# On the theory of near-term quantum advantage

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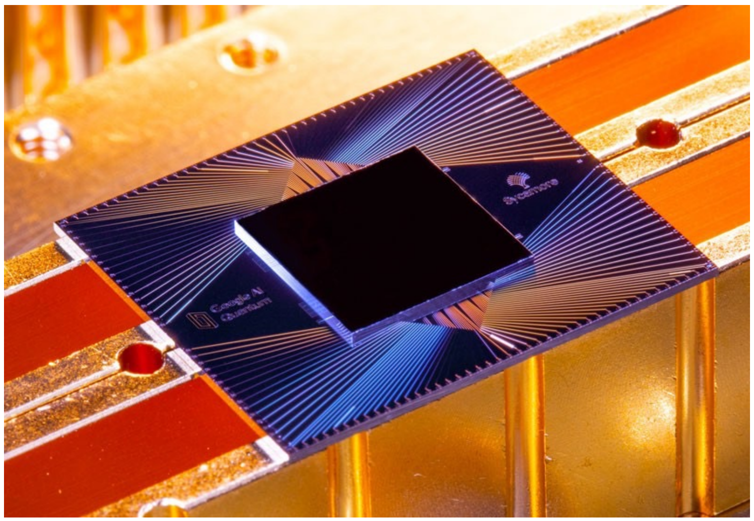
# The first “Quantum advantage” claims have now been made…



Random Circuit Sampling (Google “Sycamore”) in late 2019, USTC in 2021, Google’s second experiment in 2023…

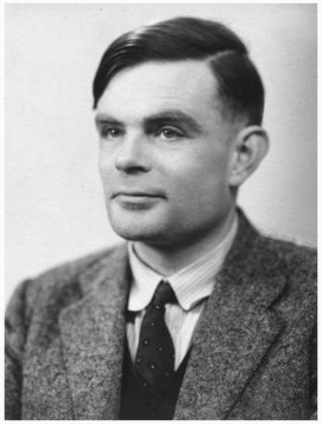
Gaussian BosonSampling – e.g., USTC “Jiuzhang” in late 2020, Xanadu’s “Borealis” in 2022…

**These talks:** the latest complexity theoretic arguments & classical algorithms to understand the power of these “random quantum circuit” experiments

# Importance of experimental quantum advantage: *foundations of computation*

- *Experimental* violation of the Extended Church-Turing thesis
  - i.e., If we want to model efficient computation, we must consider quantum mechanics!
- Complements *theoretical* evidence given by earlier speedups (e.g., [Bernstein-Vazirani '93][Simon'94][Shor '94])

Alonzo Church



Alan Turing

# Importance of experimental quantum advantage: *validating quantum physics*

- Exponential growth one of the most counter-intuitive aspect of quantum mechanics.
  - Is the exponential description of a quantum state really necessary?
- New limit in which to test physics: high complexity.
- *Difficulty*: how to verify something that's exponentially complex?

# What is the *ideal* goal of quantum advantage?

- Find a problem:

1. Can be solved efficiently using a near-term quantum experiment
2. Is classically hard to solve – can't be solved in polynomial time with a classical computer as the system size scales
3. Solution can be efficiently verified with a classical computer with minimal trust in the experiment

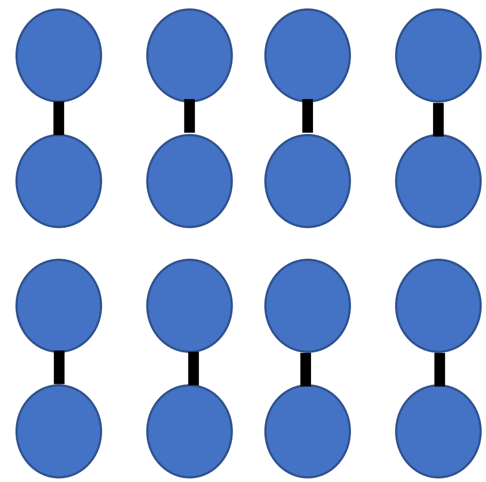
# What is the *current* goal of quantum advantage?

- Current quantum advantage experiments solve "sampling problems" in which the goal is to sample from a complicated distribution
- We have rigorous *evidence* that these problems cannot be solved classically in polynomial time
- But current experiments are ***not*** scalable!
    1. Require exponential time to verify
    2. Uncorrected noise gets worse as system size grows
- So hope is to find a ***"Goldilocks"*** system size:
    - Large enough to be classically challenging to simulate
    - *Not too large!* Otherwise effects of noise overwhelm and the experiment can't be verified
- There is optimism that current experiments have reached this size, **but classical simulation algorithms continually improve, as do quantum experiments.**
- ***Much is still unknown!!!!***

Goldilocks and the three bears

# What is Random Circuit Sampling? [e.g., Boixo et. al. 2017]

- Generate a quantum circuit C on $n$ qubits on a 2D lattice, with $d$ layers of (Haar) random nearest-neighbor gates
  - In practice use a discrete approximation to the Haar random distribution
- Start with $|0^n\rangle$ input state, apply random quantum circuit and measure all qubits in computational basis
  - i.e., Sample from a distribution $D_C$ over $\{0,1\}^n$
- Has now been implemented:
  - n = 53 qubits, d = 20 [Google, 2019]
  - n = 60 qubits, d = 24 [USTC, 2021]
  - n = 70 qubits, d = 24 [Google, 2023]
- **This will be the focus of these talks!**



(single layer of Haar random two qubit gates applied on 2D grid of qubits)

# Boson Sampling [Aaronson & Arkhipov ‘11]

- Prepare $n$ photon $m \geq n^2$-mode “Fock” state
  - i.e., $n$ identical single photons in the first of $m$ modes
- Evolve under a Haar random linear optical unitary composed of beamsplitters and phaseshifters
- Take photon number resolving measurements in each mode
- Recent experiments use similar idea with *Gaussian* input states, rather than Fock states – called “Gaussian BosonSampling”
  - Implemented with 144 modes and as many as 113 detected photons by USTC ’21
  - Implemented with as 216 modes and as many as 219 photons by Xanadu ‘22

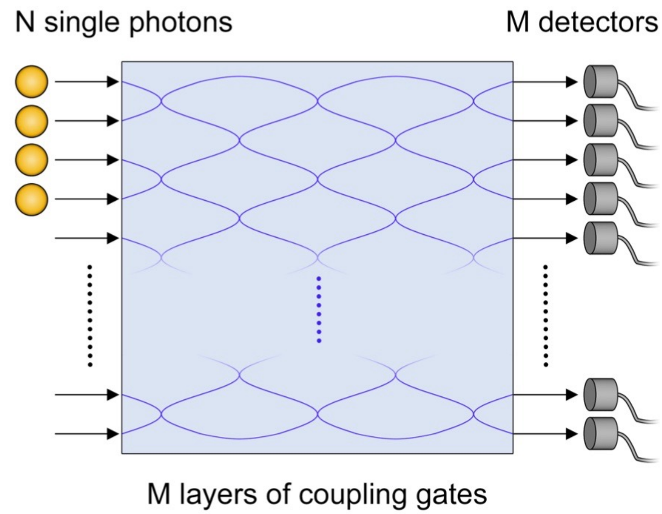



Photo credit: R. Garcia-Patron, J. Renema and V. Shchesnovich

# Agenda

1. **Hardness argument 1** (hardness of quantum sampling)
2. **Hardness argument 2** (hardness of benchmarks)
3. **Easiness argument 1** (classical algorithm for the “XQUATH” benchmark)
4. **Easiness argument 2** (classical algorithms taking advantage of uncorrected noise)

# 2. Hardness argument 1 (hardness of *worst-case* quantum circuit sampling)

# What do we mean by *quantum sampling*?

- Current quantum advantage experiments sample from the output distribution of a quantum circuit
  - i.e., on input $C$ the experiment runs $C|0^n\rangle$ and measures all $n$ qubits in computational basis to get a sample $y \in \{0,1\}^n$
- **Definition:** Let the "output probability" $p_y(C) = |\langle y|C|0^n\rangle|^2$
- **First goal:** prove impossibility of an efficient **classical sampler algorithm** $S$ that samples from the same distribution:
  - for all $C, y$ we have $\Pr_{r}[S(C,r) = y] = p_y(C)$

