Smith-Linkage-Tilting.

The linkage principle and the tilting character formula via Smith-Treumann theory

Recall: Geometric Satake equivalence

G connected reductive algebraic group, say $/\mathbb{C}$.
$Gr := G((t))/G[[t]]$. Throughout: coefficients k
For this talk, we care about $\operatorname{char} k = \ell > 0$.

Thm (Mirković-Vilonen in this generality; see notes by Baumann-Riche)

$$\mathrm{Sat}_G := (\mathrm{Perv}_{G[[t]]}(Gr), \star) \xrightarrow[S]{\sim} (\mathrm{Rep}\, G^\vee_k, \otimes)$$

- Hard! Indirect (Tannakian) connection
  Is it rewarding?
- Direct encoding of $\mathrm{Rep}\, G^\vee_k$ in Gr singularities
  $[\Delta(\lambda) : L(\mu)] = [\mathcal{P}_!(\lambda) : \mathcal{IC}(\mu)]$
  but this seems useless: blockwise would be better!

→ How to see the linkage principle geometrically?
→ Can we reprove LCF for $\ell \gg 0$? Tilting characters?

Riche-Williamson: Yes! Smith theory + Iwahori-Whittaker.

The linkage principle

$G \supset B = TU, \quad B^+ = TU^+$

$G^\vee \supset B^\vee \supset T^\vee$

$\mathfrak{X} = X_*(T) = X^*(T^\vee) \supset R$ root system

$W = W_f \ltimes \mathbb{Z}R \supset W_\ell = W_f \ltimes \ell\mathbb{Z}R$

full generality

Thm [Verma, Humphreys, Jantzen, Andersen]

$$\mathrm{Rep}\, G_k^\vee = \bigoplus_{\gamma \in \mathfrak{X}/W_\ell \bullet} \mathrm{Rep}_\gamma G_k^\vee$$

$$\mathrm{Rep}_\gamma G_k^\vee = \langle L(\lambda) \mid \lambda \in \gamma \rangle_{\text{Serre}}$$

Fixed points of $\mu_\ell \subset \mathbb{C}^\times \circlearrowright Gr$ (loop rotation)

<u>Key observation</u>: $$Gr^{\mu_\ell} = \coprod_{\gamma \in \mathfrak{X}/W_\ell} \underbrace{G((t^\ell))^\circ \cdot [t^\gamma]}_{Gr^{\mu_\ell}_{(\gamma)}}$$

$\mathfrak{X}/W_\ell \cong \overline{A_\ell} \cap \mathfrak{X}$

$Gr^{\mu_\ell}_{(\gamma)} \cong G((t^\ell))^\circ / P_{-\gamma}$

A_ℓ = fundamental ℓ-alcove

parahoric subgroup.

e.g. $Gr^{\mu_\ell}_{(0)} \simeq G((t^\ell))^\circ / G[[t^\ell]]$ affine Grassmannian

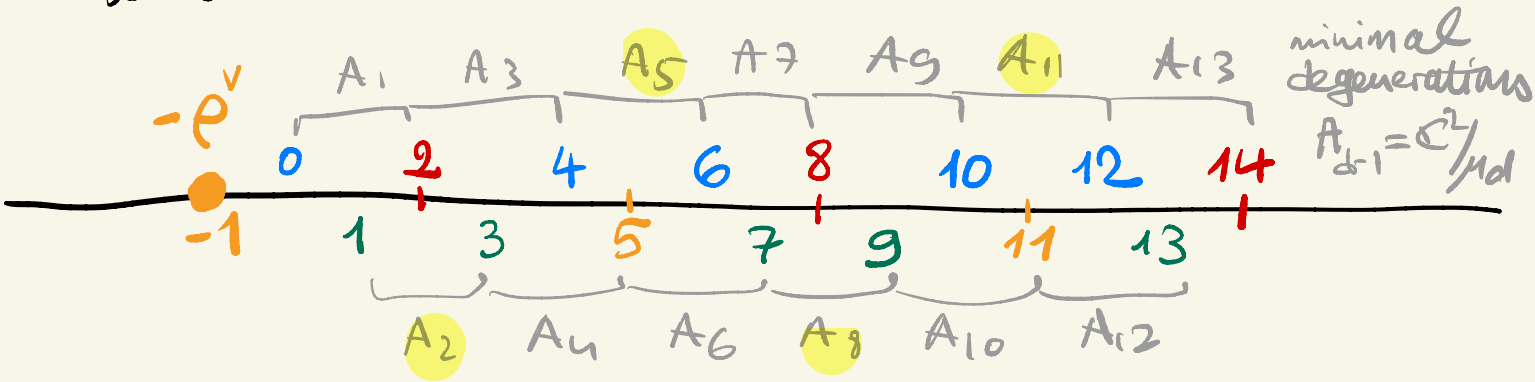
$\lambda_{reg} \Rightarrow Gr^{\mu_\ell}_{(\lambda)} \simeq G((t^\ell))^\circ / Iw_\ell$ affine flag variety

