The affine Hecke category is a monoidal colimit

James Tao

jamestao@mit.edu

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Overview

Main Idea

A monoidal category $\mathcal{C}$ with a stratification indexed by a Coxeter group W can often be expressed as the monoidal colimit of subcategories $\mathcal{C}_J \subset \mathcal{C}$ indexed by *finite type* standard subgroups $W_J \subset W$.

This talk will have two parts:

1. Colimit theorems for $\mathcal{C} = \mathcal{D}(\mathcal{L}G), \mathcal{D}(\mathbf{I}\backslash\mathcal{L}G/\mathbf{I})$.
   The category of words and bistratified descent.
   (Joint with Roman Travkin.)

2. An application-in-progress: constructing $\mathcal{H}_{\mathrm{aff}} \to \mathcal{H}_{\mathrm{fin}}$ in type A.
   Deformed affine Hecke categories.
   (Joint with Kostya Tolmachov.)

Two colimit theorems

G – algebraic group, semisimple and simply-connected.
W_I – affine Weyl group of $\mathcal{L}G$.
I – set of affine simple reflections.

For $J \subset I$, let $\mathbf{P}_J \subset \mathcal{L}G$ be the standard parahoric of type J.

Theorem (monoidal colimits)

(i) $$\mathcal{D}(\mathcal{L}G) \simeq \operatorname*{colim}_{\substack{J \subset I \\ J \text{ finite type}}} \mathcal{D}(\mathbf{P}_J)$$

(ii) $$\mathcal{D}(\mathbf{I} \backslash \mathcal{L}G / \mathbf{I}) \simeq \operatorname*{colim}_{\substack{J \subset I \\ J \text{ finite type}}} \mathcal{D}(\mathbf{I} \backslash \mathbf{P}_J / \mathbf{I})$$

Remarks. To remove the 'semisimple and simply-connected' hypothesis, change the colimit indexing diagram to Varshavsky's 'category of parahorics.'

Analogues for monodromic Hecke categories, Kac–Moody groups, ...

Motivation: generators and relations (part 1)

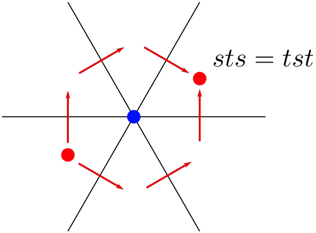
Monoidal object	**Presentation**	**Colimit thm.**	**Cat. level**
Weyl group Hecke algebra	Simple reflections 1-term relations 2-term relations	$A = \operatorname{colim}_{\substack{J \subset I \\ \lvert J \rvert \leq 2}} A_J$	sets
Weyl group (as discrete Picard grpd) Hecke category	Simple reflections 1-term relations 2-term relations 3-term relations	$A \simeq \operatorname{colim}_{\substack{J \subset I \\ \lvert J \rvert \leq 3}} A_J$	categories
Weyl group (as discrete top. group) Hecke ∞-category	??	$A \simeq \operatorname{colim}_{\substack{J \subset I \\ J \text{ f.t.}}} A_J$	∞-categories

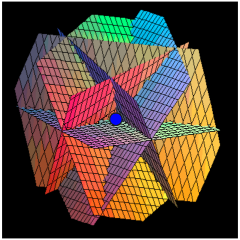
Presentation of Weyl group (as discrete Picard grpd): Thm. 1.17, *Diagrammatics for Coxeter groups*, Elias–Williamson (2017)

Presentation of Hecke category: Thm. 1.11, *Tilting modules and the p-canonical basis*, Riche–Williamson (2018)

Motivation: generators and relations (part 2)

Relation	Geometric origin	Name
1-term relations	codimension-1 faces	quadratic relations “wall crossing”
2-term relations	codimension-2 faces	braid relations
3-term relations	codimension-3 faces	Zamolodchikov relations





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Proof: The category of words (part 1)

Word is the following category:

- Objects are sequences $(w_1, \dots, w_n)$ in W_I. Each w_i is 'finite type.'
- A morphism $\varphi : (w_1, \dots, w_{n_1}) \to (w'_1, \dots, w'_{n_2})$ is an ordered map

$$\varphi_* : \{1, \dots, n_1\} \to \{1, \dots, n_2\}$$

satisfying that

$$w'_j \succeq_{\text{Bruhat}} (\text{Demazure product of } w_i \text{ for } i \in \varphi_*^{-1}(j)),$$

for all $j \in \{1, \dots, n_2\}$.