# **Starting point:** on "classical" vs "quantum" sum

- Consider two problems:
  - "**Classical" sum**: Given classical circuit computing $f: \{0,1\}^n \to \{0,1\}$ compute $\sum_{x \in \{0,1\}^n} f(x)$
  - "**Quantum" sum**: Given classical circuit computing $g: \{0,1\}^n \to \{\pm 1\}$ compute $\sum_{x \in \{0,1\}^n} g(x)$
- Both are $\mathbf{\#P}$-hard to exactly compute, since they are at least as hard as counting the number of satisfying assignments to a Boolean formula

# On *classical approximate* sum

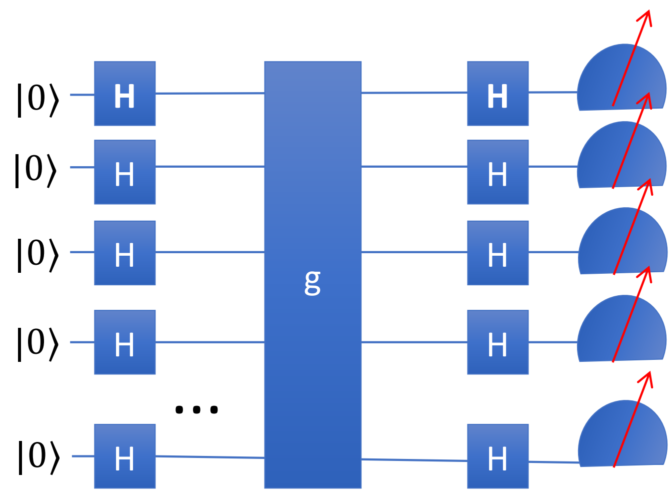
- **Classical "approximate sum":** Given $f: \{0,1\}^n \to \{0,1\}$ output *multiplicative estimate* $\alpha$ so that:
  - $(1-\epsilon)\sum_{x\in\{0,1\}^n} f(x) \le \alpha \le (1+\epsilon)\sum_{x\in\{0,1\}^n} f(x)$
- **Stockmeyer's algorithm: classical approximate sum** can be solved in classical $poly\left(n, \frac{1}{\epsilon}\right)$ time with an **NP** oracle [Stockmeyer'85]
  - In particular, it's *strictly easier* than exact case, unless **PH** collapses
- **Consequence 1:** If a classical sampler $S$ exists, then outputting a *multiplicative estimate* of probability for any outcome $y$ is *strictly easier than* **#P**
  - Because output probability is a classical sum problem!
  - i.e., define f(r)=1 if $S(C,r) = y$ and otherwise 0
  - Then $\Pr_r[S(C,r) = y] = \frac{1}{2^{|r|}}\sum_r f(r)$

# On *quantum approximate sum*

- **Quantum "approximate sum":** Given g: $\{0,1\}^n \to \{\pm 1\}$ output *multiplicative estimate* $\alpha$ so that:
  - $(1-\epsilon)\sum_{x\in\{0,1\}^n} g(x) \leq \alpha \leq (1+\epsilon)\sum_{x\in\{0,1\}^n} g(x)$
- **Claim:** Unlike the classical problem this is as hard as computing $\sum_x g(x)$ exactly!
- **Intuition:** Exponential size cancellations ("interference") make this problem much harder than **classical approximate sum**!
- **Pf sketch:** "binary search and padding"
  - **Claim:** even computing sign($\sum_{\mathrm{x}} \mathrm{g(x)}$) is **#P**-hard (and is a strictly easier problem!)
  1. "Padding": By adding dummy variables can compute $g'$ so that $\sum_{\mathrm{x}'} \mathrm{g}'(\mathrm{x}') = \sum_{\mathrm{x}} \mathrm{g(x)} - \mathrm{k}$
  2. Then compute sign i.e., is $(\sum_{\mathrm{x}'} \mathrm{g}'(\mathrm{x})) > 0$ ?
     - Then we know if $\sum_{\mathrm{x}} \mathrm{g(x)} > \mathrm{k}$
  3. Then binary search on k and repeat!
- **Exercise:** Similar argument proves it's **#P**-hard to estimate $(\sum_{\mathrm{x}} g(x))^2$
  - i.e., can run the same binary search & padding argument on $|\sum_x g(x)|$

# *Consequence 2:* estimating output probabilities of quantum circuits is #P-hard

- **Claim:** given quantum circuit $C$ estimating $p_{0^n}(C)$ is as hard as ***squared quantum approximate sum***.
- **Pf:** By "quantum Fourier sampling"
  - Given $g: \{0,1\}^n \to \{\pm 1\}$ consider the quantum circuit C that:
    - Prepares the state $|g\rangle = \sum_x g(x)|x\rangle$ then takes the Hadamard of each qubit
    - Notice that $p_{0^n}(C) = \left|\langle 0^n|H^{\otimes n}|g\rangle\right|^2 = \frac{(\sum_x g(x))^2}{2^{2n}}$
  - So multiplicative estimation of $p_{0^n}(C)$ is #**P**-hard

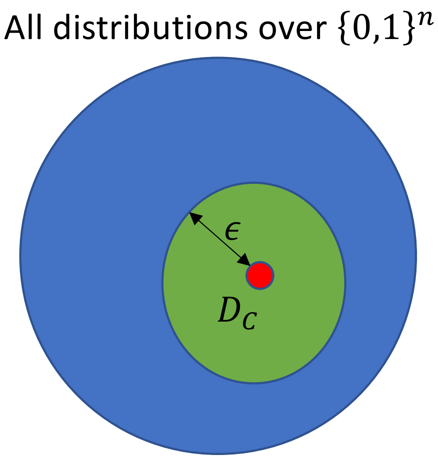


# Impossibility of *exact* sampling

- Assume, for contradiction, there is an efficient sampler $S$:
  - This means for any quantum circuit $C$:
    - $\Pr_{\mathrm{r}}[S(C,r) = y] = |\langle y|C|0^n\rangle|^2 = p_y(C)$
- By consequence 1 we know that estimating the probability $S$ outputs $0^{\mathrm{n}} = p_{0^n}$, is *strictly easier* than **#P** unless **PH** collapses
- But by consequence 2 know that estimating $p_{0^n}(C)$ is **#P**-hard, since it is as hard as ***squared quantum approximate sum***
- This is a contradiction! So there can't be such a sampler algorithm.
- Similar arguments appear in [Terhal-DiVincenzo '04, Bremner-Jozsa-Shepherd '11, Aaronson-Arkhipov '11…]

# This result is not *robust*

- The impossibility result has two major weaknesses:
  1. **Exactness assumption:** It requires that the classical algorithm samples *exactly* from the output distribution of each quantum circuit
  2. **Worst-case assumption:** It requires that the classical algorithm works *for all* quantum circuits
- **Major goal in the theory of quantum advantage:** prove impossibility of *approximate average-case* sampler
  - i.e., efficient classical algorithm $S(C, r)$ that samples from any distribution $|X - D_C|_{TV} \leq \epsilon$ whp over C
- **Note:** constant approximation in TVD is not intended to model *physical noise* but rather *classical imprecision*!