<u>Discrepancy</u>: NO DOT.

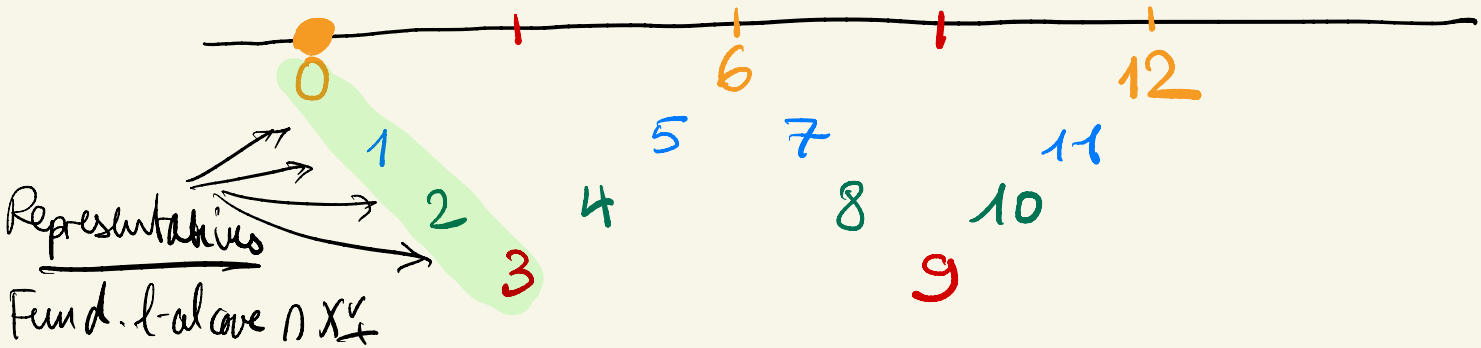
Geometric Satake: $\mathrm{Perv}_{PGL_2}(\mathrm{Gr}_{PGL_2}, k) \simeq (\mathrm{Rep}_k SL_2, \otimes)$