Proof: The category of words (part 2)

Key Idea

Word governs 'convolution' products of Schubert varieties.

For $w \in W_I$, let $\mathbf{P}_w \subset \mathcal{L}G$ be the closure of the w Bruhat cell.

- A word $\boldsymbol{w} = (w_1, \ldots, w_n)$ encodes the variety

$$\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} := \mathbf{P}_{w_1} \overset{\mathbf{I}}{\times} \cdots \overset{\mathbf{I}}{\times} \mathbf{P}_{w_n}/\mathbf{I}$$

- A morphism $\varphi : \boldsymbol{w} \to \boldsymbol{w}'$ encodes the conv. map $\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} \to \widetilde{\mathcal{F}\ell}_{\boldsymbol{w}'}$
- Example for $\widetilde{A}_2$, with simple reflections s, t, u:

$$\begin{array}{ccc} (s,t,s) & \mathbf{P}_s \overset{\mathbf{I}}{\times} \mathbf{P}_t \overset{\mathbf{I}}{\times} \mathbf{P}_s/\mathbf{I} & (g_1, g_2, g_3) \\ \downarrow & \downarrow & \downarrow \\ (s,1,sts) & \mathbf{P}_s \overset{\mathbf{I}}{\times} \mathbf{I} \overset{\mathbf{I}}{\times} \mathbf{P}_{sts}/\mathbf{I} & (g_1, 1, g_2 g_3) \end{array}$$

Proof: Monoidal colimits are amalgamated products

$$\operatorname*{colim}_{\substack{J \subset I \\ J \text{ finite type}}} \mathcal{D}(\mathbf{P}_J) \simeq \operatorname*{colim}_{(J_1,\dots,J_n)} \mathcal{D}(\mathbf{P}_{J_1}) \underset{\mathcal{D}(\mathbf{I})}{\otimes} \cdots \underset{\mathcal{D}(\mathbf{I})}{\otimes} \mathcal{D}(\mathbf{P}_{J_n})$$

$$\simeq \operatorname*{colim}_{(w_1,\dots,w_n) \in \mathbf{Word}} \mathcal{D}(\mathbf{P}_{w_1}) \underset{\mathcal{D}(\mathbf{I})}{\otimes} \cdots \underset{\mathcal{D}(\mathbf{I})}{\otimes} \mathcal{D}(\mathbf{P}_{w_n})$$

$$\simeq \operatorname*{colim}_{(w_1,\dots,w_n) \in \mathbf{Word}} \mathcal{D}\left(\mathbf{P}_{w_1} \overset{\mathbf{I}}{\times} \cdots \overset{\mathbf{I}}{\times} \mathbf{P}_{w_n}\right)$$

$$\operatorname*{colim}_{\substack{J \subset I \\ J \text{ finite type}}} \mathcal{D}(\mathbf{I} \backslash \mathbf{P}_J / \mathbf{I}) \simeq \operatorname*{colim}_{(w_1,\dots,w_n) \in \mathbf{Word}} \mathcal{D}'\left(\mathbf{I} \backslash \mathbf{P}_{w_1} \overset{\mathbf{I}}{\times} \cdots \overset{\mathbf{I}}{\times} \mathbf{P}_{w_n} / \mathbf{I}\right)$$

$\mathcal{D}'(-) := \mathcal{D}$-modules constant on each (twisted) product of cells.

Proof: The category of words (part 3)

Let $\varphi : \boldsymbol{w} \to \boldsymbol{w}'$ be a map.

- φ is a *strict embedding* if φ_* is a bijection and $\varphi \neq$ (identity).
- φ is *birational* if and only if $\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} \to \widetilde{\mathcal{F}\ell}_{\boldsymbol{w}'}$ is birational.

Let $y \in W_I$. We define full subcategories $\mathbf{Word}_{\preceq y}, \mathbf{Word}_{\prec y}$.

