K-ring of Steinberg variety

Notation: $T \subseteq B \subseteq G$ $\mathfrak{g} = \mathrm{Lie}(G)$

T: max'l torus, B: Borel, G: semisimple simply connected

W Weyl group Λ character lattice

S simple reflections

$$\tilde{\mathfrak{g}} = \{ (x, b) \in \mathfrak{g} \times G/B \mid x \in b \} \circlearrowleft G \times \mathbb{C}^*$$

$$St = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$$

$K^{G\times\mathbb{C}^*}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$ ring under convolution

$$\begin{tikzcd}
 & \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} & \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} & \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}
\end{tikzcd}$$

$$[\mathcal{F}] * [\mathcal{G}] = \pi_{23*}\left(\pi_{12}^*[\mathcal{F}] \otimes \pi_{23}^*[\mathcal{G}]\right)$$

The affine Hecke algebra

<u>Def</u> The affine Hecke algebra H is an algebra $/ \mathbb{C}[q^{\pm 1}]$ gen'd by

T_s $\quad s \in S$ simple reflection in W $\qquad e^\lambda \quad \lambda \in \Lambda$

subject to the relations.

- R1) $e^\lambda e^\mu = e^{\lambda+\mu}$, $e^0 = 1$
- R2) $(T_s + 1)(T_s - q) = 0$
- R3) $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$ $\qquad \langle \lambda, \alpha^\vee \rangle = 0$.
- R4) $T_{s_\alpha} e^{s(\lambda)} T_{s_\alpha} = q e^\lambda$ $\qquad \langle \lambda, \alpha^\vee \rangle = 1$

$\Big\}$ α simple

- R5) braid relations

basis$/\mathbb{C}[q^{\pm 1}]$ $\{ e^\lambda T_w = e^\lambda T_{s_1} \cdots T_{s_k} \}$ $\lambda \in \Lambda$, $w = s_1 \cdots s_k \in W$.

Remark 1: This is a particular presentation of the affine Hecke algebra corresponding to the decomposition $W_{aff} = W \ltimes \Lambda$.

Another presentation exists corresponding to the presentation of W_{aff} as a reflection group.

Remark 2: The affine algebra presented is usually referred to in the literature as the affine algebra for G^L

Main Result and idea of the Proof

Thm: $\exists$ an iso $K^{G\times\mathbb{C}^*}(St) \cong H$

$$K^{\mathbb{C}^*}(pt) \cong \mathbb{C}[q^{\pm 1}]$$

Idea of Proof:

$$\begin{tikzcd} J \arrow[r] \arrow[dr, "\Theta"'] & H \arrow[r] \arrow[d, dashed, "f"] & \mathrm{End}(M e_-) \arrow[d, phantom, "\cong"] \\ & K^{G\times\mathbb{C}^*}(St) \arrow[r, hook, "i"] & \mathrm{End}(K^{G\times\mathbb{C}^*}(\tilde{\mathfrak{g}})) \end{tikzcd}$$

- For generators T_s, e^λ we define objects in $K^{G\times\mathbb{C}^*}(St)$ such that the diagram commutes.
- We check that i is injective
  - $\rightsquigarrow$ this induces an algebra map f.
- We introduce filtrations on $K^{G\times\mathbb{C}^*}(St)$ and H and prove f is filtered and a graded iso.

General Facts of K-thy

Celular Fibration Lemma For a filtered space F/X with G-action

$$
\begin{tikzcd}
F \arrow[dr] & \supseteq \cdots \supseteq F_i \arrow[d] \supseteq F_{i-1} \supseteq \cdots \supseteq F_0 \arrow[dl] \\
& X
\end{tikzcd}
$$

such that F_i are G-stable.

Such that $F_i \setminus F_{i-1} \longrightarrow X$ are affine space fibrations

(1) $K^G(F_i) \subseteq K^G(F_{i+1}) \subseteq \cdots \subseteq K^G(F)$ with

$$K^G(F_i)/K^G(F_{i-1}) \cong K^G(F_i \setminus F_{i-1}) \cong K^G(X)$$

2) if $G = G' \times T$ T a torus then $R(T) = K^T(pt)$

$$K^G(F) \otimes_{R(T)} \operatorname{Frac}(R(T)) \cong K^G(F^T) \otimes_{R(T)} \operatorname{Frac}(R(T)).$$

if this holds for X.