$\ell = 3$

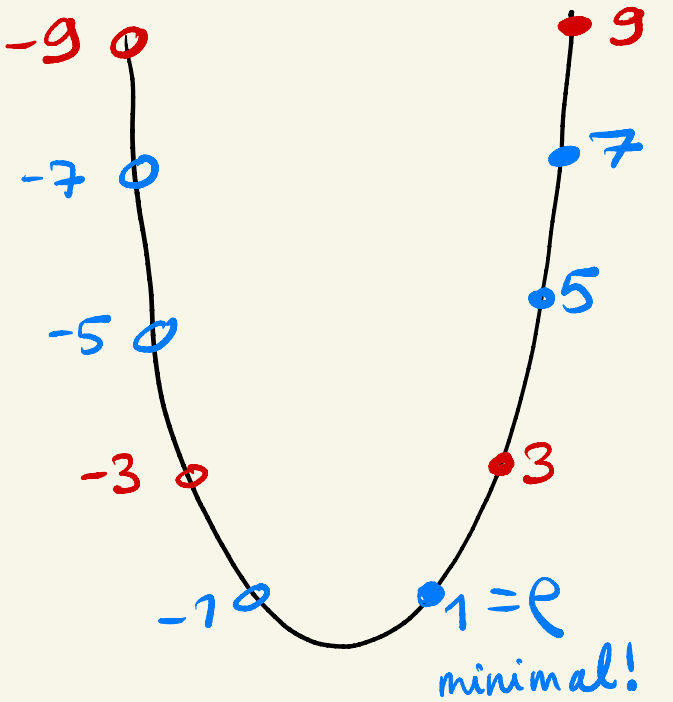
$W_{aff} \cdot e$

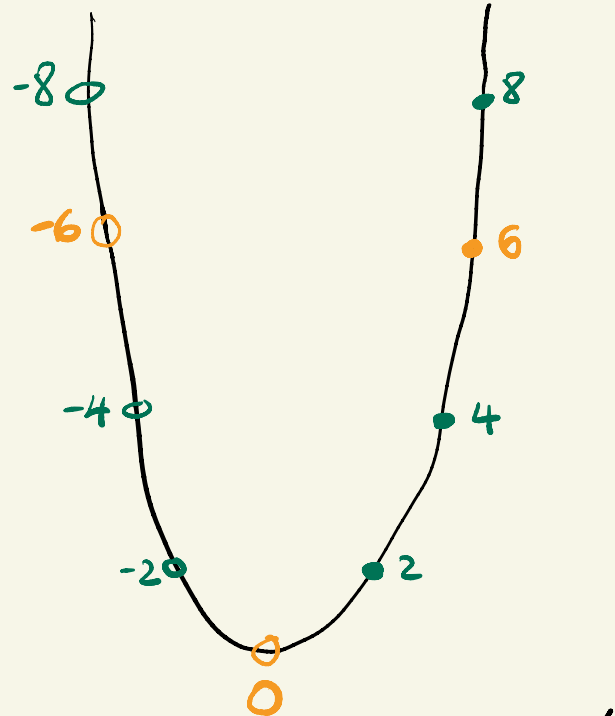


Take μ_3-fixed to get several connected components ... but no shift by $e^\vee$!



Iwahori-Whittaker model





Carry IW local systems?

Iw_u^+-orbits on Gr_{PGL_2}: taking μ_ℓ-fixed points, get several components

Iwahori-Whittaker model

<u>Recall</u>: $D^b_{IW}(Gr) \supset Perv_{IW}(Gr)$

- IW property: $(Iw^+_u, \chi^*\mathcal{L}_{AS})$-equivariance
  $\chi : Iw^+_u \to \mathbb{G}_a$, $\mathcal{L}_{AS}$ Artin Schreier sheaf
  $\rightsquigarrow$ From now on, $G/\mathbb{F}$, say $\mathbb{F} = \overline{\mathbb{F}}_p$, $p \neq \ell$.
- <u>Key fact</u>: if $\lambda \notin X_{++}$ then $Gr^+_\lambda := Iw^+_u \cdot [t^\lambda]$ supports no IW-local system. For $\lambda \in X_{++}$, there is one up to iso: let $\mathcal{L}^\lambda_{AS}$ denote this local system shifted by $d_\lambda := \dim Gr^+_\lambda = 2\langle \rho, \lambda \rangle$
- Let $j_\lambda : Gr^+_\lambda \hookrightarrow Gr$. For $\lambda \in X_{++}$,

$$\begin{tikzcd} \Delta^{IW}_\lambda := j_\lambda ! \mathcal{L}^\lambda_{AS} \arrow[rr] \arrow[dr, two heads] & & \nabla^{IW}_\lambda = j_\lambda * \mathcal{L}^\lambda_{AS} \\ & \mathcal{IC}^{IW}_\lambda \arrow[ur, hook] & \end{tikzcd}$$

affine spaces $\Rightarrow$ t-exactness

- $D^b Perv_{IW}(Gr) \xrightarrow{\sim} D^b_{IW}(Gr)$, and $Perv_{IW}(Gr)$ is a highest weight category $\rightsquigarrow$ tilting objects T^{IW}_λ, $\lambda \in X_{++}$