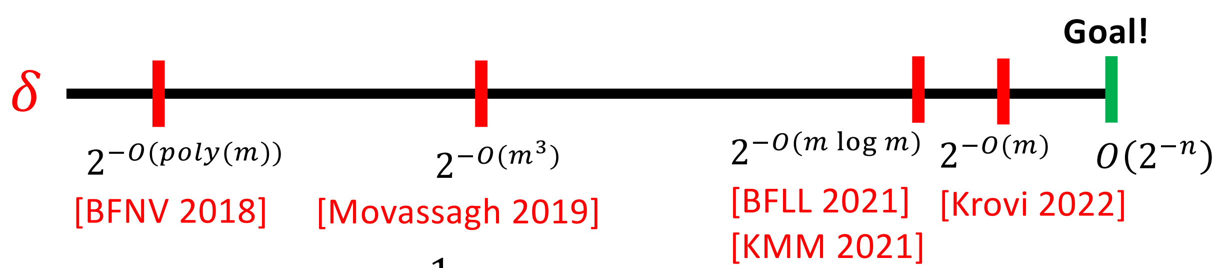


# Proving hardness of *approximate* sampling

- **Central problem** of study: $\delta$-**random circuit estimation**:

> Given as input quantum circuit C, output q so that $|q - p_{0^n}(C)| \leq \delta$ with probability 2/3 over C

- To prove hardness of *average-case approximate* sampling suffices to prove $\boldsymbol{\delta = O(2^{-n})}$ **random circuit estimation** is **#P**-hard [Stockmeyer '85][Aaronson Arkhipov '11]
- **Known hardness results** with respect to C on $n$ qubits, size $m = O(n \cdot d)$




- **Boson Sampling**: goal is $\frac{1}{e^{n \log n}}$, whereas we have hardness at $\frac{1}{e^{6n \log n}}$ [BFLL'21]

# **Inspiration:** average-case hardness of Permanent [Lipton '91]

- **Permanent** of $n \times n$ matrix is **#P**-hard in the worst-case [Valiant '79]
  - $Per[X] = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}$
- *Algebraic property*: $Per[X]$ is a degree $n$ polynomial with $n^2$ variables
- Need compute $Per[X]$ of worst-case matrix $X$
  - But we only have access to algorithm $O$ that correctly computes *most* permanents over $\mathbb{F}_p$
  - i.e., $\Pr_{Y \in_R \mathbb{F}_p^{n \times n}}\left[O(Y) = Per[Y]\right] \geq 1 - \frac{1}{poly(n)}$
- Choose $n + 1$ fixed non-zero points $t_1, t_2 \ldots, t_{n+1} \in \mathbb{F}_p$ and uniformly random matrix $R$
- Consider line $A(t) = X + tR$
  - *Observation 1 "scrambling property"*: for each $i$, $A(t_i)$ is a random matrix over $\mathbb{F}_p^{n \times n}$
  - *Observation 2*: "univariate polynomial": $Per[A(t)]$ is a degree $n$ polynomial in $t$
- But now these $n + 1$ points uniquely define the polynomial, so use polynomial extrapolation to evaluate $Per[A(0)] = Per[X]$

# [BFNV'18]: Hardness for Random Quantum Circuits

- *Algebraic property*: much like $Per[X]$, output probability of random quantum circuits has polynomial structure
  - Consider circuit $C = C_m C_{m-1} \dots C_1$
  - Polynomial structure comes from path integral:
    - $\langle 0^n | C | 0^n \rangle = \sum_{y_2, y_3, \dots, y_m \in \{0,1\}^n} \langle 0^n | C_m | y_m \rangle \langle y_m | C_{m-1} | y_{m-1} \rangle \dots \langle y_2 | C_1 | 0^n \rangle$
- This is a polynomial of degree $m$ in the gate entries of the circuit
- So the output probability $p_{0^n}(C)$ is a polynomial of degree $2m$

# First attempt at adapting Lipton's proof

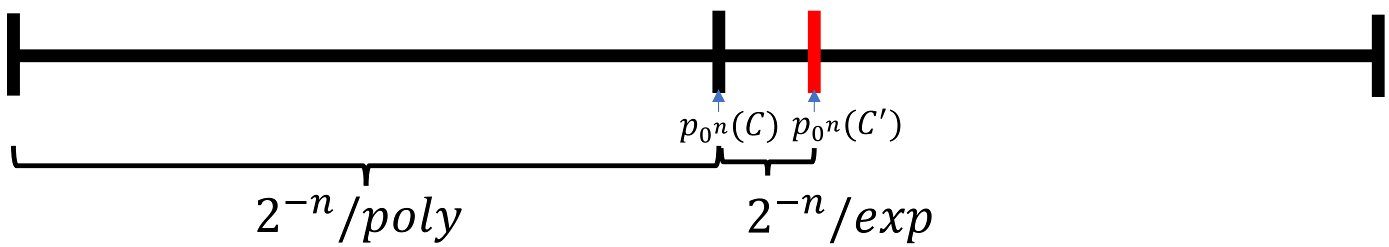
- Fix $m$ Haar random two qubit gates $\{H_i\}_{i\in[m]}$
- ***Main idea***: Implement tiny fraction of $H_i^{-1}$
  - i.e., $C_i' = C_i H_i e^{-ih_i\theta}$
  - This scrambles C if $\theta \approx small$, since each gate is close to Haar random
  - However, if $\theta = 1$ the corresponding circuit $C' = C$
- ***Strategy (in style of Lipton):*** take several non-zero but small $\theta$, compute output probabilities of "random but correlated" circuits ${C'}_{\theta_1}, {C'}_{\theta_2} \dots, C'_{\theta_{2m}}$ and apply polynomial extrapolation, evaluate at $\theta = 1$ to retrieve $p_{0^n}(C)$

# This is not quite the “right way” to scramble!

- ***Problem***: $e^{-ih_i\theta}$ is not polynomial in $\theta$
- ***Solution:*** take fixed truncation of Taylor series for $e^{-ih_i\theta}$
  - i.e., each gate of $C'_\theta$ is $C_i H_i \sum_{k=0}^{K} \frac{(-ih_i\theta)^k}{k!}$
  - So each gate entry is a polynomial in $\theta$ and so is $p_{0^n}(C'_\theta)$
  - Now extrapolate and compute $p(1) = p_{0^n}(C)$

# How to motivate the truncations?

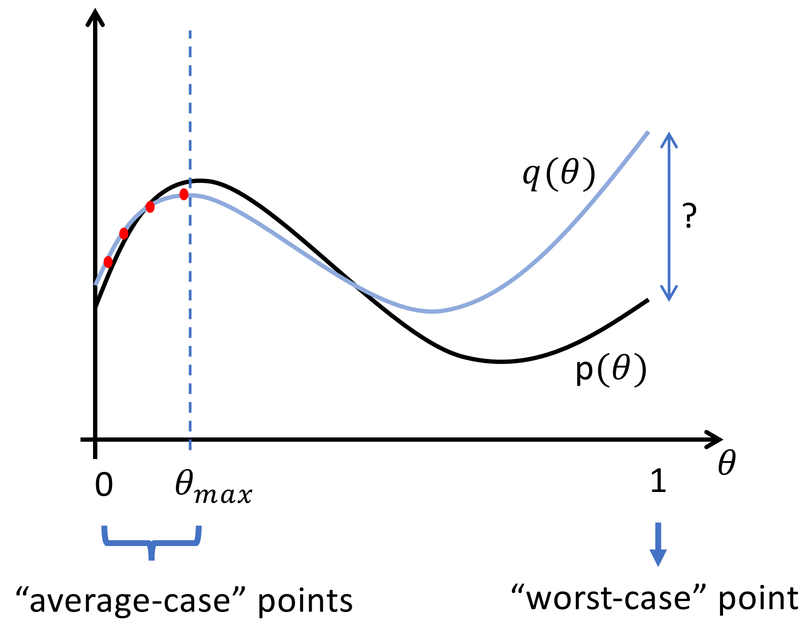
- Main technical result in [BFNV'18]: ***Estimating*** $p_{0^n}(C')$ is hard **iff** ***estimating*** $p_{0^n}(C)$ is hard
  - **Intuitively, because the "truncation error" is so much smaller than the size of the additive error we are conjecturing is hard.**




- More recently, [Movassagh'19'20] has shown a related argument (using the so called "Cayley path") that eliminates the need for these truncations

