Spanning Trees of Cayley Graphs

Diana Yang

IAS Park City Math Institute (PCMI)

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Mathematical Translation : Groups to Graphs

Definition

For a group G and a subset $S \subset G$, the Cayley Graph $\Gamma_{G,S}$ is a graph where $V(\Gamma_{G,S}) = G$. Furthermore, a directed edge $\{u, v\} \in E(\Gamma_{G,S})$ $\iff uv^{-1} \in S$

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Immediate Consequences :

- If $S = S^{-1}$ then $\Gamma_{G,S}$ is undirected.
- $\Gamma_{G,S}$ is connected $\iff \langle S \rangle \cong G$.

General Setting

Let G be finite group.

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- require that G is **ambivalent**. This means every element is conjugate to its inverse, i.e, $i(g) = i(g^{-1})$
- Examples : S_n, Q_{4n} (n even), D_n, A_n for $n \in \{1, 2, 5, 6, 10, 14\}$

Building the Graph

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- Big Idea : "Apply Matrix Tree Thm to determine $\sum_T w(T)$ to count number of spanning trees"

Constructing the Laplacian

Let $\Delta = \sum_{g \in G - \{e\}} x_{i(g)} = \sum_{i \in I - \{1\}} |C_i| \cdot x_i$

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How does it compare to our usual definition of a Laplacian?

Finding Eigenvalues of Laplacian

Proposition

Let χ be an irreducible character of group G. Then :

$$\Delta - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi(i)}{\dim(\chi)}$$

is an eigenvalue of the Laplacian M of multiplicity. $\dim(\chi)^2$

Detour into Representation Theory

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Let's consider an example:

(Left) Regular Representation

It comprises of the group ring : $V = \mathbb{C}[G]$ and the map $\pi : G \to GL(V)$ where $\pi(g)$ acts by left multiplication.
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Therefore if we consider the matrix $\pi(g)$ at the (h, k)-entry :

$$\pi(g)_{h,k} = \begin{cases} 1 & g = hk^{-1} \\ 0 & otherwise \end{cases}$$

This also means $\chi(e) = dim(V)$ and $\chi(g) = 0, \forall g \in G - \{e\}$

Important Theorems

Cor. of Maschke's Thm

Every representation of a finite group G over $\mathbb{C}$ can be expressed as a direct sum of irreducible representations.

$$V = \bigoplus_{j \in J} V_j^{\oplus^{d_j}}$$

Important Theorems

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Schur's Lemma

Given two irreps of G, (ρ_V, V) and (ρ_W, W). For any intertwining operator $\psi \in Hom_G(V, W)$ [i.e. $\psi \circ \rho_V = \rho_W \circ \psi$]

$$\psi = \begin{cases} 0 & (\rho_V, V) \ncong (\rho_W, W) \\ \alpha\psi_o & (\rho_V, V) \cong (\rho_W, W) \end{cases}$$

Where ψ_o is an isomorphism.

Proof of Proposition

Returning to the Laplacian we've previously defined :

$$m_{h,k} = \begin{cases} -x_{i(hk^{-1})} & h \neq k \\ \Delta & h = k \end{cases}$$

We can rewrite it as $M = \Delta \cdot \pi(e) - \sum_{g \in G - \{e\}} x_i(g) \cdot \pi(g)$,

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Furthermore we can block decompose M,

$$M_j = \Delta \cdot \pi_j(e) - \sum_{g \in G - \{e\}} x_{i(g)} \cdot \pi_j(g) = \Delta \cdot Id_{d_j} - \sum_{i \in I - \{1\}} \sum_{g \in C_i} \pi_j(g)$$

Proof of Proposition

$$\pi_j(h) \cdot \sum_{g \in C_i} \pi_j(g) = \pi_j(h) \cdot \sum_{g \in C_i} \pi_j(h^{-1}gh) = \sum_{g \in C_i} \pi_j(g) \cdot \pi_j(h)$$

So we see that M_j is an intertwining operator from $(\pi_j, V_j) \to (\pi_j, V_j)$

