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The Hecke category action on the principal block via Smith theory — G. Williamson 27/1/2021

(discussing work of Simon Riche-W and Josh Ciappara)

Plan of talk:

1. Philosophy of higher representation theory + examples
2. Algebraic representations + conjecture of Riche-W.
3. Smith-Treumann theory + geometric Satake
4. Translation functors + Ciappara's argument.

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1 Philosophy of higher representation theory

Classical representation theory: "consequences of linear symmetry"

$A \circlearrowright V \rightsquigarrow$ consequences for V

k-algebra, k-vector space

e.g. might decompose V into canonical pieces, imply that it is ∞-dim, imply that certain operators have certain eigenvalues...

→ also, considering representations is often a non-trivial step.
it was basically absent from 19th century mathematics,
and took over much of 20th century mathematics!

→ if you are pursuing something esoteric it is good to have apparently esoteric heritage!

Higher representation theory: "consequences of symmetries of (higher) categories"

→ linear algebra is replaced by category theory, often want k-linear, additive, abelian, triangulated

$A \circlearrowright \mathcal{C} \rightsquigarrow$ consequences for $\mathcal{C}$

k-linear category

→ has been useful for the "internal" study of categories in rep. theory, now number theory, geometry...

Examples ③

ⓐ (not particularly relevant for the rest of this talk)

S_n symmetric group on n-letters, $\mathbb{F}_p S_n$ its group algebra.

$$\bigoplus_{n\geq 0} \operatorname{Rep} \mathbb{F}_p S_n \quad \circlearrowleft \operatorname{ind}_{S_{n-1}}^{S_n} \quad \circlearrowright \operatorname{res}_{S_{n-1}}^{S_n}$$

$\rightsquigarrow$ simple highest weight representation of categorified $\widehat{\mathfrak{sl}_p}$.

Morals: How hard it is to check relations, but how powerful it is.

$\longrightarrow$ Chuang and Rouquier use this setup to prove Broué's conjecture. Their work makes it clear how powerful it can be to know even simple relations between functors.

ⓑ $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ complex semi-simple Lie algebra, Borel subalgebra, Cartan ④

W_f (finite) Weyl group.

$$\mathcal{O}_{\mathbb{Z}} = \{ V \in U(\mathfrak{g})\text{-mod} \mid \mathfrak{g} \text{ finitely-generated}, \mathfrak{b} \text{ locally-finite}, \mathfrak{h} \text{ semi-simple w/ integral eigenvalues} \}$$

"integral category $\mathcal{O}$"

$$\bigoplus_{\lambda \in \mathfrak{h}_{\mathbb{Z}}/(W_f \cdot)} \mathcal{O}_\lambda \overset{\text{central character}}{=} \mathcal{O}_{\mathbb{Z}} \circlearrowleft \operatorname{Rep}^{\text{f.d.}} \mathfrak{g}$$

semi-simple $\otimes$-category.

looks like a rather innocent situation, but there is great depth, e.g. interesting endomorphism rings etc.