- $\boldsymbol{w} \in \mathbf{Word}_{\preceq y}$ if and only if

$$\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} \to \mathcal{F}\ell := \mathcal{L}G/\mathbf{I}$$

  factors through $\mathcal{F}\ell_y$ (the y Schubert variety).

- $\boldsymbol{w}$ is *y-relevant* if this map is birational onto $\mathcal{F}\ell_y$.
- $\boldsymbol{w} \in \mathbf{Word}_{\prec y}$ if and only if this map factors through $\partial\mathcal{F}\ell_y$.

(∂ means 'boundary,' i.e. complement of open cell.)

Proof: Bistratified descent

Let $F : \mathbf{Word} \to \mathcal{E}$ be any functor. How to compute $\operatorname{colim} F$?

Theorem (bistratified descent)

Assume that, for every birational map $\boldsymbol{w} \to \boldsymbol{w}'$, the following diagram is cocartesian:

$$\begin{array}{ccc} \operatorname*{colim}_{\substack{\boldsymbol{v}\to\boldsymbol{w} \\ \text{strict emb.}}} F(\boldsymbol{v}) & \longrightarrow & F(\boldsymbol{w}) \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{\substack{\boldsymbol{v}'\to\boldsymbol{w}' \\ \text{strict emb.}}} F(\boldsymbol{v}') & \longrightarrow & F(\boldsymbol{w}') \end{array}$$

Then, for any *y-relevant* $\boldsymbol{w} \in \mathbf{Word}_{\preceq y}$, the following diagram is cocartesian:

$$\begin{array}{ccc} \operatorname*{colim}_{\substack{\boldsymbol{v}\to\boldsymbol{w} \\ \text{strict emb.}}} F(\boldsymbol{v}) & \longrightarrow & F(\boldsymbol{w}) \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{\mathbf{Word}_{\prec y}} F & \longrightarrow & \operatorname*{colim}_{\mathbf{Word}_{\preceq y}} F \end{array}$$

The conclusion is: $\operatorname{colim} F$ can be computed via a sequence of pushouts.

Proof: Applying bistratified descent

$$\begin{array}{ccc}
\operatorname*{colim}_{\substack{\boldsymbol{v}\to\boldsymbol{w} \\ \text{strict emb.}}} F(\boldsymbol{v}) & \longrightarrow & F(\boldsymbol{w}) \\
\downarrow & & \downarrow \\
\operatorname*{colim}_{\substack{\boldsymbol{v}'\to\boldsymbol{w}' \\ \text{strict emb.}}} F(\boldsymbol{v}') & \longrightarrow & F(\boldsymbol{w}')
\end{array}
\qquad
\begin{array}{ccc}
\operatorname*{colim}_{\substack{\boldsymbol{v}\to\boldsymbol{w} \\ \text{strict emb.}}} F(\boldsymbol{v}) & \longrightarrow & F(\boldsymbol{w}) \\
\downarrow & & \downarrow \\
\operatorname*{colim}_{\mathbf{Word}_{\prec y}} F & \longrightarrow & \operatorname*{colim}_{\mathbf{Word}_{\preceq y}} F
\end{array}$$

The previous diagrams correspond to *blow-up squares*:

$$\begin{array}{ccc}
\partial\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} & \longrightarrow & \widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} \\
\downarrow & & \downarrow \\
\partial\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}'} & \longrightarrow & \widetilde{\mathcal{F}\ell}_{\boldsymbol{w}'}
\end{array}
\qquad
\begin{array}{ccc}
\partial\widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} & \longrightarrow & \widetilde{\mathcal{F}\ell}_{\boldsymbol{w}} \\
\downarrow & & \downarrow \\
\partial\mathcal{F}\ell_{y} & \longrightarrow & \mathcal{F}\ell_{y}
\end{array}$$

Upshot. Bistratified descent can be applied when the sheaf theory satisfies descent w.r.t. blow-up squares. (E.g. $\mathcal{D}$-modules, ℓ-adic sheaves)

How to apply the colimit theorem (part 1)

$\mathcal{C}$ – any stable monoidal ∞-category. Assume I is irreducible.