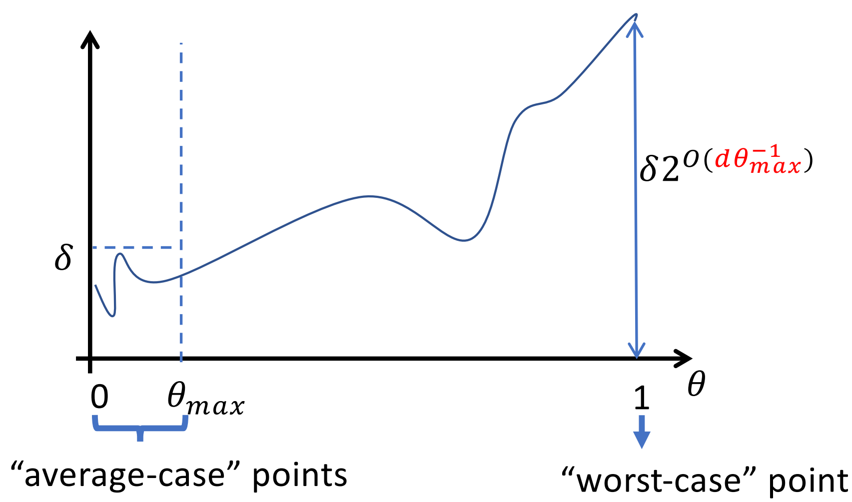
# On robustness to *imprecision*

- So far we assumed the ability to compute the output probabilities of random circuits $\{p_{0^n}(C'_{\theta_i})\}$ *exactly*
- **Actual setting:** Given $2m$ evaluation points $\{(\theta_i, y_i)\}$ so that for most $i$, $\left|y_i - p_{0^n}(C'_{\theta_i})\right| \leq \delta$
- We have two polynomials:
  - The "ideal" $p(\theta_i) = p_{0^n}\left(C'_{\theta_i}\right)$
  - The extrapolated polynomial $q(\theta_i) = y_i$
- **Our question:** How close is $q(1)$ to $p(1) = p_{0^n}(C)$ in terms of $\delta, \theta_{max}$ ?



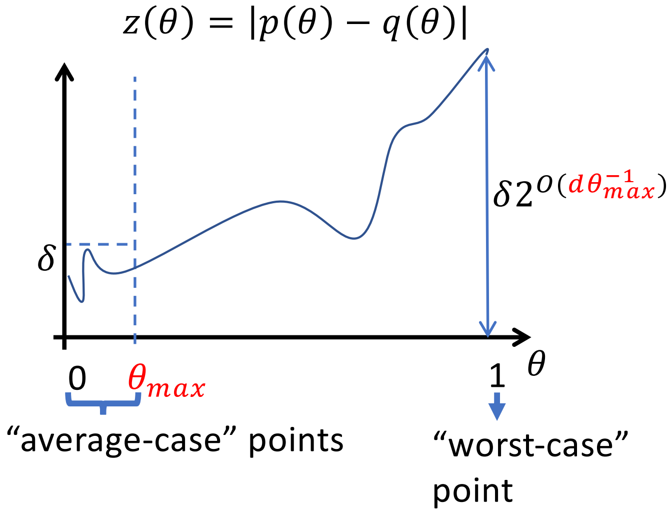
# The “Paturi picture”

- [Paturi ’92] If we have a degree $d$ polynomial $z(\theta)$ bounded on an interval $[0, \theta_{max}]$ by $\delta$ then $|z(1)| \le \delta 2^{O(d\theta_{max}^{-1})}$
- **Our case:** Consider the degree $2m$ polynomial $z(\theta) = |p(\theta) - q(\theta)|$



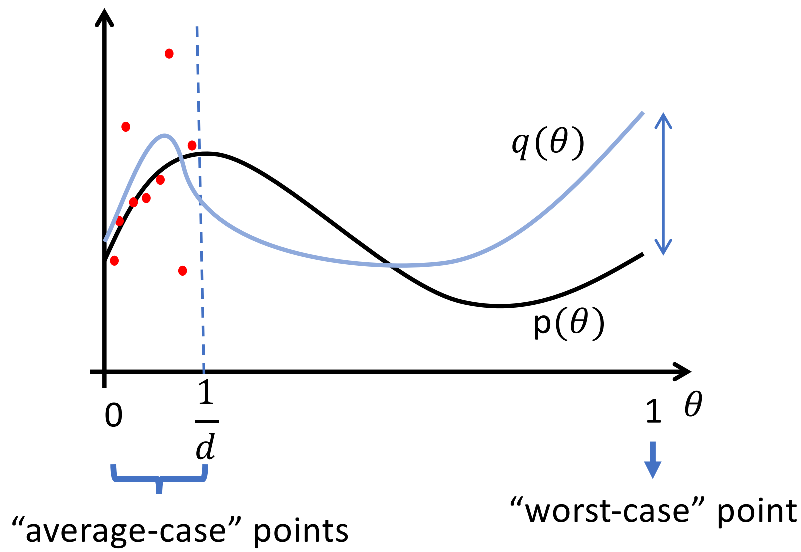
# How large can we take $\theta_{max}$?

- Lagrange extrapolation requires getting all $d = 2m$ points correct
  - So we need the algorithm to succeed wp $\geq 1 - O\left(\frac{1}{m}\right)$
- As $\theta$ gets larger $C'_\theta$ is further away from random circuit
  - i.e., Distribution of $C'_\theta$ is $O(m\theta)$-close in TVD from Haar random circuit
  - So algorithm works wp $1 - O(m\theta)$ on these points
- So need $\theta_{max} \leq \frac{1}{O(m^2)}$
- Plugging in Paturi's bound: $z(1) \leq \delta 2^{O\left(d\theta_{max}^{-1}\right)} = \delta 2^{O(m^3)}$
- So need $\delta = \frac{1}{2^{O(m^3)}}$



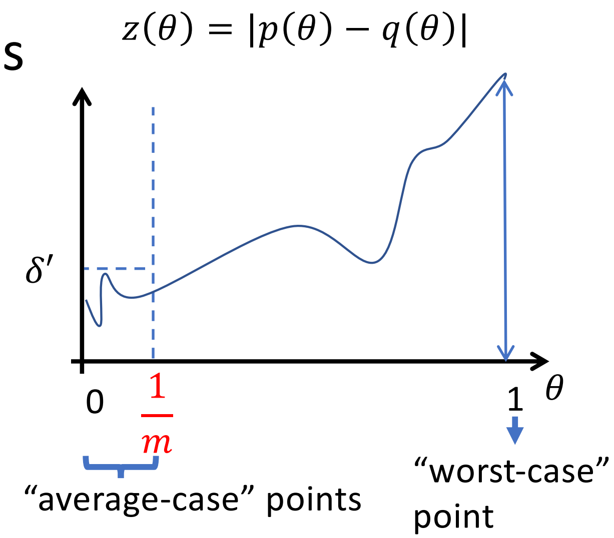
# Increasing robustness [BFLL'21] (see also [Kondo et. al.'21])

- To improve imprecision we need a new, error-robust means of polynomial extrapolation
- Will do this by oversampling – i.e., taking many more points than degree
- **"Robust Berlekamp-Welch" Thm.** Given $O(d^2)$ "faulty" evaluation points $\{(\theta_i, y_i)\}$ to $p(\theta)$ of degree $d$ so that:
    1. $\theta_i \in \left[0, \frac{1}{d}\right]$
    2. We know ***at least*** *2/3* of $y_i$ are $\delta$***-close*** to $p(\theta_i)$
- Then any polynomial $q(\theta)$ which is $\delta$***-close*** on $2/3$ fraction of the points is $\delta 2^{O(d)}$-close to $p(\theta)$ ***for all*** $\theta \in \left[0, \frac{1}{d}\right]$



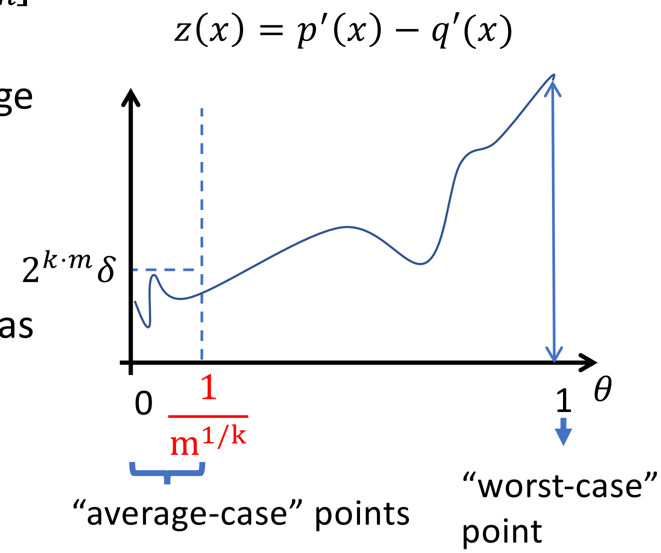
# How large can we take $\theta_{max}$ now?

