G} \qquad \mathcal{G} \in D^b_{perf,G^\vee}(\hat{St})$$

$\rightsquigarrow$ Compatibility of equivalence $\hat{D}_{IW} \cong D^b Coh(\hat{\tilde{\mathfrak{g}}})$ with monoidal actions will follow from this

The object $\hat{\Xi}$

Lemma $\hat{\Xi} = Av_{I_\circ} Av_{Iw}(\tilde{\delta})$ is the tilting object T_{w_0}

Pf: Note that $\hat{\Xi}$ is supported on $G/N \subseteq \widetilde{Fl}$.
$\pi^P : G/N \to G/P$ for some parabolic P, then
$\pi^P_*(\hat{\Xi}) = 0$. Further $j^*_{w_0} \hat{\Xi}$ is free of rank 1 in generic
degree. It follows that $\hat{\Xi}$ is indecomposable tilting T_{w_0}.

Extending from $D^b_{perf,G^\vee}(\hat{St})$ to $D^b Coh^{G^\vee}(\hat{St})$.

Note that every object $\mathcal{F}$ in $D^b Coh^{G^\vee}(\hat{St})$ has an infinite resolution $\mathcal{F}^\bullet$ by perfect objects.

<u>Basic idea</u>: We want to truncate these resolutions and use the definition for perfect complexes.

<u>Problem</u>: We can't take infinite resolutions in $\hat{D}$.

<u>Naive solution</u>: If Φ_{perf} were exact we could use the infinite resolution to determine the perverse cohomology.

<u>Better solution</u>: If Φ_{perf} has finite amplitude wrt the t-structure

Extending from $D^b_{perf, G^\vee}(\hat{St})$ to $D^b Coh^{G^\vee}(\hat{St})$.

Prop i) $\exists\, d>0$ st if $\mathcal{F} \in \hat{D}$ is st $Hom(\Phi_{perf}(\mathcal{O}(\lambda,\nu)\otimes V), \mathcal{F}[i])=0$ for $i \in [-d, d]$ then $H^{P,O}(\mathcal{F})=0$.

ii) $\exists\, \partial$ st $\forall\, \mathcal{F} \in \hat{P}$ $\forall\, \lambda, \nu \in \Lambda, V \in Rep\, G$.

$$Hom(\Phi_{perf}(\mathcal{O}(\lambda,\nu)\otimes V), \mathcal{F}[i])=0 \qquad i \notin [-\partial, \partial].$$

Main input: $\mathcal{F}, \mathcal{G} \in \hat{P}$ then $\mathcal{F} * \mathcal{G} \in \hat{D}^{[-\dim G/B, \dim G/B]}$ $(*)$

Extending from $D^b_{perf,G^\vee}(\hat{St})$ to $D^b Coh^{G^\vee}(\hat{St})$.

Proof of i): By the result $(*)$ ETS result for $J_\lambda * \mathcal{F}$ for $\lambda >> 0$. And to check $H^{p,0}(J_\lambda * \mathcal{F}) = 0$ it's ETS $Hom(j_{w!}, J_\lambda * \mathcal{F}[i]) = 0$ for $i \in [-d', d']$ for some $d' > 0$. $\forall w$

Claim $Hom(\hat{\Xi} * j_{w!}, J_\lambda * \mathcal{F}[i]) = Hom(j_{w!}, J_\lambda * \mathcal{F}[i])$ for $\lambda >> 0$

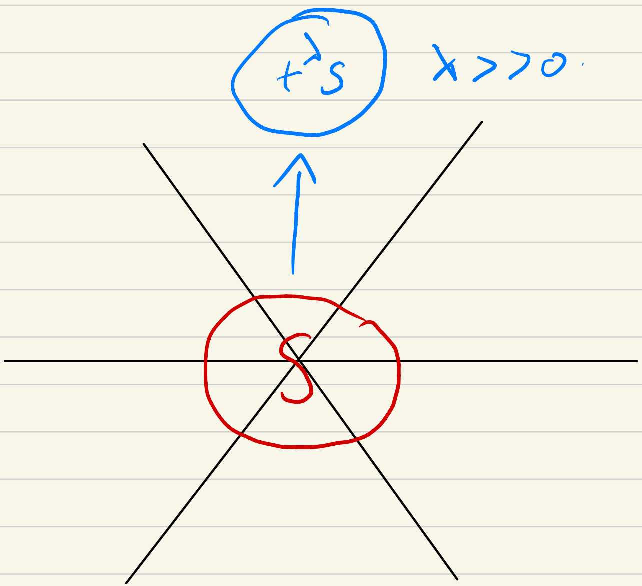
This follows as $\hat{\Xi}$ has a filtration by $j_{x!}$ $x \in W_f$ and for $\lambda >> 0$ $J_\lambda * \mathcal{F}$ only has costalks on minimal $W_f \backslash W$ representatives

Using this can reduce to checking $\Phi_{IW}(Av_{IW}(J_\lambda * \mathcal{F}))$ has 0 cohomology in degrees $[-d'', d'']$, which follows from the assumptions

costalks of $\mathcal{J}_\lambda * \mathcal{F}$

For a sheaf $\mathcal{F}$ $\exists$ a finite subset $S \subseteq W$ st

$$\mathcal{J}_\lambda * \mathcal{F} \text{ has only non zero costalks on } t^\lambda S$$



For $\lambda >> 0$ $t^\lambda S$ in the dominant cone, so is minimal in $W_f \backslash W$

Monoidal structure

From $Av_{I_0} A_{Iw}(\mathcal{F}) \cong \hat{\Xi} * \mathcal{F}$

$$\pi_2^* \pi_2(\mathcal{G}) \cong \mathcal{O}_{\hat{St}} * \mathcal{G}$$

we get the two actions of $D^b_{perf, G^\vee}(\hat{St})$ on $\hat{D}_{Iw} \cong D^b Coh(\hat{\tilde{g}})$ agree for $\mathcal{O}_{\hat{St}} \leftrightarrow \hat{\Xi}$

It follows from definitions that they agree and with extra work that they agree for $D^b Coh^{G^\vee}(\hat{St}) \cong \hat{D}$

Now use that the action on this module is "faithful". in analogy with the decategorified case.