How to construct a monoidal triangulated functor $F : \mathcal{H}_{\mathrm{aff}} \to \mathcal{C}$?

1. For $i \in I$, choose $F_i \in \mathrm{Fun}^{\mathrm{mon}}(\mathcal{H}_{I \smallsetminus \{i\}}, \mathcal{C})$
2. For $i, j \in I$, choose $F_i|_{\mathcal{H}_{I \smallsetminus \{i,j\}}} \xrightarrow[\sigma_{ij}]{\sim} F_j|_{\mathcal{H}_{I \smallsetminus \{i,j\}}}$ in $\mathrm{Fun}^{\mathrm{mon}}(\mathcal{H}_{I \smallsetminus \{i,j\}}, \mathcal{C})$
3. For $i, j, k \in I$, ensure commutativity in $\mathrm{Fun}^{\mathrm{mon}}(\mathcal{H}_{I \smallsetminus \{i,j,k\}}, \mathcal{C})$:

$$\begin{array}{ccccc} & & F_j|_{\mathcal{H}_{I \smallsetminus \{i,j,k\}}} & & \\ & {\scriptstyle \sigma_{ij}|_{\mathcal{H}_{I \smallsetminus \{i,j,k\}}}} \nearrow & & \searrow {\scriptstyle \sigma_{jk}|_{\mathcal{H}_{I \smallsetminus \{i,j,k\}}}} & \\ F_i|_{\mathcal{H}_{I \smallsetminus \{i,j,k\}}} & & \xrightarrow{\sigma_{ik}|_{\mathcal{H}_{I \smallsetminus \{i,j,k\}}}} & & F_k|_{\mathcal{H}_{I \smallsetminus \{i,j,k\}}} \end{array}$$

4. (higher associativity constraints)

How to apply the colimit theorem (part 2)

Choose a t-structure on $\mathcal{C}$. Restrict attention to functors $F : \mathcal{H}_{\mathrm{aff}} \to \mathcal{C}$ which send all tilting generators into $\mathcal{C}^{\heartsuit}$.

To construct these functors, one only needs the "1-categorical colimit theorem," which follows from the Elias–Williamson presentation.

This is because of 'truncatedness':

$$\mathrm{Hom}^i_{\mathcal{C}}(F(\mathcal{T}_1), F(\mathcal{T}_2)) = 0 \quad \text{for } i < 0.$$

This corresponds to vanishing of some $\pi_1, \pi_2, \ldots$, because we are using cohomological indexing.

To construct more general functors, one needs the "∞-categorical colimit theorem."

How to apply the colimit theorem (part 3)

Problem: In a general ∞-category, it's hard to check Step **4**.

In an ordinary category, however, Step **4** automatically follows.

Key Idea

Choose a t-structure on $\mathcal{C}$. The subcategory of $\mathrm{Fun}^{\mathrm{mon}}(\mathcal{H}_{I \setminus \{i\}}, \mathcal{C})$ which sends all tilting generators into $\mathcal{C}^{\heartsuit}$ is an ordinary category.

This trick may work even when there is no t-structure on $\mathcal{C}$ such that the desired functor $\mathcal{H}_{\mathrm{aff}} \to \mathcal{C}$ sends all tilting generators into $\mathcal{C}^{\heartsuit}$.

Indeed, we may now use a different t-structure for each $i \in I$.

Tolmachov's thesis: $\mathrm{Perf}^G(St) \to \mathcal{H}_{\mathrm{fin}}$ in type A (part 1)

From now on, $I = \widetilde{A}_{n-1}$, with vertices $\{0, \dots, n-1\}$, and $G := \mathrm{GL}_n$.

Question

Is there a (monoidal) functor $\mathcal{H}_{\mathrm{aff}} \to \mathcal{H}_{I \setminus \{0\}}$ which is compatible with the following map of braid groups $\mathbb{B}_{\mathrm{aff}} \to \mathbb{B}_{\mathrm{fin}}$?