- **Input:** faulty points to polynomial $p(\theta)$: $(\theta_1, y_1), (\theta_2, y_2) \dots (\theta_{O(m^2)}, y_{O(m^2)})$
- Ask **NP** oracle to give us a polynomial $q(\theta)$ that is $\delta$-close to 2/3 of these points
  - This can easily be checked by evaluating $q$ at each $\theta_i$
- Robust Berlekamp-Welch theorem tells us:
  - $|p(\theta) - q(\theta)| \leq \delta' = \delta 2^{O(m)}$ **for all** $\theta \in \left[0, \frac{1}{m}\right]$
- Then Paturi tells us:
  - $|p(1) - q(1)| = |z(1)| \leq \delta' 2^{O\left(d\theta_{max}^{-1}\right)} = \delta 2^{O(m^2)}$
  - So we need to take $\delta \sim \frac{1}{2^{O(m^2)}}$



# Getting to robustness $2^{-O(m \log m)}$

- Given faulty points $(\theta_1, y_1), (\theta_2, y_2) \dots (\theta_{O(m^2)}, y_{O(m^2)})$ with $\theta_i \in \left[0, \frac{1}{m}\right]$
- Trick! Rather than asking the **NP** oracle for the approximating polynomial q of degree $m$, replace the variable $\theta$ with $\theta^k$ for some large $k$ and ask for this new poly q′
  - This rescaling *increases* the degree to $km$!
  - But it "stretches" unit interval near 0 and "compresses" near 1
  - So for fixed value of $\theta_{max} = \frac{1}{m}$ the corresponding value of $\theta_{max}$ has increased, it's now $\frac{1}{m^{1/k}}$
- Plugging in Paturi's bound: $z(1) \leq \delta' 2^{O(km \cdot m^{1/k})}$
- Setting $k = \log(m)$ we have $z(1) \leq \delta 2^{O(m \cdot \log(m))}$
- So we need to set $\delta \sim 2^{-O(m \cdot log(m))}$



# Comments & Open Directions

- Main open question in the theory of quantum advantage: ***improve*** the ***additive imprecision*** of these average-case hardness results to $O(2^{-n})$ from $2^{-O(m)}$ for RCS or $\frac{1}{e^{n \log n}}$ from $\frac{1}{e^{6n \log n}}$ for Boson Sampling
- Current hardness results have improved dramatically but we've also discovered ***barriers*** implying that new techniques will be needed to improve them further (e.g., [AA'2011][Napp et. al. '19][BFLL'21])

# 3. Hardness argument 2 (hardness of benchmarks)

# Limitations of total variation distance

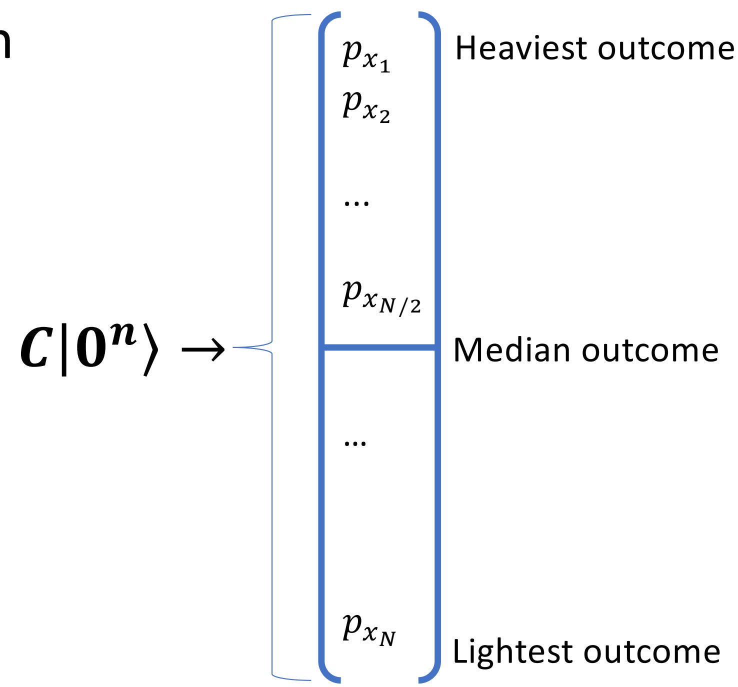
- Total variation distance is difficult to measure!
  - There are well-known exponential lower bounds for sample complexity, even for "merely" testing closeness to the uniform distribution e.g., [Valiant & Valiant'17]
- Closeness in total variation distance is not a reasonable model of uncorrected physical noise
  - i.e., system size increases, having TVD remain a small constant isn't realistic without error mitigation
- Is there a "quantum signal" that is easier to verify and implement?

# Candidates for verifiable "quantum signals"

- Many candidates rely on the "Porter-Thomas property" of random quantum circuits
  - Each output probability is exponentially distributed
  - i.e., $\Pr_C\left[|\langle x|C|0^n\rangle|^2 = \frac{q}{2^n}\right] \sim e^{-q}$
  - True for Haar random unitaries
  - Conjectured to be true even for shallow depth random circuits
- This Porter-Thomas property implies that the output distribution of a random but fixed circuit is somewhat "flat" but not uniform whp
- **Observation:** Easy to sample from the output distribution with a quantum computer and observe many "heavy" outcomes – how difficult is this to do classically?

# Heavy Output Generation [Aaronson & Chen ’17]

- **Definition:** With respect to a circuit $C$ call an outcome $x \in \{0,1\}^n$ *heavy* if $p_x$ is greater than *median* in the output distribution of $C$
- **HOG**: Given random circuit $C$ output strings $x_1, x_2, \ldots, x_k$ so that at least 2/3 are *heavy*
- **Claim:** Quantumly can solve **HOG** simply by repeatedly running $C|0^n\rangle$ and measuring
  - Why? Because whp over $C$, the sum of probabilities that are above median in output distribution is $\geq 0.7$
    - Using Porter-Thomas property!
  - Then use Chernoff bound to prove 2/3 of outputs are heavy whp



# Quantum Threshold Assumption (QUATH)

- **HOG** still seems like a sampling task – why should this be hard classically?
- [Aaronson and Chen'17]: **HOG** is classically hard assuming **QUATH**
- **QUATH:** No efficient classical algorithm takes input random $C$ with $m \gg n$ gates and decides if $p_{0^n}$ is heavy with probability $\frac{1}{2} + \Omega\left(\frac{1}{2^n}\right)$
  - Where probability is over both $C$ and internal randomness of classical algorithm
- **Motivation:** QUATH seems closer to problems we understand, since it involves estimation of $p_{0^n}$
- Key point is that the bias scales exponentially in $n$ rather than size $m$
  - Not hard to show classical algorithm with a bias that scales exponentially in $m$
  - e.g., randomly guessing a small number of Feynman paths and comparing to a threshold

# QUATH implies HOG is hard

- **Pf. (Intuition):** By contrapositive assume there's an algorithm for HOG. We want to solve QUATH.
  - On input $C$ use HOG algorithm to output list of mostly heavy strings in output distribution of $C$
  - Output "heavy" if $0^n$ is on the list.
- **Pf. (More formal analysis):**
  - Easier to consider a uniform outcome $z \in \{0,1\}^n$ rather than the $0^n$ outcome
    - But it doesn't matter by a property of random circuits called "hiding"
    - i.e., Let $C'$ be the circuit chosen by taking $C$ and appending Pauli X gates to each $i$-th qubit if $z_i = 1$
    - Notice that new circuit, $C'$, has property that $p_{0^n} = |\langle 0^n|C|0^n\rangle|^2 = |\langle z|C'|0^n\rangle|^2$ and $C'$ is still random circuit
  - Strategy is same as the intuition: use **HOG** algorithm on $C'$ to output list $z_1, \ldots, z_k$ so that $2/3$ of $z_i$ are heavy, then choose uniform element of list, call it $z_{i^*}$
    - **If $\boldsymbol{z = z_{i^*}}$ output "heavy"**
    - **If $\boldsymbol{z \neq z_{i^*}}$ output "heavy" wp ½, "light" wp ½**
  - The probability this algorithm is correct on heaviness of $p_{0^n}(C)$ is at least:
    - $\Pr[z_{i^*} = z] \cdot \frac{2}{3} + \Pr[z_{i^*} \neq z] \cdot \frac{1}{2} = \frac{1}{2^n} \cdot \frac{2}{3} + (1 - 2^{-n}) \cdot \frac{1}{2} = \frac{1}{2} + \Omega\left(\frac{1}{2^n}\right)$

