Joint Spectra, Self-similarity, and Julia Sets

N. Goncharuk, M. Putinar, Z. Sunik, R. Yang, T. Zheng

(Activity Report)

Contents

1. Program Proposal	1
2. Summary of Activities	2
3. Summary of Questions, Ideas, and Findings	3
3.1. Julia Sets and Semi-Conjugacy	3
3.2. Self-Similarity	5
3.3. Actions at the Finite Level	5
3.4. The Grigorchuk Group	7
3.5. Numerical Range	8
3.6. Hyponormal Operators	9
4. Other Topics	11
References	11

1. Program Proposal

This IAS summer research project intends to explore a newly emerged connection between three distinct areas of mathematics: spectral theory, representation theory, and holomorphic dynamics. Despite their differences, these fields converge in a surprising way through self-similar group representations. Joint spectra of noncommuting linear operators and Julia sets of holomorphic maps are traditionally difficult to compute due to their complex nature. However, in the past two decades, several breakthroughs have shown that self-similar group representations naturally link these two subjects, revealing new pathways for the computation of the two types of sets. This project aims to leverage these links to develop new computational techniques and theoretical insights. To achieve these goals, we are bringing together some of the world's leading experts in related fields to foster collaboration and stimulate groundbreaking research in this interdisciplinary domain.

Self-similar group representations. The first example of groups of intermediate growth, the Grigorchuk group $\mathcal{G}$, was constructed by using four length-preserving actions, say a, b, c, and d, on the set of dyatic subintervals in $[0, 1)$. These actions induce a canonical unitary representation of $\pi : \mathcal{G} \to U(L^2[0, 1))$. An interesting new phenomenon was observed: there exists a unitary operator $W : L^2[0, 1) \to L^2[0, 1) \oplus L^2[0, 1)$ such that all entries of the 2×2 block matrices $W\pi(x)W^*, x \in \mathcal{G}$, are from $\pi(\mathcal{G})$. This phenomenon is referred to as the *self-similarity* of π. It was discovered later that many other groups have such representations, such as the dihedral group D_∞, the basilica group, the lamplighter group, and some monodromy groups of analytic functions. The self-similarity provides new machinery for studying the spectral properties of these groups.

Projective spectrum. For decades, spectral theory has been a cornerstone in operator theory, index theory, differential equations, and mathematical physics. However, a proper definition for *noncommuting* operators remained elusive until 2009. Given elements $A_0, ..., A_n$ in a Banach algebra $\mathcal{B}$, their *projective spectrum* is defined as

$$p(A) := \{z \in \mathbb{P}^n \mid A(z) := z_0A_0 + \cdots + z_nA_n \text{ is not invertible}\}.$$

Indeed, linear pencils of operators are subjects or study for many decades [3, 8]. This simple and straightforward definition of joint spectrum has inspired an extensive study by many scholars, with results expanding into a wide range of mathematical disciplines.

Renormalization maps. Consider the linear pencil $A_\pi(z) := z_0 I + z_1\pi(a) + z_2\pi(b) + z_3\pi(c) + z_4\pi(d)$ associated with the group $\mathcal{G}$ and the representation π mentioned above. Then the projective spectrum $p(A_\pi)$ is the collection of $z \in \mathbb{P}^4$ such that $A_\pi(z)$ is not invertible. The self-similarity of π leads to the polynomial map

$$F_{\mathcal{G}}(z) = [z_0\alpha - z_1^2(z_0 + z_4) : z_1^2(z_2 + z_3) : z_4\alpha : z_2\alpha : z_3\alpha], \quad z \in \mathbb{P}^4, \tag{1}$$

where $\alpha(z) := (z_0 + z_4)^2 - (z_2 + z_3)^2$. Such self maps in the projective space, referred to as the *renormalization maps*, can be derived for many other groups with self-similar representations. Remarkably, in the case of the dihedral group D_∞, it is shown that the Julia set of F_{D_∞} coincide with the projective spectrum, giving rise to a rare example of multivariable Julia set that can be described by algebraic equations.