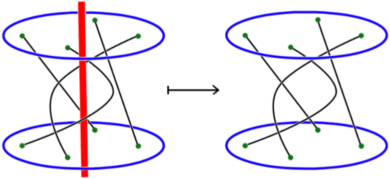


Image source: Tolmachov's thesis

Bezrukavnikov's equivalence states that $\mathcal{H}_{\mathrm{aff}} \simeq D^b(\mathrm{Coh}^G(\mathrm{St}))$.
Tolmachov's thesis constructs a functor $\mathrm{Perf}^G(\mathrm{St}) \to \mathcal{H}_{I \setminus \{0\}}$.

Tolmachov's thesis: $\mathrm{Perf}^G(St) \to \mathcal{H}_{\mathrm{fin}}$ in type A (part 2)

Key Idea for Bezrukavnikov's equivalence

$\mathrm{Perf}^G(\mathrm{St})$ is generated by the vector bundles $\mathcal{O}(\lambda,\mu) \otimes_{\mathbb{C}} V$ and maps (highest weight arrows, monodromy endomorphisms) subject to some relations. The object $\mathcal{O}(\lambda,\mu) \otimes V$ corresponds to $J_\lambda \star \mathcal{Z}_V \star \Xi \star J_\mu \in \mathcal{H}_{\mathrm{aff}}$.

Question 1: Where should $J_\lambda \star \mathcal{Z}_V \star \Xi \star J_\mu$ map to?

Weight decomposition $V = \oplus_\lambda V_\lambda$.
Recall that $\mathcal{Z}_V$ is an iterated extension, in which J_λ occurs $\dim V_\lambda$ times.

$\mathbb{B}_{\mathrm{aff}} \to \mathbb{B}_{\mathrm{fin}}$ forces $J_\lambda \mapsto \mathbb{L}_\lambda$ (Jucys–Murphy sheaves).

Anything in $\mathbb{B}_{\mathrm{fin}}$ convolved with Ξ yields Ξ.

Answer: $J_\lambda \star \mathcal{Z}_V \star \Xi \star J_\mu \mapsto \mathbb{L}_\lambda \star (V \otimes_{\mathbb{C}} \Xi) \star \mathbb{L}_\mu$.

Tolmachov's thesis: $\mathrm{Perf}^G(St) \to \mathcal{H}_{\mathrm{fin}}$ in type A (part 3)

To get maps and relations, need to 'take apart' $V \otimes_{\mathbb{C}} \Xi$. Thus, we ask:

Question 2: Under $\mathcal{H}_{\mathrm{aff}} \to \mathcal{H}_{\mathrm{fin}}$, where should $\mathcal{Z}_V$ map to?

Answer: For V_{std}, it's an "averaged" parabolic Springer sheaf.

Let P be the parabolic which fixes a line. $G \backslash (G \overset{P}{\times} U_P) \simeq P \backslash U_P$

$$\begin{array}{ccc} & & \frac{G}{U} & \longrightarrow & U \backslash G / U \\ & & \downarrow & & \\ \frac{U_P}{P} & \xrightarrow{\pi} & \frac{G}{G} & & \end{array}$$

Parabolic Springer sheaf: $\mathrm{Spr}_P := \pi_* \underline{\mathbb{C}}_{\frac{U_P}{P}}[2 \dim U_P]$

Pull-push Spr_P, then force it to be T-monodromic.

Tolmachov's thesis: $\mathrm{Perf}^G(St) \to \mathcal{H}_{\mathrm{fin}}$ in type A (part 4)

What about $\wedge^k V_{\mathrm{std}}$?

λ_k – partition of n given by the 'hook' $(k, 1, \dots, 1)$.

IC_{λ_k} – IC-complex of the unipotent orbit in $G = \mathrm{GL}_n$ given by λ_k.

Main Theorem of Tolmachov's thesis

(a) $\wedge^k \mathrm{Spr}_P \simeq \mathrm{IC}_{\lambda_k} \oplus \mathrm{IC}_{\lambda_{k+1}}$ for $1 \leq k \leq n-1$.

(b) $\wedge^n \mathrm{Spr}_P \simeq \mathrm{IC}_{\lambda_n}$.

(c) IC_{λ_n} becomes invertible after some averaging.