//[BGS]

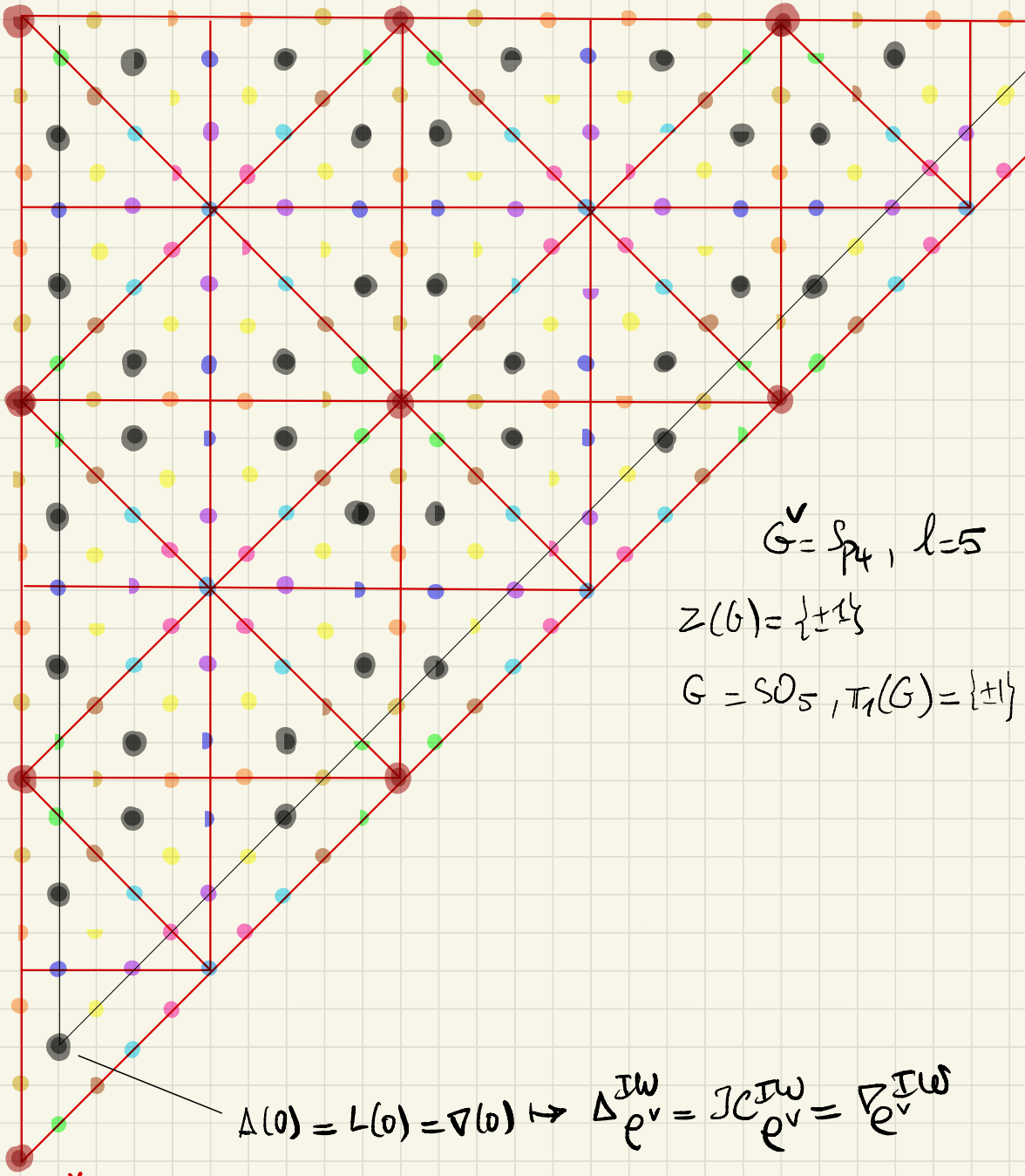
- Moreover, d_λ of fixed parity on each connected component
  $\Rightarrow$ The T^{IW}_λ are parity sheaves, $T^{IW}_\lambda = \mathcal{E}^{IW}_\lambda$
- $D^b_{IW}(Gr) \circlearrowleft D^b_{G[[t]]}(Gr)$, preserves parity and perversity.