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So we see that M_j is an intertwining operator from $(\pi_j, V_j) \to (\pi_j, V_j)$

By Schur's lemma, $M_j = \alpha_j \cdot Id_{dim(V_j)}$

$$tr(M_j) = \alpha_j \cdot dim(V_j)$$

$$\alpha_j = \frac{tr(M_j)}{dim(V_j)}$$

$$\alpha_j = \Delta - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi_j(i)}{dim(\chi_j)}$$

With multiplicity $dim(\chi_j)^2$

Applications

For a connected graph Γ, its Laplacian has eigenvalue 0 of multiplicity 1.

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$$\Delta - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi_{triv}(i)}{dim(\chi_{triv})} = \Delta - \sum_{i \in I - \{1\}} x_i \cdot |C_i| = 0$$

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Thus the number of spanning trees of $\Gamma_{G,S}$ is given by

$$\frac{1}{|G|} \cdot \prod_{\chi \in Irrep(G) - \{\chi_{triv}\}} \left(\Delta - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi(i)}{dim(\chi)} \right)^{dim(\chi)^2}$$

$$\text{Where } x_i = \begin{cases} 1 & C_i \subset S \\ 0 & else \end{cases}$$

Source

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The number of spanning trees of the Bruhat graph

Richard Ehrenborg

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, United States of America

Onto D_n

Here are the irreducible representations of D_n

1 dimensional Irreps

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1

n odd

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ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

n even

2 dimensional Irreps

$$\rho^h(r^k) = \begin{pmatrix} w^{hk} & 0 \\ 0 & w^{-hk} \end{pmatrix}, \qquad \rho^h(sr^k) = \begin{pmatrix} 0 & w^{-hk} \\ w^{hk} & 0 \end{pmatrix}.$$

for $0 < h < \frac{n}{2}$, $w = e^{\frac{2\pi i}{n}}$.

$\chi_h(r^k) = 2cos(\frac{2\pi hk}{n})$ and $\chi_h(sr^k) = 0$

Onto D_n

Conjugacy Classes for D_n

For n odd:

$$\{1\}, \{r, r^{-1}\}, \ldots, \{r^{\frac{n-1}{2}}, r^{\frac{1-n}{2}}\}, \{s, sr^1, \ldots, sr^{n-1}\}$$

For n even:

$$\{1\}, \{r, r^{-1}\}, \ldots, \{r^{\frac{n}{2}}\}, \{s, sr^2, \ldots, sr^{n-2}\}, \{sr^1, sr^3, \ldots, sr^{n-1}\}$$

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Since we want $\Gamma_{G,S}$ connected, we need S to generate D_n : it has to have the conjugacy class(es) of "flips"

Observation

If $S = \{s, sr^1, ..., sr^{n-1}\}, \Gamma_{D_n,S}$ is complete bipartite graph $K_{n,n}$

Sanity Check : Cayley's Tree Formula

Since $S = \{s, sr^1, ..., sr^{n-1}\}$, $\Delta = n$

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We will first start with n odd :

1-D Irrep : $\lambda_2 = n - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi_2(i)}{dim(\chi_2)} = n - (-n) = 2n$

2-D Irrep : $\lambda_h = n - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi_h(i)}{dim(\chi_h)} = n - 0 = n$

There are $\frac{n-1}{2}$ 2-D Irreps, and each eigenvalue shows up $2^2 = 4$ times.

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For n even, there are $\frac{n}{2} - 1$ 2-D Irreps, however 2 more 1-D Irreps.

$$\lambda_3 = n - \left(\frac{n}{2} - \frac{n}{2}\right) = n \text{ and } \lambda_4 = n - \left(-\frac{n}{2} + \frac{n}{2}\right) = n$$

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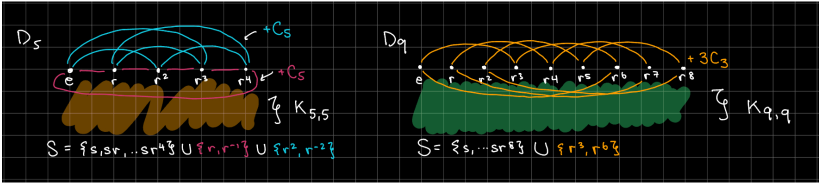
Thus in either case, we get n^{n-2} spanning trees. (Very Nice :) !)