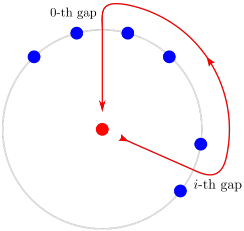
(d) $\wedge^{n+1} \mathrm{Spr}_P = 0$.

Finally, bootstrap from $\wedge^k V_{\mathrm{std}}$ to all GL_n-reps via a Tannakian argument.

Constructing $\mathcal{H}_{\mathsf{aff}} \to \mathcal{H}_{\mathsf{fin}}$ via the colimit theorem (part 1)

To apply the colimit theorem, we need to do Steps **1**, **2**, **3** from before.

1. $F_i : \mathcal{H}_{I \setminus \{i\}} \to \mathcal{H}_{I \setminus \{0\}}$ is conjugation by a specific element $b_i \in \mathbb{B}_{\mathsf{aff}}$.



2. $\sigma_{ij} : F_i|_{\mathcal{H}_{I \setminus \{i,j\}}} \xrightarrow{\sim} F_j|_{\mathcal{H}_{I \setminus \{i,j\}}}$ expresses that $b_j^{-1} b_i$ centralizes $\mathcal{H}_{I \setminus \{i,j\}}$.

(Use centrality of $\Delta^2_{w_0}$, Prop. 5.4, *Monodromic model for Khovanov–Rozansky homology*, Bezrukavnikov–Tolmachov)

Constructing $\mathcal{H}_{\mathrm{aff}} \to \mathcal{H}_{\mathrm{fin}}$ via the colimit theorem (part 2)

3. Want to check $\sigma_{jk} \circ \sigma_{ik} \simeq \sigma_{ik}$ in $\mathrm{Fun}^{\mathrm{mon}}(\mathcal{H}_{I \setminus \{i,j,k\}}, \mathcal{C})$.
Right now, we do not know if this is true.

What if Step **3** fails?

That is, what if $\sigma_{ik}^{-1} \circ \sigma_{jk} \circ \sigma_{ik}$ is not the id. natural transformation?

Key Idea (deformed affine Hecke category)

One may define a new category $\mathcal{H}_{\mathrm{aff}}^{(\alpha)}$ by deforming $\mathcal{H}_{\mathrm{aff}}$ using the 'cocycle' $\sigma_{ik}^{-1} \circ \sigma_{jk} \circ \sigma_{ik}$. By construction, there will be a monoidal functor

$$\mathcal{H}_{\mathrm{aff}}^{(\alpha)} \to \mathcal{H}_{I \setminus \{0\}}.$$

Deformed affine Hecke category (part 1)

Interpret the colimit diagram in $\mathcal{H}_{\mathrm{aff}} \simeq \mathrm{colim}_{J \subsetneq I} \mathcal{H}_J$ as follows:

1. For $i \in I$, write down the category $\mathcal{H}_{I \smallsetminus \{i\}}$.
2. For $i, j \in I$, write down the identity functor

$$\begin{array}{ccc} \mathcal{H}_{I \smallsetminus \{i,j\}} & \hookrightarrow & \mathcal{H}_{I \smallsetminus \{i\}} \\ \mathrm{Id} \downarrow & & \\ \mathcal{H}_{I \smallsetminus \{i,j\}} & \hookrightarrow & \mathcal{H}_{I \smallsetminus \{j\}} \end{array}$$

3. For $i, j, k \in I$, write down the trivial commutativity natural iso

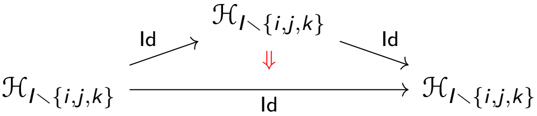
$$\begin{array}{cccc} & \mathcal{H}_{I \smallsetminus \{i,j,k\}} & \hookrightarrow & \mathcal{H}_{I \smallsetminus \{i\}} \\ & \mathrm{Id} \downarrow & & \\ \mathrm{Id} \quad \swarrow & \mathcal{H}_{I \smallsetminus \{i,j,k\}} & \hookrightarrow & \mathcal{H}_{I \smallsetminus \{j\}} \\ & \mathrm{Id} \downarrow & & \\ & \mathcal{H}_{I \smallsetminus \{i,j,k\}} & \hookrightarrow & \mathcal{H}_{I \smallsetminus \{k\}} \end{array}$$

Deformed affine Hecke category (part 2)

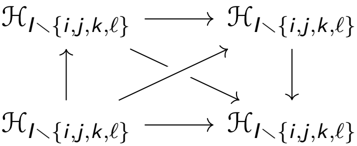
Define $\mathcal{H}_{\text{aff}}^{(\alpha)} := \operatorname{colim}_{J \subsetneq I} \mathcal{H}_J$ using a modified colimit diagram:

Steps **1** and **2** are the same as before.