Basic Geometry of Steinberg variety

We have a bijection

$$W \longleftrightarrow \{ G\text{-orbits on } G/B \times G/B \}$$
$$w \longmapsto Y_w = G(1, w)$$

$\mathfrak{b} \cap {}^w\mathfrak{b}$

Let

$$\begin{tikzcd} Z_w \arrow[r] \arrow[d, hook] & Y_w \arrow[d, hook] \\ \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = St \arrow[r] & G/B \times G/B \end{tikzcd}$$

this is a vector bundle / Y_w and $\overline{Z}_w$ give all the components of St.

Note $Y_w \xrightarrow{\pi_1} G/B$ is an affine space bundle, thus so is $Z_w \to Y_w \xrightarrow{\pi_1} G/B$

$$\begin{tikzcd} Z_1 \arrow[r, hook] \arrow[d, phantom, "\cong"] & St \\ \tilde{\mathfrak{g}} \arrow[ur, "\Delta \text{ diagonal}"'] & \end{tikzcd}$$

we also have that Z_1 is the diagonal component.

<u>What does St look like?</u>

$$\tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{t}$$
$$(x,b) \mapsto x \in \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \cong \mathfrak{t}$$

In fact

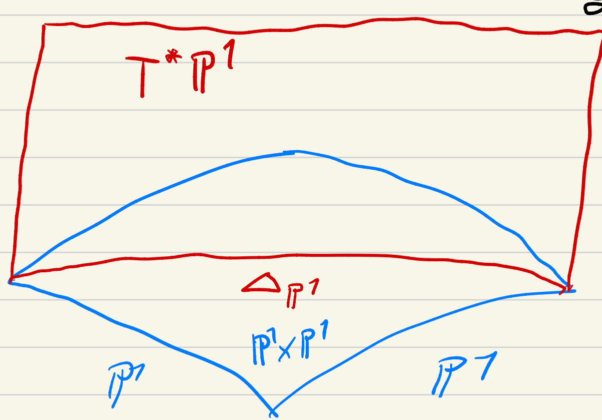
$$R\Gamma(\mathcal{O}_{\tilde{\mathfrak{g}}}) = \mathcal{O}_{\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{g}}$$

over $x \in \mathfrak{t}^{rs}$ the fiber of $St \xrightarrow{\pi_1} \tilde{\mathfrak{g}} \to \mathfrak{t}$ is $\coprod_{w \in W} G/T$

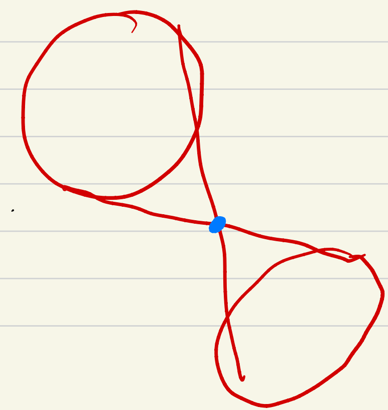
The fiber over $0 \in \mathfrak{t}$ is $\bigcup T^*_{Y_w}(G/B)^2$ the union of conormals.

$$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$$

<u>Example of SL_2</u>:



$\longrightarrow$



Generators in K-thy

We will define the assignment of the generators of H by

for $\lambda \in \Lambda$ $\quad \theta : e^{\lambda} \mapsto [\Delta_* \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)]$, $\quad K^{G \times \mathbb{C}^*}(St) \circlearrowleft K^{\mathbb{C}^*}(pt)$

for s simple reflection $\quad \theta : T_s \mapsto -[\mathcal{O}_{\overline{Z_s}}]$ $\quad = \mathbb{C}[q^{\pm 1}]$

$$\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_s = \{ (x, P) \in \mathfrak{g} \times G/P_s \mid x \in P \}.$$

$$Z_1 \cup Z_s = \tilde{\mathfrak{g}} \underset{\tilde{\mathfrak{g}}_s}{\times} \tilde{\mathfrak{g}} \subseteq St$$

Exact Sequences and relations

We have the following $G \times \mathbb{G}_m$-equivariant exact sequences.