Proposed problems. The participants will study the following topics while gathering at IAS.

1. The dynamical properties of the renormalization maps of $F_{\mathcal{G}}$ and some monodromy groups.

2. The projective spectra of these groups. Specifically, can the spectra be described? How are the spectra related to the Julia sets of the above maps?

3. Study whether the spectra or the Julia sets reflect the growth of the group.

2. Summary of Activities

Team meetings took place daily from 9am to 12pm and then continued with free discussions after lunch. Free discussions also extended into the weekend (July 26 & 27). The team discussed several directions of research related to spectral properties of self-similar groups and related pencils of operators.

1. The team suggested the notion of the spectrum for a tuple of non-commuting operators. This new notion is strongly related to the notion of projective spectrum used before, and is known to be related to the classical notion of the numerical range. Projected results include: computations of the spectrum for actions of self-similar groups; general treatment of the spectral measure that is supported on the spectrum; development of suitable functional calculus; application of these general results to self-similar groups, their spectra and spectral measures.

2. It is already known from the work of Bartholdi and Grigorchuk that the spectrum of the infinite-dimensional operator is equal to the closure of the union of spectra of the finite-dimensional representations on levels when the operator is self-adjoint and the Screier graph of the action is amenable. What we added during and after the meeting are the following three observations.

(a) Even in the non-self-adjoint case, as long as the Schreier graph is amenable, the closure of the union of the spectra on the levels is part of the spectrum of the infinte-dimensional operator.

(b) Moreover, under the same conditions (amenability), the closure of the union is part of the essential spectrum.

(c) The last remarks implies that, in the self-adjoint case, the entire spectrum is essential, that is, the discrete part of the spectrum is empty.

We expect to obtain more general results, or a general methods that apply to many self-similar groups, particularly in the non-selfadjoint case.

3. Schur complement relates the spectrum of the group to the dynamics of a rational function. The team suggests to develop methods to handle this type of rational functions. This includes:

(a) using algebraic methods to find an algebraic reduction via semi-conjugacy to a one-dimensional map, if it exists.

(b) developing dynamical methods to find a continuous reduction to a one-dimensional map, if algebraic reduction does not exist. The group obtained partial progress for the Tangled Odometers

Group, and expects many new results for similar groups. Potentially, the methods will allow us to handle the full pencil $A_\pi(z) := z_0 I + z_1\pi(a) + z_2\pi(b) + z_3\pi(c) + z_4\pi(d)$ for the Grigorchuk group $\mathcal{G}$. Group $\mathcal{G}$ is a key subject in the theory of self-similar groups, and new results on its spectrum are of high importance.

(c) Studying relations between this rational function and gap labeling for discrete Schrodinger operators, in cases when Schreier graphs are linear and hence spectral problems are related to the spectra of Schrodinger operators. In particular, this holds for group $\mathcal{G}$. New methods here can lead to new results for spectra of one-dimensional Schrodinger operators.

(d) Studying geometry of the Julia sets of these rational functions, their relations to the spectra, and their behavior under semi-conjugacy. Partial progress was made towards general result on the Julia sets of the renormalization maps.

4. The team identified several examples of self-similar groups that can serve as models for general results mentioned above. This includes the Grigorchuk group $\mathcal{G}$ and the Grigorchuk-Erschler group; Hanoi towers group; Lamplighter group; Tangled Odometers group; and several iterated mondromy groups.

5. The team discussed the following open question: is the general definition of self-similar presentation of the group (via isometry between the Hilbert space and its powers) equivalent to its definition via actions on infinite trees? Proof or counterexamples to this conjecture are necessary for the development of the theory and can lead to discovery of new types of self-similar groups and their actions.