indecomposable summands = "projective functors" of Bernstein-Gelfand

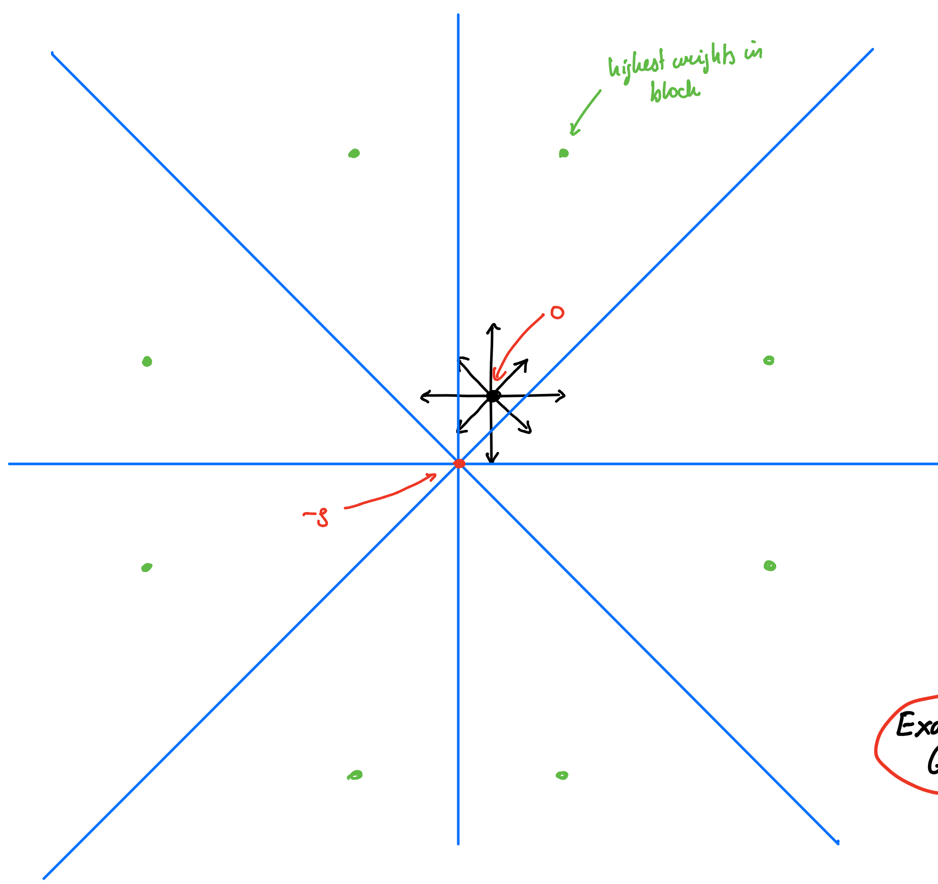
$$\mathcal{O}_0 \circlearrowleft \text{Hecke category},$$

"Soergel bimodules", cat$^{\underline{n}}$ of $\mathbb{Z}W_f$ and (eventually) of its Hecke algebra.

$\rightarrow$ "taking matrix coefficients yields an interesting algebra."

$\rightarrow$ all blocks are "strongly cyclic modules" over Hecke algebra. *

$\longrightarrow$ action + thms on Hecke category $\Rightarrow$ KL conjecture.

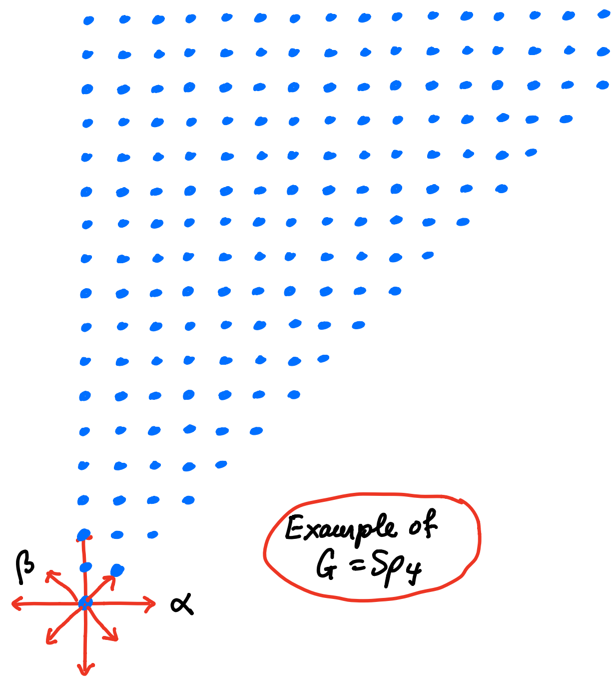


Example of $G = \mathfrak{sp}_4$

Algebraic representations + conjecture of Riche – W.