What if we add rotations into S?

Let's see some examples:



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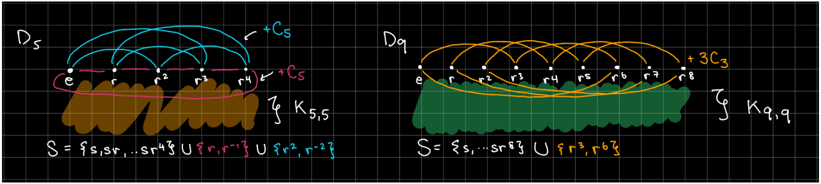
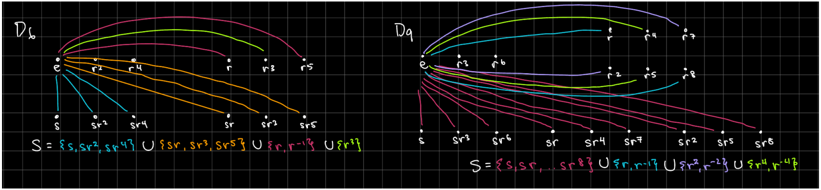


addition of cycles

It gets more interesting :



Turan Graphs

What if we add rotations into S?

Observation

We can count the number of spanning trees for $T_{2k}(2n)$ whenever $k|n$ using this method.

These will be the partitions : Letting $d = \frac{n}{k}$
$\{e, r^k, ..., r^{k(d-1)}\}, \{r, r^{k+1}, ..., r^{k(d-1)+1}\}, ...\{r^{k-1}, r^{2k-1}, ...r^{dk-1}\}$,
$\{$s, sr$^k, ..., sr^{k(d-1)}\}, ..., \{sr^{k-1}, sr^{2k-1}, ...sr^{dk-1}\}$
Where $S = D_n - \{e, r^k, ..., r^{k(d-1)}\}$

Getting Rid of the Pesky Factor of 2

Disjoint Union of Graphs

Consider $G \sqcup H$

Then, its Laplacian $L_{G \sqcup H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}$

Thus its eigenvalues are $\{\lambda_j^G\}_j \cup \{\lambda_k^H\}_k$

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Consider $\Gamma_{D_n,S}$ where $sr^i \notin S, \forall i \in [n]$ By picking the "correct" rotations to stick in S, we get two disjoint Turan graphs.

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Thus by picking the correct S, we can express the number of spanning trees of

$$T_k(n) = \frac{1}{n} \cdot \prod_{\chi \in Irrep(D_n) - \{\chi_{triv}, \chi_2\}} \left(\Delta - \sum_{i \in I - \{1\}} \frac{x_i \cdot |C_i| \cdot \chi(i)}{dim(\chi)} \right)^{\frac{dim(\chi)^2}{2}}$$

$$\text{Where } x_i = \begin{cases} 1 & C_i \subset S \\ 0 & else \end{cases}$$

Example with D_6

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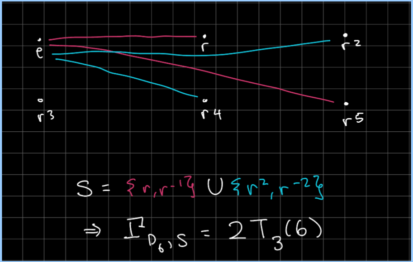

Observation

Thus to get $T_k(n)$ from D_n, we let $S = \{r^i\} - \{r^{ki}\}$ where $i \in [n]$

The End / Thank You

Questions?

Other Semidirect Products?