6. The team studied the projective spectra $p(T)$ of the pencil $T(z) = z_0 I + z_1 T + z_2 T^*$, where T is a hyponormal operator, and showed that

$$p(T) = \bigcup_{\lambda \in \sigma(T)} \{z \in \mathbb{P}^2 : z_0 + z_1\lambda + z_2\overline{\lambda} = 0\}.$$

3. Summary of Questions, Ideas, and Findings

3.1. **Julia Sets and Semi-Conjugacy.** Given a domain $\Omega \subset \mathbb{C}^n$, a sequence of holomorphic function $\{f_k\}$ on Ω is said to *converge normally* to f if the convergence is uniform on every compact subset of Ω. Cauchy integral formula then implies that f is also holomorphic. A sequence of vector-valued holomorphic maps $\{F_k : \Omega \to \mathbb{C}^m\}$ is said to converge normally if each component converges normally.

Definition 3.1. A family of holomorphic maps $\{F_\lambda : \Omega \to \mathbb{C}^m \mid \lambda \in \Lambda\}$ is called a *normal family* if every sequence in the family contains a normally convergent subsequence.

For a holomorphic map $H = (H_1, ..., H_n) : \Omega \to \Omega$, we shall use the notation H^k to denote the kth iteration of H. For instance, the notation $H^2(z)$ stands for $H(H(z))$. Complex dynamics studies various issues concerning the convergence of the iteration sequence $\{H^k\}_{1=1}^\infty$. The interest is primarily on the case $\Omega = \mathbb{C}^n$ with H being a polynomial or rational map.

Definition 3.2. Given a non-constant rational map $H : \hat{\mathbb{C}}^n \to \hat{\mathbb{C}}^n$, its *Fatou set* $\mathcal{F}(H)$ is the maximal open subset of $\hat{\mathbb{C}}^n$ on which the sequence $\{H^k\}_{k=1}^\infty$ is a normal family. The *Julia set* $\mathcal{J}(H)$ is the complement $\hat{\mathbb{C}}^n \setminus \mathcal{F}(H)$.

Evidently, the Julia set is closed, and both $\mathcal{J}(H)$ and $\mathcal{F}(H)$ are invariant under the map H.

In the case a map $H = (H_0, ..., H_n) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is homogeneous polynomials of degree $d \geq 2$, it induces a map $H' : \mathbb{P}^n \to \mathbb{P}^n$ through the commuting diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} & \xrightarrow{H} & \mathbb{C}^{n+1} \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{P}^n & \xrightarrow{H'} & \mathbb{P}^n, \end{array} \tag{2}$$

where ϕ is the canonical projection. Conversely, if $H' : \mathbb{P}^n \to \mathbb{P}^n$ is a holomorphic map, then there exists a homogeneous polynomial map $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that the above diagram commutes. We should be careful that the map H' is not well-defined at the common zeros of $H_0, ..., H_n$ because $\mathbb{P}^n$ contains no origin. Thus the set $\phi(H^{-1}(0))$ consists of singular points for H'. If there are only a finite number of such singular points, the map H' is called a *meromorphic map*. For $k \geq 1$, we set $I_k(H') = \phi\big(H^{-k}(0)\big)$ and call it the *indeterminacy set* of H'^k. Clearly, $I_k(H') \subset I_{k+1}(H')$ for each k. We set $I_\infty(H') = \cup_{k=1}^{\infty} I_n(H')$. For convenience, in the sequel we shall write H' as H, and respectively $I_k(H')$ by I_k and $I_\infty(H')$ by I_∞.

Definition 3.3. Given a meromorphic map $H : \mathbb{P}^n \to \mathbb{P}^n$, its extended indeterminacy set is defined as $E_H = \overline{I_\infty}$.

It is often rather technical to determine E_H, and this fact can add many (and often unnecessary) complications to the study of dynamics in the projective space. It is customary to consider E as part of the Julia set $\mathcal{J}(H)$. Thus, a point $p \in \mathbb{P}^n \setminus E$ is a Fatou point if it has a neighborhood $U \subset \mathbb{P}^n \setminus E$ on which the family $\{H^k \mid k = 1, 2, ...\}$ is normal.

Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$ and resp. $\mathbb{C}^m$. Two holomorphic self maps $F : \Omega_1 \to \Omega_1$ and $f : \Omega_2 \to \Omega_2$ are said to be *semi-conjugate* if there exists a holomorphic map $\Phi : \Omega_1 \to \Omega_2$ such that $f \circ \Phi = \Phi \circ F$. In the case such Φ is biholomorphic, we say F and f are *conjugate*. Since the semi-conjugacy implies $f^k \circ \Phi = \Phi \circ F^k, k = 1, 2, ...,$ the two maps have similar dynamical properties.

In the cases we discuss below, Ω_1 is either $\mathbb{C}^n$ or $\mathbb{P}^n$, $\Omega_2 = \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$, and the map Φ is often a rational function $p(z)/q(z)$, where p and q are coprime. Then the set of singularities of Φ is $\Lambda := \{p(z) = 0\} \cap \{q(z) = 0\}$. The team considered the following question.

Question 3.4. Assume $\Lambda \subset \mathcal{J}(F)$. Does Φ map $\mathcal{F}(F)$ to $\mathcal{F}(f)$?

Observe that if the answer to this question is positive, then we have

$$\bigcup_{x \in \mathcal{J}(f)} \Phi^{-1}\{x\} \subseteq \mathcal{J}(F). \tag{3}$$

It is known that for the dihedral group D_∞ with self-similar action (see the next subsection) on the rooted binary tree, we have $f(z) = 2z^2 - 1$, whose Julia set is $[-1, 1]$, and the equality in (3) holds [15].

The Hanoi Tower group G [6] is generated by three elements a, b, c, and their self-similar action on a ternary regular rooted tree is described by the recursion

$$a \cong \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \; b \cong \begin{pmatrix} 0 & 0 & I \\ 0 & b & 0 \\ I & 0 & 0 \end{pmatrix}, \; c \cong \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

If we consider the pencil $A(x, y) = xI + \pi(a+b+c) + y(t+t^{-1}), x, y \in \mathbb{C}$, where π is the associated Koopman representation, and

$$t = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix},$$

then the renormalization map is given by

$$F(x, y) = \left(x + \frac{2y^2(-x^2 + x + y^2)}{(x - 1 - y)(x^2 - 1 + y - y^2)}, \; \frac{y^2(x - 1 + y)}{(x - 1 - y)(x^2 - 1 + y - y^2)} \right).$$

It is known that F is semi-conjugate to $f(z) = z^2 - z - 3, z \in \mathbb{C}$ via the rational map $\Phi(x, y) = (x^2 - 1 - xy - 2y^2)/y$, whose singular set $\Lambda = \{(\pm 1, 0)\}$. It is not hard to check that $\Lambda \subset \mathcal{J}(F)$. However, the Julia set of this f is more complicated, and it is worth studying whether the inclusion (3) holds in this case. Furthermore, the following questions are appealing.

Question 3.5. Consider the general pencil $A(z) = z_0 + \pi(z_1 a + z_2 b + z_3 c) + z_4(t + t^{-1})$.
(a) Can the renormalization map F be computed?
(b) If so, is F semi-conjugate to a one-variable polynomial?
(c) Does F preserve the projective spectrum $p(A)$?

3.2. **Self-Similarity.** Let $X = \{x_1, x_2, \cdots, x_d\}$ be a finite alphabet of cardinality $d \geq 2$ and X^* be the set of all finite words over X. Then

$$X^* = \bigcup_{n=0}^{\infty} X^n, \tag{4}$$

where X^n is the set of words of length n. A self-similar action of a group G is an action of G on X^* such that for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w) \tag{5}$$

for all $w \in X^*$. Observe that such an action preserves the length of the words, so each X^n is G-invariant. This, in particular, implies that G has a representation (the induced Koopman representation) of dimension d^n. A group G is called self-similar if for some X it has a faithful self-similar action on X^*.