(1) $$0 \to \Delta_* \mathcal{O}_{\tilde{\mathfrak{g}}}\langle 1 \rangle \hookrightarrow \mathcal{O}_{\tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}}_s} \tilde{\mathfrak{g}}} \to \mathcal{O}_{\tilde{Z}_s} \to 0.$$

(2) $$0 \to \mathcal{O}_{\tilde{Z}_s}(-\rho, \rho - \alpha) \hookrightarrow \mathcal{O}_{\tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}}_s} \tilde{\mathfrak{g}}} \to \Delta_* \mathcal{O}_{\tilde{\mathfrak{g}}} \to 0$$

First note that $\Delta_* \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)$ acts under left convolution as $\mathcal{O}_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(\lambda, 0) \otimes$ and as right convolution as $\mathcal{O}_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(0, \lambda) \otimes$

Exact sequences and relations

The convolution with $\mathcal{O}_{\tilde{\mathcal{Y}} \times_{\tilde{\mathcal{Y}}_s} \tilde{\mathcal{Y}}}$ on the left of $\tilde{\mathcal{Y}} \times_{\mathfrak{g}} \tilde{\mathcal{Y}}$

corresponds to $\pi_s^* \pi_{s*}$ for $\pi_s : \tilde{\mathcal{Y}} \to \tilde{\mathcal{Y}}_s$.

as $\pi_s^* \pi_{s*} [\mathcal{O}_{\tilde{\mathcal{Y}}}(\lambda)] = (1+q)[\mathcal{O}_{\tilde{\mathcal{Y}}}(\lambda)]$ whenever $\langle \lambda, \alpha_s^\vee \rangle = 0$. $(*)$

It follows from $(*)$ and SES(1):

$$(\Theta(T_s) - 1)(\Theta(T_s) + q) = 0$$

Also by $(*)$ for $\langle \lambda, \alpha_s^\vee \rangle = 0$

$$\Theta(T_s)\,\Theta(e^\lambda) = \Theta(e^\lambda)\,\Theta(T_s)$$

Exact sequences and relations

From the exact sequence (2) we get

$$\theta(T_s)\,\theta(e^{-\rho})\,\theta(T_s) = q\,\theta(e^{\alpha-\rho})$$

This is a particular case of R4) when $\langle \lambda, \alpha_s^\vee \rangle = 1$, but the general result of R4) follows now from R3) checked before

Commutativity

$$e_- = \sum (-q)^{\ell(w)} T_w$$

$$T_s\, e_- = -e_-$$

$$\begin{tikzcd}
\mathcal{J} \arrow[r] \arrow[dr, "\Theta"'] & H \arrow[r] & \mathrm{End}(He_-) \arrow[d, phantom, "\wr\wr\, f"] \\
& K^{G\times\mathbb{C}^*}(\tilde{g}\times_g \tilde{g}) \arrow[r] & \mathrm{End}(K^{G\times\mathbb{C}^*}(\tilde{g}))
\end{tikzcd}$$

Recall $He_- \cong R(T)[q^{\pm 1}]$ so has a basis $e^\lambda e_-$

Similarly $K^{G\times\mathbb{C}^*}(\tilde{g}) \cong K^{G\times\mathbb{C}^*}(G/B) \cong K^{B\times\mathbb{C}^*}(pt) \cong R(T)[q^{\pm 1}]$.

so we have an isomorphism $e^\lambda e_- \leftrightarrow [\mathcal{O}_{\tilde{g}}(\lambda)]$.

The action of e^λ and $[\Delta_* \mathcal{O}_{\tilde{g}}(\lambda)]$ are clearly compatible.