$\star$

Theorem [Bezrukavnikov-Gaitsgory-Mirković-Riche-Rider]

$$\mathcal{IC}_{\rho^\vee} \star - : Perv_{G[[t]]}(Gr) \to Perv_{IW}(Gr)$$

is an equivalence of highest weight categories, $X_+ \to X_{++}$, $\lambda \mapsto \lambda + \rho$



$G^\vee = Sp_4$, $\ell = 5$

$Z(G) = \{\pm 1\}$

$G = SO_5$, $\pi_1(G) = \{\pm 1\}$

$\Delta(0) = L(0) = \nabla(0) \mapsto \Delta^{IW}_{\rho^\vee} = IC^{IW}_{\rho^\vee} = \nabla^{IW}_{\rho^\vee}$

$-\rho^\vee \rightsquigarrow 0$

$\boxed{Gr}$ | $\boxed{Gr^{\mu_\ell} = \coprod_\gamma Fl_\gamma}$ $\quad G_m \to G_m' = G_m/\mu_\ell \circlearrowright Gr^{\mu_\ell}$

✓ $K^b Tilt \xrightarrow{\sim} D^b Perv \xrightarrow{\sim} D^b$

✗ $K^b Tilt \not\xrightarrow{\sim} D^b Perv \xrightarrow[\text{NOT}]{} D^b$

[BGMRR]

2-periodic: $[0]=[2]$

$$
\begin{tikzcd}
D^b_{sph} \arrow[r, "\mathcal{I}_\ell * -"] & D^b_{IW} & D^b_{IW,G_m} \arrow[l, "For"'] \arrow[r, "i^{!*}"] & Sm_{IW_\ell,G_m} & D^b_{IW_\ell,G_m} \arrow[l, "Q"'] \arrow[r, "For"] & D^b_{IW_\ell} \\
Perv_{sph} \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & Perv_{IW} \arrow[u, phantom, "\cup"] & Perv_{IW,G_m} \arrow[u, phantom, "\cup"] \arrow[l, "For", "\sim"'] & & Perv_{IW_\ell,G_m} \arrow[u, phantom, "\cup"] \arrow[r, "For"] & Perv_{IW_\ell} \arrow[u, phantom, "\cup"] \\
Tilt_{sph} \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & Tilt_{IW} \arrow[u, phantom, "\cup"] & Tilt_{IW,G_m} \arrow[u, phantom, "\cup"] \arrow[l, "For", "\sim"'] \arrow[r, "i^{!*}", "\sim"'] & Sm^\circ_{IW_\ell,G_m} \arrow[u, phantom, "\cup"] & Tilt_{IW_\ell,G_m} \arrow[u, phantom, "\cup"] \arrow[r, "For"] & Tilt_{IW_\ell} \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

✗ ✓ ✗ ✗ ✓

$Tilt_{sph} = Even^\circ$ if ℓ good [JMW, MR, BGMRR]

$Tilt_{IW} = Even^\circ_{IW}$ $\quad Tilt_{IW,G_m} = Even^\circ_{IW,G_m}$

Here $Even^\circ$ means: even objects with the right shift so that they are self-dual (i.e. perverse)

parity: $\mathbb{D}(\lambda) = d\lambda$

Here parity sheaves may fail to be perverse!