G/k split reductive group
$\cup$
B Borel subalgebra
$\cup$
T maximal torus

$\mathfrak{X}$ character lattice
$\cup$
$\mathfrak{X}_+$ dominant weights



Example of $G = Sp_4$

Chevalley Theorem algebraic

$$\left\{ \begin{matrix} \text{simple} \\ G\text{-modules} \end{matrix} \right\} /_{\cong} \;=\; \mathfrak{X}_+$$

$$L_\lambda \longleftarrow\!\shortmid \; \lambda$$

highest weight λ

ℓ = char k

Linkage Principle

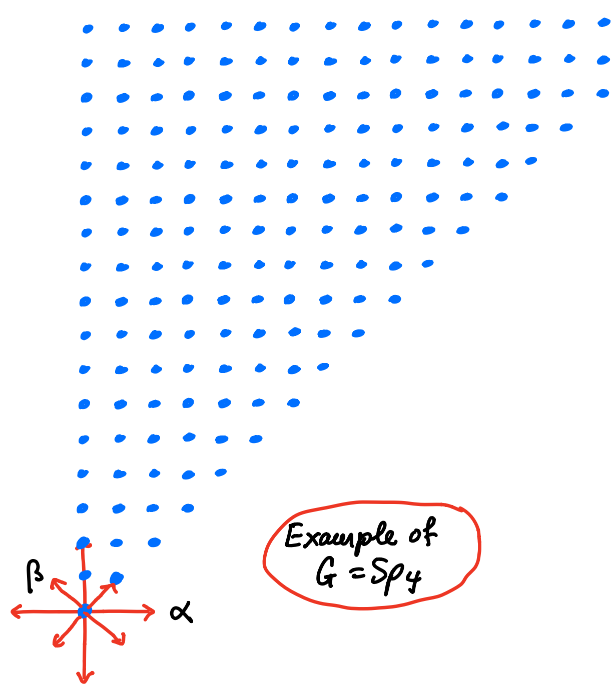
affine Weyl group (of DUAL group) $W = W_f \ltimes \mathbb{Z}\Phi$

$$\cup$$

$$W_\ell = W_f \ltimes \ell\mathbb{Z}\Phi$$

Dot action: $x \cdot \lambda = x(\lambda+\rho) - \rho$

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$$



Example of $G = Sp_4$

Linkage Principle

$$\operatorname{Rep} G = \bigoplus_{\gamma \in \mathfrak{X}/(W_\ell \cdot)} \operatorname{Rep}_\gamma G$$

where

$$\operatorname{Rep}_\gamma G = \langle L_\lambda \mid \lambda \in \gamma \cap \mathfrak{X}_+ \rangle$$

$\ell = \operatorname{char} k$

Linkage Principle

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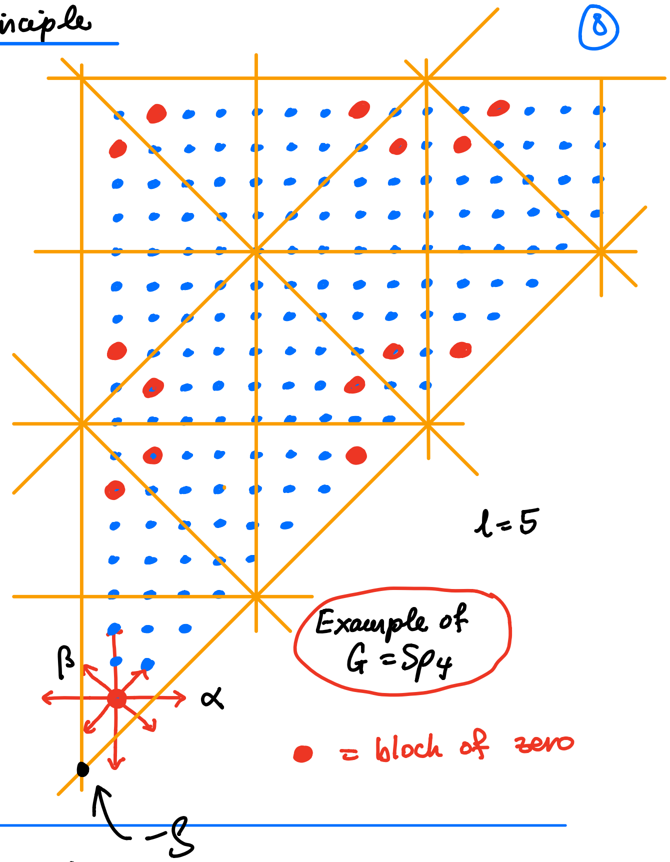