# Linear Cross-Entropy (XEB) [Boixo et. al. '16] [Arute et. al. '19]

- An alternative measure of heaviness is **XEB**:
  - $\mathbf{XEB}(p_{exp}, p_{ideal}) = 2^n \sum_x \left(p_{exp}(x) p_{ideal}(x)\right) = 2^n E_{x \sim p_{exp}(x)}[p_{ideal}(x)]$
  - If $p_{exp} = p_{ideal}$ then $\mathbf{XEB}(p_{exp}, p_{ideal}) = 2$ but $\mathbf{XEB}(U, p_{ideal}) = 1$
- *XEB can be **well-approximated in few device samples** via concentration of measure arguments, but **requires exponential time to compute** ideal output probabilities of observed samples*
  - *i.e., observe experimental outcomes $z_1, \ldots, z_k$ and compute* $\frac{2^n \sum_i p_{ideal}(z_i)}{k}$

# Why is scoring well on XEB classically hard? [Aaronson & Gunn '19]

- **XHOG** ("Linear Cross Entropy Heavy Output Generation")
  - Given $C$, output $k$ distinct samples $z_1, z_2, \dots, z_k$ so that $E_i[|\langle z_i|C|0^n\rangle|^2] \geq \frac{\boldsymbol{b}}{2^n}$
  - Where $b = 1 + \epsilon$
- By repeatedly running a *noiseless* circuit we'd be able to achieve $\boldsymbol{b} = 2$
- *Noise* can cause the experiment to have considerably different values for $\boldsymbol{b}$
  - E.g., Google scores $b = 1.002$ on its 53 qubit RCS experiment
- Still seems like a sampling task – why should this be hard classically?

# The XQUATH assumption [Aaronson & Gunn '19]

- **XHOG** is hard assuming **XQUATH**
- **XQUATH**: No efficient classical algorithm, given random $C$, produces estimate, $p$, to $p_{0^n} = |\langle 0^n|C|0^n\rangle|^2$ so that:
  - $2^{2n}\left(E_C\left[\left(p_{0^n} - \frac{1}{2^n}\right)^2\right] - E_C[(p_{0^n} - p)^2]\right) = \Omega(2^{-n})$
- i.e., No classical algorithm can achieve a **mean squared error** at estimating an output probability of a random circuit, that's slightly better than the trivial algorithm that always outputs $2^{-n}$
- **XQUATH** implies **XHOG** is hard by very similar reduction!
  - i.e., assume there's an **XHOG** algorithm that outputs samples $z_1, z_2, \ldots, z_k$ so that $E_i\left[p_{z_i}\right] = \frac{b}{2^n}$ then output $\frac{b}{2^n}$ if $0^n$ is on the list and else output $\frac{1}{2^n}$

# Comments & Open Directions

- This is a very "lossy" reduction! Even scoring well (e.g., constant $b > 1$) on **XHOG** gives rise to $\exp(-n)$ bias for **XQUATH**. Can this be improved?
- Under certain assumptions about the noise, the **XEB** score well-approximates the fidelity of the noisy experiment. Hence it can be useful for benchmarking (see e.g., [Boixo et. al. '17] and our work [Liu et. al. '21] for more details).

4. Easiness argument 1 (XQUATH is false at sublinear depth) [Gao et. al. ‘21][Aharonov et. al. ‘22]

# Revisiting the intuition for XQUATH

- Recall **XQUATH**: No efficient classical algorithm, given random $C$, produces estimate, $p$, to $p_{0^n} = |\langle 0^n|C|0^n\rangle|^2$ so that:
  - XScore = $2^{2\mathrm{n}}\left(E_C\left[\left(p_{0^n} - \frac{1}{2^n}\right)^2\right] - E_C[(p_{0^n} - p)^2]\right) = \Omega(2^{-n})$
- Intuition is that the best classical algorithm for estimating $p_0$ for a random circuit $C = C_m C_{m-1} \dots C_1$ is to sample the path integral in the computational basis:
  - $p_{0^n} = \left(\sum_{y_2,y_3,\dots,y_m \in \{0,1\}^n} \langle 0^n|C_m|y_m\rangle\langle y_m|C_{m-1}|y_{m-1}\rangle \dots \langle y_2|C_1|0^n\rangle\right)^2$
  - There are $\exp(n \cdot d)$ paths with uniform value, so it's unclear how to achieve an advantage that scales as $2^{-n}$
- **Observation:** Turns out this isn't true! If we consider the path integral in the Pauli basis the values of the paths are highly non-uniform!

# Pauli path integrals

- Rather than thinking of quantum circuit as applying unitary gates to vectors, think about it as applying *unitary channels* to *density matrices*
- Denote the normalized Pauli operators $P_n = \left\{\frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}}\right\}^{\otimes n}$
- Can write an $n$-qubit density matrix $\rho = \sum_{t \in P_n} \alpha_t \cdot t$ with $\alpha_t = Tr[t\rho]$
- Recall in the "computational basis" path integral we express:
  - $\langle x|U|\psi\rangle = \sum_{y \in \{0,1\}^n} \langle x|U|y\rangle\langle y|\psi\rangle$
- Analogously, in Pauli basis $Tr\left[sU\rho U^\dagger\right] = \sum_{t \in P_n} Tr\left[sUtU^\dagger\right] Tr[t\rho]$
  - We call $Tr\left[sUtU^\dagger\right]$ the "transition amplitude"

# Expressing $p_x$ as a Pauli path integral

- Now we can express any output probability as a Pauli path integral, in analogy to what we are accustomed to in the computational basis
- Let $C = C_d C_{d-1} \dots C_1$ where each layer $C_i$ acts on $n$ qubits
- $p_x = |\langle x|C|0^n\rangle|^2$
- $= \sum_{s \in P_n^{d+1}} \mathrm{Tr}(|x\rangle\langle x|s_d)\mathrm{Tr}(s_d C_d s_{d-1} C_d^\dagger) \dots \mathrm{Tr}(\mathrm{s}_1 C_1 s_0 C_1^\dagger)\mathrm{Tr}(\mathrm{s}_0|0^\mathrm{n}\rangle\langle 0^n|)$
- $= \sum_{s \in P_n^{d+1}} f(C, s, x)$ *(we define $f(C, s, x)$ as the "value" of path $s$)*

# Two important facts

- The **XQUATH** algorithm relies on two facts which both follow from elementary properties of Haar random gates
- **Fact 1** (e.g., [HL'09]) Let $U$ be a Haar random 2 qubit gate and $p, q \in P_2$,
  - Then $E_U\left[\mathrm{Tr}\left[pUqU^\dagger\right]^2\right] = \begin{cases} 1,\ if\ p = q = \frac{I^{\otimes 2}}{2} \\ 0, if\ p = \frac{I^{\otimes 2}}{2}\ and\ q \neq \frac{I^{\otimes 2}}{2} \\ 0, if p \neq \frac{I^{\otimes 2}}{2}\ and\ q = \frac{I^{\otimes 2}}{2} \\ \frac{1}{15}, otherwise \end{cases}$
- **Fact 2** ("orthogonality of Pauli paths") Let $C$ be a random circuit (with Haar gates) and $s \neq s' \in P_n^{d+1}$ be any two different paths and any $x \in \{0,1\}^n$
  - Then $E_C[f(C,s,x)f(C,s',x)] = 0$
  - **Corollary:** for any path $s \neq I_n^{\otimes d+1}$, $E_c[f(C,s,x)] = 0$
    - *since the* $I_n^{\otimes d+1}$ *path has value* $\frac{1}{2^n}$ *so* $E_C\left[f(C,s,x)f\left(C, I_n^{\otimes d+1}, x\right)\right] = \frac{1}{2^n}E[f(C,s,x)] = 0$