Hom between $\mathcal{E}, \mathcal{F}$ same parity

$H^\bullet_{G_m}(pt) = k[x]$ $\quad 1 \leftarrow\!\shortmid x$ $\quad (\deg x = 2)$ $\quad x \mapsto 0$

Yellow part $\Rightarrow$ Linkage principle

Blue part $\Rightarrow$ Tilting character formula (all $\lambda \in \mathfrak{X}_+$, all ℓ)

$\Rightarrow$ Simple character formula ($\Rightarrow$ LCF for $\ell \gg 0$)

Riche-Williamson (cf Andersen $\ell \geq 2h-2$, cf Sobaje all ℓ)

Thm: Q and For preserve indec. parity

Reminder on Smith-Treumann theory

étale setting!

Crucial lemma: $\mu_\ell \circlearrowright X$, $X^{\mu_\ell} \xrightarrow{i} X \xleftarrow{j} U = X - X^{\mu_\ell}$

[for $\mathcal{F} \in D^b_{\mu_\ell, c}(X)$, the cone of $i^!\mathcal{F} \to i^*\mathcal{F}$ has perfect geom. stalks.

$$\Rightarrow D^b_c(X) \underset{i^*}{\overset{i^!}{\rightrightarrows}} D^b_{\mu_\ell, c}(X^{\mu_\ell}) \xrightarrow{Q} D^b_{\mu_\ell, c}(X^{\mu_\ell})/(\text{perf}) =: Sm(X^{\mu_\ell})$$

$\| $ Tate(X) last week

the two compositions give isomorphic functors

$\rightsquigarrow$ Smith localization $i^{!*}$, commutes with all operations! operations

Better: If $\mu_\ell \subset \mathbb{G}_m \circlearrowright X$, use $Sm_{\mathbb{G}_m}(X^{\mu_\ell}) := \dfrac{D^b_{\mathbb{G}_m, c}(X^{\mu_\ell})}{(\mu_\ell\text{-perf})}$

Key fact: $\mathrm{End}^\bullet_{Sm_{\mathbb{G}_m}(pt)}(k_{pt}) = k[x, x^{-1}]$

where $\deg x = 2$ $\Rightarrow$ Parity sheaves will make sense.

[Treumann: Tate spectrum; Leslie-Lonergan: $\mathbb{Z}_\ell \rightsquigarrow \mathbb{F}$]

Proof of the linkage principle

$$\begin{tikzcd}
\mathrm{Rep}\, G^\vee & \mathrm{Perv}_{sph} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{Perv}_{IW} & \mathrm{Perv}_{IW,\mathbb{G}_m} \arrow[l] \\
\mathrm{Tilt}\, G^\vee \arrow[u, phantom, "\cup"] & \mathrm{Tilt}_{sph} \arrow[l] \arrow[r, "\sim"] \arrow[u, phantom, "\cup"] & \mathrm{Tilt}_{IW} \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\shortparallel"] & \mathrm{Tilt}_{IW,\mathbb{G}_m} \arrow[l] \arrow[u, phantom, "\cup"] \arrow[r, "i^{!*}", "\sim"'] \arrow[d, phantom, "\shortparallel"] & \mathrm{Sm}^0_{IW_\ell,\mathbb{G}_m} \\
 & & \mathrm{Even}^0_{IW} & \mathrm{Even}^0_{IW,\mathbb{G}_m} &
\end{tikzcd}$$

Theorem 1: Smith localization $i^{!*}: \mathrm{Tilt}_{IW,\mathbb{G}_m} \xrightarrow{\sim} \mathrm{Sm}^0_{IW_\ell,\mathbb{G}_m}$ is an equivalence.

Consequence: $\mathrm{Hom}(T(\lambda), T(\mu)) \neq 0 \Rightarrow \mathrm{Hom}(\mathcal{E}^{IW}_{\lambda+\rho}, \mathcal{E}^{IW}_{\mu+\rho}) \neq 0$

$\Rightarrow W_\ell(\lambda+\rho) = W_\ell(\mu+\rho) \Rightarrow W_\ell \bullet \lambda = W_\ell \bullet \mu$

indeed, $i^{!*}(\mathcal{E}_\nu)$ stays indecomposable by theorem 1, hence it is supported on a single component of Gr^Θ.