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$\operatorname{Rep} G \circlearrowleft (\operatorname{Tilt} G, \otimes)$ ← additive tensor category of tilting G-modules

$\cup$

"principal block" $\operatorname{Rep}_0 G$

Conjecture (Riche-W. 2013)

The action of $\operatorname{Tilt} G$ on $\operatorname{Rep} G$ induces an action of the affine Hecke category $\mathcal{H}^\vee$ for Langlands dual group on principal block.

Riche-W: Conjecture implies:

→ $\operatorname{Rep}_0 G$ is "strongly cyclic" for $\mathcal{H}$

→ description of Rep_0 in terms of $\mathcal{H}^\vee$ "anti-spherical module"

→ "p-KL conjecture" e.g. Lusztig conj. for large p.

History: GL_n : RW ✓ 2015, Elias-Losev ✓ 2017, "p-KL conjecture": Achar-Riche + A-Makisumi-R-W (2018) ✓

Full conjecture: Bezrukavnikov-R ✓ 2020, Ciappara ✓ 2021

(no constructible sheaves)

(no coherent sheaves! Very simple and IMHO beautiful proof)

BREAK

③ Smith–Treumann theory + geometric Satake

From now on work on Langlands dual side.

$LG = G((t))$, $L^+G = G[[t]]$ work in étale setting over $\overline{\mathbb{F}}_p$, $\ell \neq p$.

$G^\vee$ = dual group over k, char $k = \ell$.

$\mathcal{G}r = LG / L^+G$ affine Grassmannian.

$$\begin{array}{ccc} \mathrm{Perv}_{IW}(\mathcal{G}r, k) & \circlearrowleft & (\mathrm{Perv}_{L^+G}(\mathcal{G}r, k), *) \\ \downarrow S & & \downarrow S \\ \mathrm{Rep}\, G^\vee & \circlearrowleft & (\mathrm{Rep}\, G^\vee, \otimes) \end{array}$$

"Iwahori-Whittaker model" cf. Tony's talk in Term 1.

"geometric Satake"

very important for us that this is an equivalence of right module categories!

Loop rotation fixed points

$\mathbb{G}_m$ ↻ LG, L^+G, $\mathcal{G}r$ via loop rotation.