Consider $C_7 \rtimes C_3$

- $\alpha : C_3 \to Aut(C_7) \cong C_6$
- $\phi : C_7 \to C_7$, $\phi(1) = 4$ (order 3 automorphism)
- $(a, b) * (c, d) = (a + 4^b c, b + d)$
- $C_7 \rtimes C_3 \cong \langle a, b | a^7, b^3, a^4 b = ba \rangle = \{b^i a^j | i \in [3], j \in [7]\}$
- However fails to be ambivalent

Conjugacy Class of a

$$(b^i a^j) a (a^{-j} b^{-i}) = b^i a b^{-i} = a^{4i} b^{i-i} = a^{4i}$$

Conjugacy class for a is $\{a, a^4, a^2\}$. Notably, a^{-1} is not an element.

Generalized Quaternion Group (Order $4n$)

$$Q_{4n} = \frac{C_{2n} \rtimes C_4}{\langle (n,2) \rangle} = \langle a, b | a^{2n}, b^4, bab^{-1} = a^{-1}, b^2 = a^n, \rangle$$

Conjugacy Classes of Q_{4n}

Representative	e	a	a^2	$\cdots$	a^{n-1}	a^n	b	ab
Size	1	2	2	$\cdots$	2	1	n	n

There are $n+3$ conjugacy classes.

Representations of Q_{4n}, n even

4 1-D Irreps

Representation	a^k $(0 \le k \le 2n-1)$	ba^k $(0 \le k \le 2n-1)$
ψ_1	1	1
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Generalized Quaternion Groups

Representations of Q_{4n}, n even

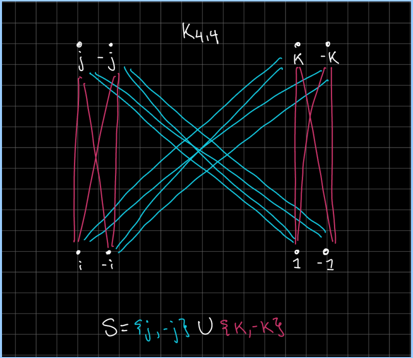
2-D Irreps, $w = e^{\frac{\pi i}{n}}$

Representation	a^k	ba^k
ϕ_j $(1 \leq j \leq n-1$, j odd$)$	$\begin{pmatrix} w^{kj} & 0 \\ 0 & w^{-kj} \end{pmatrix}$	$\begin{pmatrix} 0 & w^{-kj} \\ w^{-kj} & 0 \end{pmatrix}$
η_h $(1 \leq h \leq \frac{n-2}{2})$	$\begin{pmatrix} w^{2kh} & 0 \\ 0 & w^{-2kh} \end{pmatrix}$	$\begin{pmatrix} 0 & w^{-2kh} \\ w^{-2kh} & 0 \end{pmatrix}$

Cayley Graphs for Generalized Quaternions

Let $S = [b] \cup [ab]$

Example on Q_8



Complete Bipartite Graph

Cayley Graphs for Generalized Quaternions

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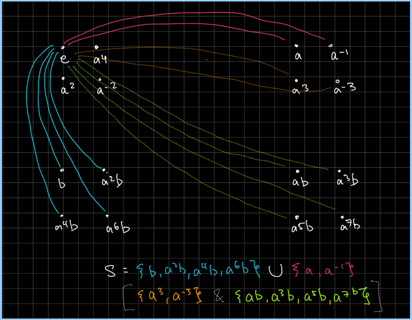
Example on Q_{16}



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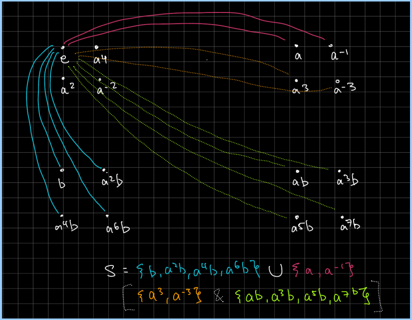

$Q_{4n}/Z(Q_{4n}) \cong D_n$, probably won't get anything interesting.