3′. For $i, j, k \in I$, write down a nontrivial commutativity natural iso



4. For $i, j, k, \ell \in I$, the following tetrahedron must commute:



All six maps are Id, all four triangles come from step **3′**.

Deformed affine Hecke category (part 3)

Key Idea (deformed affine Hecke category)

Use the natural iso's $\sigma_{ik}^{-1} \circ \sigma_{jk} \circ \sigma_{ik}$ to define natural iso's for Step $\mathbf{3}'$.
The resulting category $\mathcal{H}_{\mathrm{aff}}^{(\alpha)}$ admits a monoidal functor to $\mathcal{H}_{I \smallsetminus \{0\}}$.

We expect that $\mathcal{H}_{\mathrm{aff}}^{(\alpha)}$ can be (noncanonically) obtained from $\mathcal{H}_{\mathrm{aff}}$ by altering the 3-term (and higher) associativity constraints for the monoidal structure.

Hence, the centers of $\mathcal{H}_{\mathrm{aff}}^{(\alpha)}$ and $\mathcal{H}_{\mathrm{aff}}$ are equivalent (as categories), but their (braided) monoidal structures are different.

Next steps. Describe the categories $\mathcal{H}_{\mathrm{aff}}^{(\alpha)}$ more explicitly.
Investigate the center of $\mathcal{H}_{\mathrm{aff}}^{(\alpha)}$ and compare with the center of $\mathcal{H}_{I \smallsetminus \{0\}}$.

References

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Appendix: How to construct $\mathbb{B}_{\mathrm{aff}} \to \mathcal{H}_{\mathrm{aff}}$ (part 1)

"Reduced lift" presentation of braid monoid $\mathbb{B}^+_{\mathrm{aff}}$:

$$\mathbb{B}^+_{\mathrm{aff}} \simeq \left\langle t_w \text{ for } w \in W_{\mathrm{aff}} \;\middle|\; \begin{array}{c} t_{w_1} t_{w_2} = t_{w_1 w_2} \text{ whenever} \\ \ell(w_1) + \ell(w_2) = \ell(w_1 w_2) \end{array} \right\rangle$$

Valid even when $\mathbb{B}^+_{\mathrm{aff}}$ is viewed as a discrete topological monoid!

Finite type: Thm. 1.7, *Action du groupe des tresses sur une catégorie*, Deligne (1997)
Arbitrary type: Generalize Deligne's proof, or apply Thm. 5.1, *Configuration spaces of labeled particles*, Dobrinskaya (2006)

Define the monoidal functor $\mathbb{B}^+_{\mathrm{aff}} \to \mathcal{H}_{\mathrm{aff}}$ via $t_w \mapsto (j_w)_! \underline{\mathbb{C}} = \Delta_w$.

Appendix: How to construct $\mathbb{B}_{\text{aff}} \to \mathcal{H}_{\text{aff}}$ (part 2)

Thm. 5.2 of Dobrinskaya (2006)

I – any Coxeter–Dynkin diagram

The homotopy groupification of $\mathbb{B}_I^+$ is the (discrete) braid group $\mathbb{B}_I$ if and only if $K(\pi, 1)$ conjecture holds for $\mathbb{B}_I$.

Paolini and Salvetti (2020) proved the $K(\pi, 1)$ conjecture for affine I.

Universal property of homotopy groupification:
$(j_w)_! \underline{\mathbb{C}} = \Delta_w$ invertible $\Longrightarrow$ Get a monoidal functor $\mathbb{B}_{\text{aff}} \to \mathcal{H}_{\text{aff}}$.