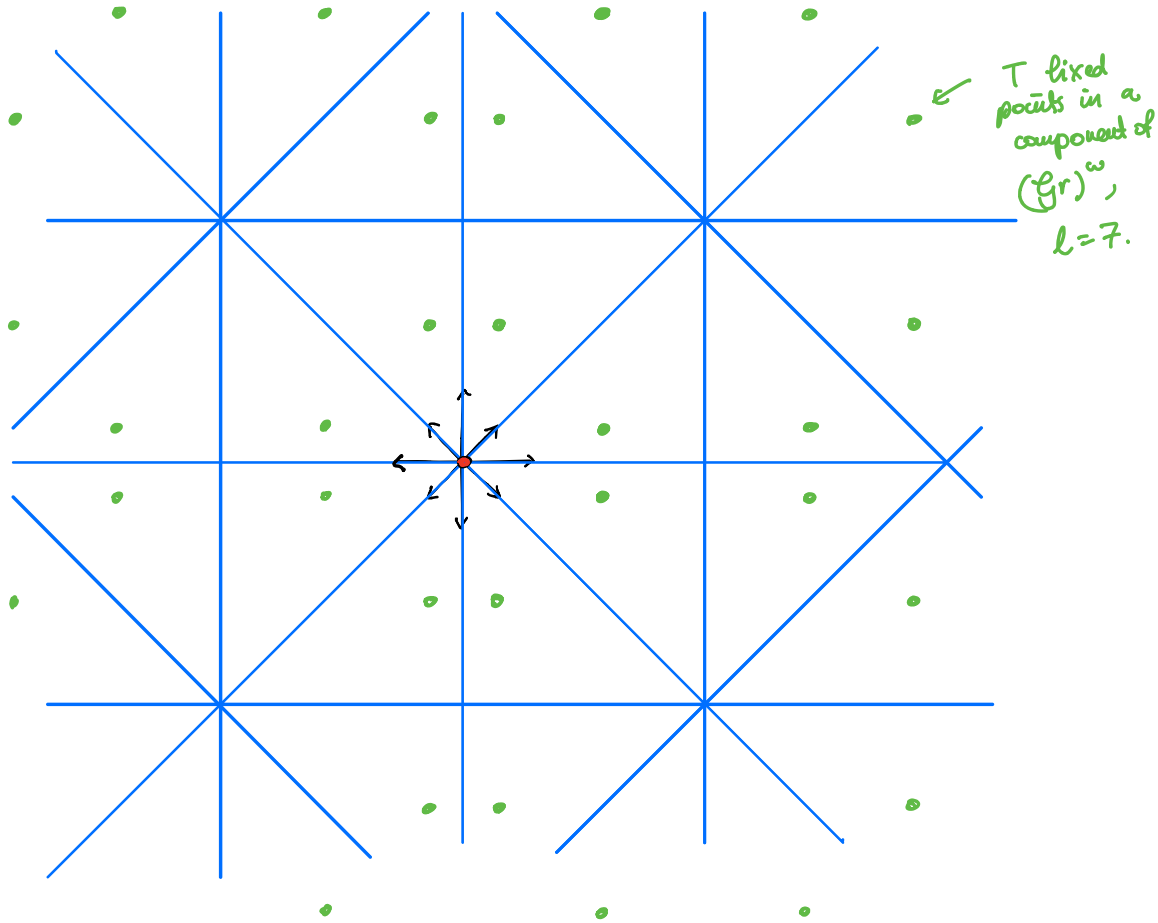
$\varpi \subset \mathbb{G}_m$ ℓ^{th} roots of unity

Example: $(LG)^{\mathbb{G}_m} = G$, $(LG)^{\varpi} =: L_\ell G := G((t^\ell))$

Key facts: ⓐ $(\mathcal{G}r)^{\varpi} = \bigsqcup_{\gamma \in \mathfrak{X}/W} \mathcal{G}r_{(\gamma)}$

ⓑ Each $\mathcal{G}r_{(\gamma)}$ is (isomorphic to) a partial affine flag variety for $(L_\ell G)^\circ$

Examples: $\mathcal{G}r_{(0)} = G((t^\ell))^\circ / G[[t^\ell]]^\circ$, $\mathcal{G}r_{(\lambda)} = G((t^\ell))^\circ / \text{Iwahori}$. (regular)


T fixed points in a component of $(Gr)^{\omega}$, $\ell = 7$.