# XQUATH algorithm (part 1)

- **Claim:** Given a random circuit $C$ outputting $p = \frac{1}{2^n} + f(C, s^*, 0^n)$ achieves $XScore$ of $\left(\frac{1}{15}\right)^d$ where $s^* = \left(\frac{1}{\sqrt{2^n}} Z \otimes I^{\otimes n-1}\right)^{\otimes d+1}$
- **Recall:** $XScore = 2^{2n}\left(E_C\left[\left(p_{0^n} - \frac{1}{2^n}\right)^2\right] - E_C[(p_{0^n} - p)^2]\right)$
- **Proof:** $XScore = 2^{2n} E_C\left[\frac{1}{2^{2n}} - \frac{2}{2^n} p_{0^n} - p^2 + 2p \cdot p_{0^n}\right]$ *(by algebra)*
  - $= 2^{2n} E_C[-\frac{1}{2^{2n}} - p^2 + 2p \cdot p_{0^n}]$ *(using that $E_C[p_{0^n}] = \frac{1}{2^n}$)*
  - $= 2^{2n} E_C[-\frac{2}{2^{2n}} - f(C, s^*, 0^n)^2 + 2p \cdot p_{0^n}]$ *(by def. of $p$ & by cor. Fact 2 cross terms = 0)*
  - $= 2^{2n} E_C[-\frac{2}{2^{2n}} - f(C, s^*, 0^n)^2 + \frac{2p_{0^n}}{2^n} + 2f(C, s^*, 0^n) p_{0^n}]$ *(by def. of $p$)*
  - $= 2^{2n} E_C[-f(C, s^*, 0^n)^2 + 2f(C, s^*, 0^n)^2]$ *(using that $E_C[p_{0^n}] = \frac{1}{2^n}$ & orthogonality)*
  - $= 2^{2n} E_C[f(C, s^*, 0^n)^2]$ *(by algebra)*

# XQUATH algorithm (part 2)

- *Recall* $C = C_d C_{d-1} \dots C_1$ and the path $s^* = \left(\frac{1}{\sqrt{2^n}} Z \otimes I^{\otimes n-1}\right)^{\otimes d+1}$
  - where each layer $C_i$ consists of two qubit gates $C_i^{(1)}, C_i^{(2)}, \dots, C_i^{(n/2)}$
- So far we have: $XScore = 2^{2n} E_C[f(C, s^*, 0^n)^2]$
  - $= 2^{2n} \mathrm{E_C}\left[\mathrm{Tr}(|x\rangle\langle x| s_d^*)^2 \cdot \mathrm{Tr}\left(s_d^* C_d s_{d-1}^* C_d^\dagger\right)^2 \cdot \dots \cdot \mathrm{Tr}\left(s_1^* C_1 s_0^* C_1^\dagger\right)^2 \cdot \mathrm{Tr}(s_0^* |0^n\rangle\langle 0^n|)^2\right]$
    - First and the last terms are $\left(\frac{1}{\sqrt{2^n}}\right)^2$ which cancels the $2^{2n}$ term in front
  - $=$ Product of $d$ squared transition amplitudes each of the form:
    - $= E_{C_i}\left[Tr\left[\left(Z \otimes I^{\otimes n-1}\right) C_i \left(Z \otimes I^{\otimes n-1}\right) C_i^\dagger\right]^2\right]$ (using that each $s_j^* = (Z \otimes I^{\otimes n-1})$)
    - $= E_{C_i^{(1)}}\left[Tr\left[(Z \otimes I) C_i^{(1)} (Z \otimes I) C_i^{(1)\dagger}\right]^2\right] \cdot E_{C_i^{(2)}}\left[Tr\left[(I \otimes I) C_i^{(2)} (I \otimes I) C_i^{(2)\dagger}\right]^2\right] \cdot \dots$
      - Grouping the *two qubit* gates that act on each pair of qubits together and $Tr[A \otimes B] = Tr[A] \cdot Tr[B]$
- **By Fact 1**, all of these expectations except the first are 1, the first is $\frac{1}{15}$
  - So the total score is $\sim \frac{1}{15^d}$

# Consequences of XQUATH algorithm

- Notice that the classical algorithm simply computes value of single path in the Pauli basis (takes time $O(n \cdot d)$)
- Algorithm achieves $\mathrm{X}S\mathrm{core}$ of $\frac{1}{2^{O(d)}}$
- If circuit depth is sublinear, then this is a higher score than $\frac{1}{2^n}$ contradicting **XQUATH**!

# Comments & Open Directions

- A similar algorithm achieves a score of $2^{-O(d)}$ on **XEB** but this algorithm is not yet practical i.e., it doesn't spoof current experiments – can we improve this?
- How hard is achieving a sufficiently large constant score on **XEB** for random quantum circuits with super-constant depth? Recall this is what a *noiseless* random quantum circuit achieves by sampling!
- There's an alternative spoofing method due to [Pan-Chen-Zhang '21], which uses a clever tensor contraction method to simulate Google's 53 qubit **XEB** score on supercomputer in a reasonably short amount of time but takes considerably longer for the USTC 60 qubit experiment

5. **Easiness argument 2** (classical algorithms taking advantage of uncorrected noise)

# Uncorrected noise *defines* the NISQ era

- Without error-correction noise eventually overwhelms
  - e.g., Google's RCS experiment ~0.2% signal and 99.8% noise
- Can uncorrected noise help us to classical simulate near-term quantum experiments?
- That is, consider fixing a noise model and for RCS a first reasonable choice is *depolarizing* noise
  - e.g., Each layer of random gates is followed by layer of single qubit depolarizing noise channel with ***constant noise*** strength $\gamma$:
  - $\mathcal{E}(\rho) = (1-\gamma)\rho + \frac{\gamma I}{2} Tr[\rho]$
    - Note that $\mathcal{E}$(I)=I but $\mathcal{E}(P) = (1-\gamma)P$ for $P \in \{X, Y, Z\}$
- Note: having only depolarizing noise is a simplification!

# Quantifying the effects of uncorrected noise

- Intuitively, uncorrected **depolarizing** noise increases entropy. As our circuit gets deeper the output distribution converges to uniform
- **Main question:** how quickly does this happen?
- We've known since the late 90's that the **noisy quantum circuit distribution** with depth $d$ and the **uniform distribution** are $\leq 2^{-\gamma d}$ close in TVD [Aharonov et. al. '96]
- This rules out scalable noisy quantum advantage at *super-logarithmic depth*
- What about random circuits? Could the convergence be faster?
  - Numerical evidence that convergence to uniform happens faster [Boixo et. al. '17]
  - i.e., TVD upper bounded by $\leq 2^{-\gamma \cdot d \cdot n}$ whp over $C$
    - This would rule out scalable noisy quantum advantage *at any depth*!

# How much depth is required for quantum advantage?

- ***Anticoncentration*** is one ingredient of current hardness of sampling arguments that requires sufficiently deep random circuits (with Haar random gates)
- A distribution over circuits **anticoncentrates** if:
  - There exists constants $\alpha \in (0,1], c > 0$ so that $\Pr_C \left[ p_{0^n}(C) \geq \frac{\alpha}{2^n} \right] \geq c$
  - Notice this is *not sufficient* for hardness e.g., the uniform distribution anticoncentrates!
  - Rather it's a *sanity check* that $\pm O(2^{-n})$ additive estimates to $p_{0^n}$ aren't trivial!
- Until recently, we only knew **anticoncentration** for 2D circuits (with Haar random gates) happened at depth $\geq \sqrt{n}$ [Harrow & Mehraban '18]
- This is too deep for scalable noisy quantum advantage!
  - i.e., we know that the output distributions are $\leq 2^{-\gamma d} \sim 2^{-\sqrt{n}}$ close to uniform

# Is there any hope for *fully scalable*, noisy quantum advantage from RCS?

- Consequently until last year, there was little optimism that we could get such an advantage
  - Rather we hope for "Goldilocks" system sizes to keep the system from getting too noisy
- Then two results rekindled some hope at $\log(n)$ depth…
  1. Anticoncentration at $\log(n)$ depth [Barak et. al. '21][Dalzell et. al. '22]
  2. TVD between noisy random circuit distribution and uniform is *lower bounded* by $2^{-O(d)}$ whp [Deshpande et. al. '22]
     - Matches the Aharonov et. al. '96 upper bound and rules out faster convergence rates