Let $T = T(X)$ be a d-regular rooted tree with set V of vertices X^* and the set $E = \{(w,\ wx)|\ w \in X^*,\ x \in X\}$ of edges. For a vertex $v \in V$, T_v is the subtree of T with root at v. It is easy to see that the self-similar action $(G,\ X^*)$ induces an action of G on $T(X)$ by automorphisms. Then the root (corresponding to empty word) and the levels $V_n = X^n$ of the tree are G-invariant, and the maximum of transitivity that may occur is the level transitivity (i.e. the transitivity of action on each level). For each fixed $n \in \mathbb{N}$ and $g \in G$ one has a decomposition $g = (g_1,\ g_2,\ \cdots,\ g_{d^n})\sigma$, where $\sigma \in Sym(d^n)$ and $g_i \in Aut(T_i)$. Here $T_i,\ 1 \leq i \leq d^n$, is the subtree of T with root at the ith vertex of level n. Elements $g_1,\ g_2,\ \cdots,\ g_{d^n}$ are called sections of g in corresponding vertices and the action $(G,\ X^*)$ is self-similar if and only if for every $n \in \mathbb{N},\ 1 \leq i \leq d^n$ the section g_i belongs to G after canonical identification of T_i with T. It is not hard to see that $(G,\ X^*)$ is self-similar if and only if sections $g_1,\ g_2,\ \cdots,\ g_d$ in vertices of the first level belong to G (after identification $T_i \cong T,\ i = 1, 2, \cdots, d$).

On the other hand, a general notion of self-similar representation can be defined abstractly, without referring to any measure-preserving group actions. Consider a unitary group representation $\pi : G \to U(\mathcal{H})$, where $U(\mathcal{H})$ stands for the group of unitary operators on a Hilbert space $\mathcal{H}$. For simplicity, we denote such a representation by $(\pi, \mathcal{H})$.

Definition 3.6. A unitary representation $(\pi, \mathcal{H})$ of a group G is said to be *self-similar* (or *d-similar*) if there exists an integer $d \geq 2$ and a unitary operator $W : \mathcal{H} \to \mathcal{H}^d$ such that every entry of the $d \times d$ block matrices $\hat{\pi}(g) = W\pi(g)W^*, g \in G$, is either equal to 0 or of the form $\pi(x)$ for some $x \in G$.

In this case, since $\hat{\pi}(g)$ is a unitary operator on $\mathcal{H}^d$ and each of its nonzero entries is a unitary operator on $\mathcal{H}$, every row or column of $\hat{\pi}(g)$ has precisely one nonzero entry. Note that this definition is valid for uncountable groups and Lie groups. It is not hard to verify that if a finitely generated group G has a self-similar action on a d-regular rooted tree, then the induced Koopman representation π is d-similar according to Definition 3.6. But it is not obvious whether the converse is true. Thus the team proposed the following question.

Question 3.7. Suppose a finitely generated group G has a d-similar representation. Does it necessarily admit a self-similar action on a d-regular rooted tree?

3.3. **Actions at the Finite Level.** Suppose group G has a self-similar action on X^* with the associated Koopman representation π. Let π_m denote the representation associated with this action on level X^m. For a finite set $\{g_i\} \subset G$, consider the pencil $A(z) = \pi(z_0 + \sum_{j=1}^{n} z_j g_j)$ and the pencil

$A_m(z)$ similarly associated with π_m. Set $Q_m(z) = \det A_m(z)$. In many cases, Schur reduction implies the relation $Q_{m+1}(z) = q_m(z)Q_m(F(z))$ for some homogeneous polynomial q_m. Hence, if $z \in \tau_m := \{z \in \mathbb{P}^n : Q_m(z) = 0\}$, then $z \in \tau_{m+1}$. We will call F the renormalization map. In this case τ_m is an increasing net of varieties in $\mathbb{P}^n$, and F maps one variety to another one in the net. It is worth noting that the presence of the map F depends on the group G and the pencil $A(z)$, and it is therefore not universal. In any case, set $\tau = \overline{\cup_{m=0}^{\infty}\tau_m}$. The team made the following observation.