Linkage principle and loop rotation fixed points

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Notation: $(\mathcal{G}r)^{\varpi} \xhookrightarrow{i} \mathcal{G}r \xhookleftarrow{i_\lambda} \mathcal{G}r_{(\lambda)}$

Thm (RW) $\quad i^{!*} : \mathrm{Tilt}_{IW, \mathbb{G}_m}(\mathcal{G}r) \xrightarrow{\sim} \mathrm{Sm}^{\mathbb{M}}_{IW, \mathbb{G}_m}((\mathcal{G}r)^{\varpi})$

(cf. Daniel's talk)

Remarkable consequence: $\quad \mathrm{Tilt}^{\lambda}_{IW, \mathbb{G}_m}(\mathcal{G}r) \xrightarrow{\sim} \mathrm{Sm}^{\mathbb{M}}_{IW, \mathbb{G}_m}(\mathcal{G}r_{(\lambda)}) \quad (*)$

$$\begin{array}{c} \mathrm{Tilt}^{\lambda}_{IW, \mathbb{G}_m}(\mathcal{G}r) \\ \wr\!\!| \\ \mathrm{Tilt}^{\lambda\text{-}s}(G^\vee) \end{array}$$

Our conjecture is [basically] equivalent to wall crossing functors corresponding to push-pull functors under $(*)$!

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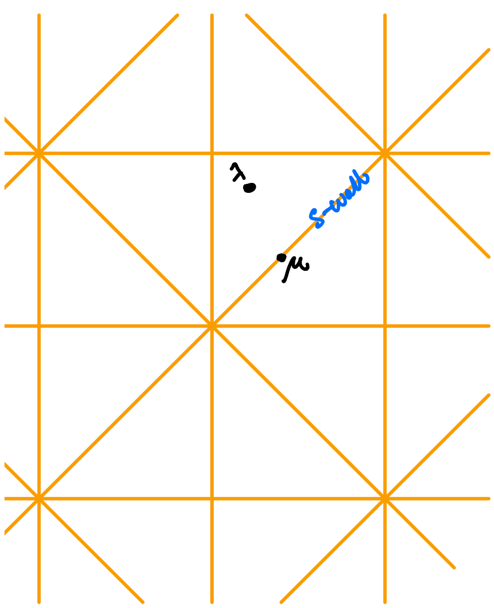


What makes translation functors work?

For simplicity: λ regular, μ s-singular

$\gamma :=$ unique dominant weight in $W_f(\mu - \lambda)$.

V G-module with highest weight γ

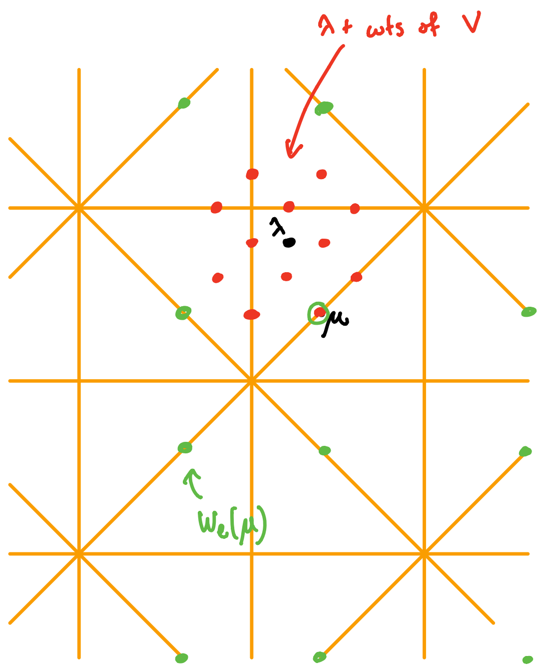
($\Leftrightarrow$ extremal weight $\mu - \lambda$)

Define:

$$T^{\mu}_{\lambda} = \underset{\text{onto } \mathrm{Rep}_{\mu}}{\text{project}} \circ (-) \otimes V \circ \underset{\mathrm{Rep}_\lambda \hookrightarrow \mathrm{Rep}}{\text{include}}$$

"translation onto s-wall"

Key combinatorical fact: $\quad (\lambda + \text{ weights of } V) \cap W_\ell(\mu) = \{\mu\}$



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Key combinatorical fact: $\boxed{(\lambda + \text{weights of } V) \cap W_\ell(\mu) = \{\mu\}}$

(14)

Conjecture is an easy consequence of:

Thm (Ciaparra) The following squares commute up to natural isomorphism:

$$\begin{tikzcd} \mathrm{Tilt}^{\lambda}_{IW,\mathbb{G}_m}(\mathcal{G}r) \arrow[r, "T_\lambda^\mu"] \arrow[d, "i_\lambda^{!*}"] & \mathrm{Tilt}^{\mu}_{IW,\mathbb{G}_m}(\mathcal{G}r) \arrow[r, "T_\mu^\lambda"] \arrow[d, "i_\mu^{!*}"] & \mathrm{Tilt}^{\lambda}_{IW,\mathbb{G}_m}(\mathcal{G}r) \arrow[d, "i_\lambda^{!*}"] \\ \mathrm{Sm}_{IW,\mathbb{G}_m}(\mathcal{G}r_{(\lambda)}) \arrow[r, "q_{s*}"] & \mathrm{Sm}_{IW,\mathbb{G}_m}(\mathcal{G}r_{(\mu)}) \arrow[r, "q_s^*"] & \mathrm{Sm}_{IW,\mathbb{G}_m}(\mathcal{G}r_{(\lambda)}) \end{tikzcd}$$

where $q_s : \mathcal{G}r_{(\lambda)} \simeq L_\ell G/P_\phi \to \mathcal{G}r_{(\mu)} \simeq L_\ell G / P_s$

is the natural quotient map.

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Suppose that $\lambda, \mu \in$ fundamental ℓ-alcove, γ = dominant in $W_f(\mu-\lambda)$

$$\begin{tikzcd} & \mathcal{Y}_\gamma := LG\cdot(t^\lambda, t^\mu) \subset \mathcal{G}r \times \mathcal{G}r \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{G}r & & \mathcal{G}r \end{tikzcd}$$

("lattices in relative position γ")

First observation: ("translation functors are given by an easy correspondence")

The following commutes up to natural isomorphism:

$$\begin{tikzcd} \mathrm{Tilt}^{\lambda}_{IW,\mathbb{G}_m}(\mathcal{G}r) \arrow[r, "(-)*T_\gamma"] \arrow[d, "q_* p^*"'] \arrow[dr, "T_\lambda^\mu"] & \mathrm{Tilt}_{IW,\mathbb{G}_m}(\mathcal{G}r) \arrow[d, "i_\mu^{!*}"] \\ \mathcal{D}_{IW,\mathbb{G}_m}(\mathcal{G}r) \arrow[r, "i_\mu^{!*}"] & \mathrm{Sm}^{\natural}_{IW,\mathbb{G}_m}(\mathcal{G}r_{(\mu)}) \end{tikzcd}$$

Remember that "Smith restriction commutes with all operations".

$$\begin{tikzcd}
\mathcal{G}r & \mathcal{Y}_\gamma \arrow[l, "q"'] \arrow[r, "p"] & \mathcal{G}r \\
(\mathcal{G}r)^\omega \arrow[u, hook] & (\mathcal{Y}_\gamma)^\omega \arrow[l] \arrow[r] \arrow[u, hook] & (\mathcal{G}r)^\omega \arrow[u, hook] \\
\mathcal{G}r_{(\lambda)} \arrow[u, hook] & \Gamma^\mu_\lambda \arrow[l] \arrow[r] \arrow[u, hook] & \mathcal{G}r_{(\mu)} \arrow[u, hook]
\end{tikzcd}$$

Second observation: Suppose in addition that λ is regular and that μ is S-singular.

Then Γ^μ_λ = graph of quotient map

$$\begin{array}{ccc} L_\ell G/P_\emptyset & \longrightarrow & L_\ell G/P_S \\ \wr\mathrm{l} & & \wr\mathrm{l} \\ \mathcal{G}r_{(\lambda)} & & \mathcal{G}r_{(\mu)} \end{array}$$

Proof: $\mathbb{G}_m$-fixed point argument + "key combinatorial fact" from earlier.