Goldilocks and the three bears

# Can a classical algorithm beat uniform sampling at depth $\log n$?

- For $\mathrm{d} = \mathrm{O}(\log(n))$ depth noisy circuits we know that the uniform distribution is $2^{-O(d)} = \frac{1}{\mathrm{n}^{\mathrm{c}}}$ close in TVD to the output distribution by [Aharonov et. al. '96] upper bound
- But it was possible that quantum advantage persists for sampling from a distribution $\frac{1}{n^{c\prime}}$ -close in TVD to the noisy output distribution for some sufficiently large constant $c' > \mathrm{c}$
- This possibility has recently been ruled out by very recent work of [Aharonov et. al. '22]

# The [Aharonov, Gao, Landau, Liu, Vazirani'22] algorithm

- [Aharonov et. al. '22] give a classical algorithm for sampling from a distribution $\epsilon - close$ to the distribution of noisy random quantum circuits in $poly\left(n, \frac{1}{\epsilon}\right)$ time modulo several caveats
- This hides a factor of $n^{1/\gamma}$ with noise-rate $\gamma$, **which keeps the algorithm from being competitive with near-term experiments**
- Also algorithm **requires anticoncentration, so is only efficient and useful (i.e., beats uniform sampling) at $\mathbf{log}(\boldsymbol{n})$ depth**
- Finally, algorithm requires certain constraints on the gate set (satisfied e.g., by Haar random gates)

# Main ideas of [Aharonov et. al. ’22]

- **Key observation** [Gao & Duan’18][Aharonov et. al. ‘22]: Output probabilities (and marginals) of noisy random quantum circuits in Pauli basis have most mass on a small number of paths, rest of the paths are exponentially suppressed
- **Recall notation:** in Pauli basis $p_x(C) = \sum_{s \in P_n^{d+1}} f(C, s, x)$
- Then by definition of depolarizing noise, the noisy output probability: $\tilde{p}_x = \sum_{s \in P_n^{d+1}} (1-\gamma)^{|s|} f(C, s, x)$
  - Where $|s|$ is the Hamming weight, or number of non-Identity Paulis in path
- **Main idea:** To compute $p_x$ simply throw away high-weight Pauli terms and exactly compute the low weight terms!
- i.e., for appropriate cutoff, $\ell$, compute $\overline{q_x} = \sum_{s:|s| \leq \ell} (1-\gamma)^{|s|} f(C, s, x)$

# Analysis of the [Aharonov et. al. '22] algorithm

- Recall the algorithm works by truncating the Pauli path integral of each noisy output probability, then computing each truncated probability *path by path*
- Analysis in two steps:
  1. Upper bound the TVD, $|\tilde{p} - \bar{q}|_1$ as a function of the truncation parameter $\ell$
  2. Upper bound the running time of the algorithm as a function of $\ell$

# Step 1: How to set cutoff $\boldsymbol{\ell}$ to bound TVD

- Goal is to obtain upper bound on $|\tilde{p} - \bar{q}|_1 = \Delta$
- $E_C[\Delta^2] \leq 2^n E_C\left[\sum_{x\in\{0,1\}^n} (\tilde{p}_x - \bar{q}_x)^2\right]$ *(by Cauchy-Schwarz)*
- $= 2^n E_C\left[\sum_x\left(\sum_{s:|s|>\ell}(1-\gamma)^{|s|} f(C,s,x)\right)^2\right]$ *(by definition of $\tilde{p}_x$ and $\bar{q}_x$)*
- $= 2^n E_C\left[\sum_x \sum_{s:|s|>\ell}(1-\gamma)^{2|s|} f(C,s,x)^2\right]$ *(orthog. of Pauli paths, Fact 2)*
- $= \sum_{k>\ell}(1-\gamma)^{2k} W_k$ *(rewriting, where $W_k$ is "Fourier weight")*
- $\leq (1-\gamma)^{2\ell} \sum_{k>\ell} W_k$ *(since $k > \ell$)*
- $\leq e^{-2\gamma\ell} \cdot O(1)$ *(nontrivial upper bound on $W_k$ follows from anticoncentration)*

- So can take $\ell \approx \frac{1}{\gamma} \cdot \log\left(\frac{1}{\epsilon}\right)$ to obtain $\Delta \leq \epsilon$ with high probability by Markov

# Step 2: How to compute truncated prob., $\bar{q}_x$?

- Algorithm works by computing value of each path in truncated probability
- How many terms in $\bar{q}_x = \sum_{s:|s|\leq\ell}(1-\gamma)^{|s|} f(C,s,x)$ ?
- Number of paths with Hamming weight at most $\ell$ is $\leq \ell \cdot \binom{n(d+1)}{\ell} \cdot 3^\ell$
  - Since each path has $n(d+1)$ Pauli operators and we're choosing $\ell$ to be non-identity & there are $3^\ell$ different sequences of operators $\{X,Y,Z\}^\ell$
  - Takes $O(n \cdot d)$ time to compute each path
- Total time dominated by # of paths $\sim (n \cdot d)^{O(\ell)} \sim n^{\frac{1}{\gamma}\log\left(\frac{1}{\epsilon}\right)}$ if $\ell = \frac{1}{\gamma}\log\left(\frac{1}{\epsilon}\right)$
- Can improve dependence to $2^{O(\ell)}$ by being be more clever – uses anticoncentration and the fact that many paths contribute 0 to the path integral.
  - Notice by choice of $\ell$ that this is exponential in $\frac{1}{\gamma}$ as well

# Comments & Open Directions

- This algorithm applies to constant noise rates. For $\gamma = \tilde{O}\left(\frac{1}{n}\right)$ there's evidence for hardness of sampling [Dalzell et. al. '21]
- This algorithm doesn't spoof near-term RCS experiments due to scaling of runtime with noise rate – can we improve this dependence?
- Can we generalize the Aharonov et. al. algorithm to other noise models besides depolarizing?
  - Our very recent work suggests this result is quite sensitive to *unital noise* (Ghosh et. al., arXiv: 2306.16659)! Real world experiments have both unital and non-unital noise channels!
- Can we generalize the Aharonov et. al. algorithm to gate sets that are very far from Haar random?
  - E.g., See our work with [Haferkamp et. al. '19] for a candidate architecture that anticoncentrates at constant depth…
- How hard are noisy random circuits with *sublogarithmic depth* and Haar random gates?
  - Not covered by this algorithm because of anticoncentration is known to fail here [Dalzell et. al. '21][Deshpande et. al. '22]!
- **Most generally, is fully scalable quantum advantage possible without error mitigation, for any experiment?**

# More work I hope you check out!

- Random circuits with non-unital noise do not anticoncentrate at any depth
  - Our work: Ghosh et. al., arXiv: 2306.16659
- Hardness of *Gaussian* Boson Sampling experiments: e.g.,
  - Our work on this [Deshpande et. al. '21, arXiv: 2102.12474]
  - "Bipartite GBS" [Grier et. al.'21, arXiv: 2110.06964]
- Verifying and spoofing current Boson Sampling experiments
  - Efficiently distinguishing Boson Sampling distribution from uniform [Aaronson Arkhipov '13, arXiv:1309.7460]
  - Our very recent work classically simulates the largest current size Gaussian Boson Sampling [Oh et. al. '23, arXiv:2306.03709]
    - Tensor network that takes advantage of photon loss!
- Useful applications of quantum advantage experiments? e.g.,
  - Molecular vibronic spectra problem via Boson Sampling
    - See original proposal of [J.Huh et. al., arXiv: 1412.8427]
    - See our quantum inspired classical algorithm for this problem, as well as alternative quantum chemistry problems that still might be classically hard [Oh et. al., arXiv: 2202.01861]
  - Certified random number generation from Random Circuit Sampling
    - see proposal of Aaronson and Hung (e.g., arXiv: 2303.01625)
    - our work providing evidence for this proposal [Bassirian et. al. '22, arXiv: 2111.14846])

# Thanks!