We have checked that $\Theta(T_s)$ and $\Theta(e^\lambda)$ satisfy the same commutation relation as T_s and e^λ, so it is enough to check $\Theta(T_s)[\mathcal{O}_{\tilde{g}}] = f(T_s e_-)$, which follows from $(*)$

Injectivity

We need to check injectivity of

$$K^{G\times\mathbb{C}^*}(\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}}) \hookrightarrow \operatorname{End}(K^{G\times\mathbb{C}^*}(\tilde{\mathfrak{g}})).$$

Recall that $\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}}$ has a filtration by Z_w, so

$$K^{G\times\mathbb{C}^*}(\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}}) \text{ is flat } / \ R(T)[q^{\pm 1}]$$

and so injectivity can be checked at the generic point of $\mathbb{C}[q^{\pm 1}]$ and thus this reduces by the localization thm to.

$$K^{G\times\mathbb{C}^*}(G/B\times G/B) \cong K^{G\times\mathbb{C}^*}(G/B) \underset{K^{G\times\mathbb{C}^*}(pt)}{\otimes} K^{G\times\mathbb{C}^*}(G/B)$$

$$\hookrightarrow \operatorname{End} K^{G\times\mathbb{C}^*}(G/B).$$

Filtration and graded isomorphism

H has a filtration by $H_{\leq w} = \{e^{\lambda} T_{w'} \mid w' \leq w\}$.

$K^{G\times\mathbb{C}^*}(St)$ has a filtration induced by the filtration Z_w of St, which satisfies the assumptions of the Cellular Filtration Lemma.

$$e^{\lambda} \in H_{\leq 1} \longrightarrow K^{G\times\mathbb{C}^*}(St)_{\leq 1} \cong K^{G\times\mathbb{C}^*}(\underbrace{\tilde{g}_{\Delta}}_{= Z_1})$$

$$T_s \in H_{\leq s} \longrightarrow K^{G\times\mathbb{C}^*}(St)_{\leq s} \cong K^{G\times\mathbb{C}^*}(\tilde{g} \times_{\tilde{g}_s} \tilde{g}).$$

by construction

Filtration and graded isomorphisms

Now for $w \in W$ let $w = s_{\alpha_1} \cdots s_{\alpha_n}$ a reduced expression.

Note that $[\Delta_* \mathcal{O}_{\tilde{g}}(\lambda)] * [\mathcal{O}_{\bar{Z}_{s_{\alpha_1}}}] * \cdots * [\mathcal{O}_{\bar{Z}_{s_{\alpha_n}}}]$ is supported on the image of

$$\tilde{g} \times_{\tilde{g}_{s_{\alpha_1}}} \tilde{g} \cdots \times_{\tilde{g}_{s_{\alpha_n}}} \tilde{g} \xrightarrow{\pi_{1,n+1}} \tilde{g} \times_{g} \tilde{g}$$

which is easily seen to be $\coprod_{w' \leq w} Z_{w'}$

Thus as $H_{\leq w}$ spanned by $e^{\lambda} T_{w'}$, we get that indeed

$$H_{\leq w} \hookrightarrow K^{G \times \mathbb{C}^*}(St)_{\leq w}.$$

thus it is filtered

$$H_{\leq w} / H_{< w} = \langle e^{\lambda} T_w \rangle \cong R(T)[q^{\pm 1}] T_w.$$

$$K^{G \times \mathbb{C}^*}(St)_{\leq w} / K^{G \times \mathbb{C}^*}(St)_{< w} \cong K^{G \times \mathbb{C}^*}(Z_w) \cong R(T)[q^{\pm 1}][\mathcal{O}_{Z_w}]$$

The graded isomorphism follows from

$$Z_{s_1} * \cdots * Z_{s_k} = Z_w$$

and the intersections in

$$\begin{tikzcd} \pi_{12}^{-1}(Z_{s_1}) \cap \pi_{23}^{-1}(Z_{s_2}) \cdots \cap \pi_{k\,k+1}^{-1}(Z_{s_k}) \arrow[d, "\pi_{1\,k+1}"', "\wr"] \\ Z_w \end{tikzcd}$$

are transversal, so the associated graded has

$$T_w \mapsto c_w [\mathcal{O}_{Z_w}] \qquad \text{with } c_w \text{ an invertible element.}$$