$\Rightarrow$ Linkage principle.

<u>Sketch</u>: (i) $i^{!*}$ preserves parity because it commutes with all operations

(ii) full faithfullness: reduce to Δ_λ^{IW}, ∇_μ^{IW}, then by adjunction $\mathcal{L}_{AS}^\lambda$, $\mathcal{L}_{AS}^\lambda$ on gr_λ^+, then k_{pt}, k_{pt} on pt.

(iii) $i^{!*}(\mathcal{E}^{IW}_{\lambda,\mathbb{G}_m})$ is indecomposable by (ii), $(gr_\lambda^+)^\circ$ open in support $\Rightarrow = \mathcal{E}_\lambda^{Sm}$

($\Leftrightarrow$ existence!)

Tilting character formula

$$\mathrm{Sm}_{IW_\ell}(Gr^{H_\ell}) \xleftarrow[Q]{} D^b_{IW_\ell, \mathbb{G}_m}(Gr^{H_\ell}) \xrightarrow[For]{} D^b_{IW_\ell}(Gr^{H_\ell})$$

Theorem 2: Q and For preserve indecomposable parity complexes.

Key steps: For parity complexes $\mathcal{E}, \mathcal{F} \in D^b_{IW_\ell, \mathbb{G}_m}(Gr^{H_\ell})$ of the same parity:

a) $\mathrm{Hom}^\bullet(For\,\mathcal{E}, For\,\mathcal{F}) = \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{F}) \otimes_{x \mapsto 0} k$

b) $\mathrm{Hom}(Q\mathcal{E}, Q\mathcal{F}) = \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{F}) \otimes_{x \mapsto 1} k$

equivariant formality for parity complexes

$[0] = [2]$ in Sm

$\mathrm{End}^\bullet_{\mathrm{Sm}}(k_{pt}) = k[x, x^{-1}]$

a) $\Rightarrow$ For preserves indec. parity.

Note: The $\mathbb{G}_m$-action on Gr^{H_ℓ} factors through $\mathbb{G}_m \to \mathbb{G}'_m = \mathbb{G}_m / H_\ell$

$\rightsquigarrow$ indistinguishable from trivial action, with k coefficients.

c) $\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{F}) = k[x] \otimes_k \mathrm{Hom}^\bullet(For\,\mathcal{E}, For\,\mathcal{F})$

d) $\mathrm{Hom}(Q\mathcal{E}, Q\mathcal{F}) = \mathrm{Hom}^\bullet(For\,\mathcal{E}, For\,\mathcal{F})$

$\mathcal{E}$ indec parity
$\Rightarrow \mathrm{End}(For\,\mathcal{E})$ local
$\Rightarrow \mathrm{End}(Q\mathcal{E}) = \mathrm{End}^\bullet(For\,\mathcal{E})$ too

Use: A fin. dim. graded ring whose degree 0 part is local is itself local [Gordon–Green].

Thm 3: For $\lambda \in \overline{A_\ell^\bullet} \cap \mathcal{X}$ $w \in W^{(\lambda)} = \left\{ w \in W \,\middle|\, \begin{array}{l} w \text{ max in } wW_\lambda \\ \text{min in } W_P w \end{array} \right\}$,

$$\mathrm{ch}(T(w \bullet_\ell \lambda)) = \sum_{y \in W^{(\lambda)}} \underbrace{{}^\ell n_{y,w}(1)}_{\substack{\text{antispherical} \\ \ell\text{-Kazhdan-Lusztig} \\ \text{polynomial}}} \cdot \underbrace{\chi(y \bullet_\ell 0)}_{\text{WCF}}$$

$\Rightarrow$ Simple character formula,

in particular this reproves LCF for $\ell \gg 0$

and one can also compute simple characters for small ℓ.

[About checking LCF in small rank: cf computations of Jensen-Scheinmann + work in progress of Jensen]