Proposition 3.8. *If a finitely generated group G has a self-similar action on X^*, then $\tau \subseteq p(A)$.*

The following conjecture was proposed.

Conjecture 3.9. In the setting above, we have

$$p(A) = \tau \cup \{z \in \mathbb{P}^n \setminus \tau : \ \|A_m^{-1}(z)\| \text{ unbounded w.r.t } m\}.$$

As an example, the team studied the following group.

Definition 3.10. The Tangled Odomenters group [7] with generators a, b is defined by the self-similar action on a rooted regular ternary tree with self-similar relation

$$a \cong \begin{pmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b \cong \begin{pmatrix} 0 & 0 & b \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Grigorchuk, Nekrshevych and Sunic determined the spectrum of the self-adjoint operator $\pi(a + a^{-1} + b + b^{-1})$. Sunic and some of his students determined the closure of the union of the spectra on the levels for the non-self-adjoint operator $\pi(a + b)$. Both work are unpublished. The case of $\pi_m(a + b)$ was found to be related to the one-dimensional rational function $f(z) = \frac{z^2}{1-z}, z \in \mathbb{C}$. We anticipate that the following holds.

Theorem 3.11. *For the Tangled Odometers group, we have $\tau = \sigma(\pi(a + b))$.*

Proof of the equality is likely to be complicated, since operators are not self-adjoint. In the projected proof, the dynamical properties of the map F will enable us to construct the inverse to $\pi(a + b)$.

For the Tangled Odometers group, the renormalization map associated with the pencil $I + \pi(xa + yb)$ is $F(x, y) = (\frac{x^2}{1-y}, \frac{y^2}{1-x})$, and it is not known to be semi-conjugate to any one variable map. However, its dynamics on the invariant set $\{x = y\}$ is well-understood. Since non-reducible components of projective spectra are curves that intersect $x = y$ but do not intersect each other except at $(\pm 1, 0)$, $(0, \pm 1)$, the dynamics of F on $\cup \tau_n \setminus \{(\pm 1, 0), (0, \pm 1)\}$ is continuously semi-conjugate to the dynamics of its restriction to $x = y$. This leads to the following.

Theorem 3.12. *For the real pencil $A(x, y) = 1 + \pi(xa + yb), x, y \in \mathbb{R}$ of the Tangled Odometers group, the projective spectrum $p(A)$ is a countable union of algebraic curves of increasing degree, only accumulating onto a union of an interval and an arc of an ellipse E. Intersection of $p(A)$ with the line $x = y$ coincides with the real part of the Julia set of the rational function $F|_{x=y}$.*

This is a first example of spectrum computation for the case when there is no algebraic reduction to a one-dimensional map. Since the spectrum is a union of algebraic curves of growing degree, it was thus not computed explicitly. However, we can fully describe the numerical range (to be introduced later).

Theorem 3.13. *For the Tangled Odometers group, the numerical range is bounded by algebraic arcs that are projective duals of: the line $x + y + 1 = 0$, the arc of the ellipse E, and the arc of the degree-3 algebraic curve $F^{-1}(E)$.*

Projective duals will be computed explicitly.

Conjecture 3.14. For the Tangled Odometers group, currents supported on τ_n converge in the weak*-topology to a limit current that is a natural generalization of the spectral measure for projective spectra.

This is a good starting example towards the general definition of the spectral measure (that is to be supported on the limit of the projective duals of τ_n).

The following conjecture motivates our exploration of the Tangled Odometers group.

Conjecture 3.15. The projective spectrum $p(A)$ of the Tangled Odometers group is related to that of the Basilica group, since the two groups have homeomorphic limit sets.

The nature of the relation between spectra is not clear. The full projective spectrum of the basilica group is not computed.

3.4. **The Grigorchuk Group.** The Grigorchuk group $\mathcal{G}$ is generated by the action of four involutions a, b, c, d on the binary tree, with the self-similarity

$$a \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, b \cong \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, c \cong \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, d \cong \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

The properties of $\mathcal{G}$ have been studied extensively in the literature. In particular, it is the first known example of a finitely generated group that has intermediate growth [4, 5]. Since b, c, d pairwise commute, if we set $u = (b+c+d-1)/2$ then $u^2 = 1$. It is shown that the group $< a, u >$, which is a subset of the group algebra $\mathbb{C}(\mathcal{G})$, is isomorphic to D_∞. Hence the projective spectrum $p(A_\pi)$ of the pencil $A_\pi(z) = \pi(z_0 + z_1 a + z_2 b + z_3 c + z_4 d)$ is fully computed for the case $z_2 = z_3 = z_4$. For distinct real values x_1, x_2, x_3, x_4, the classical spectrum $\sigma(\pi(x_1 a + x_2 a + x_3 c + x_4 d))$ is known to be a Cantor set (this follows from the relation with one-dimensional Schrodinger operators). The following is a program to investigate the Julia set and the full projective spectrum $p(A_\pi)$.

Since the map $F_{\mathcal{G}}(z)$ defined in (1) permutes z_2, z_3, z_4, its power $F_{\mathcal{G}}^3$ fixes the three variables. We define a sequence of rational maps $R_k : \mathbb{P}^4 \to \mathbb{P}^1$ by the equation

$$(F_{\mathcal{G}}^3)^k(z) = [R_k(z) : z_2 : z_3 : z_4], \quad k = 1, 2, \ldots.$$

Hence the dynamical properties of the sequence $(F_{\mathcal{G}}^3)^k(z), k = 0, 1, \ldots$ hinges on the sequence R_k. More precisely, we have the Fatou set

$$\mathcal{F}(F_{\mathcal{G}}^3) = \{z \in \mathbb{P}^4 : \{R_k\} \text{ is a normal family in a neighborhood of } z\}.$$

This suggests a new direction of study.

Question 3.16. Consider the map $F_{\mathcal{G}}$.
(a) Can $\mathcal{J}(F_{\mathcal{G}}^3)$ be determined?
(b) How is $\mathcal{J}(F_{\mathcal{G}}^3)$ related to $p(A_\pi)$?
(c) What is the relation between $\mathcal{J}(F_{\mathcal{G}}^3)$ and $\mathcal{J}(F_{\mathcal{G}})$?

We tend to believe that the following hold.

1. The projective spectrum $p(A_\pi)$ is locally homeomorphic to the product of the Cantor set with the interval everywhere except the 3 points in the indeterminacy locus of $F_{\mathcal{G}}$.

2. The numerical range of the real pencil $x_1 a + x_2 b + x_3 c + x_4 d$ is bounded by two intervals and a projective dual of a single dynamically defined analytic curve (to be computed numerically or explicitly).

The theory developed for Schrodinger operators implies that gaps in the Cantor-like projective spectrum occur at certain levels for cumulative spectral density, called gap labels. Finding gap labels for Grigorchuk group should be possible.

Conjecture 3.17. The renormalization map $F_{\mathcal{G}}$ defines an action on gap labels.

Potentially, this conjecture could be generalized to a wider class of one-dimensional Schrodinger operators. The conjecture is related to studying the action of $F_{\mathcal{G}}$ on the currents that are supported on τ_n and their convergence to the limit current. The following is high anticipated.

Conjecture 3.18. For the Grigorchuk group, the Julia set of $\mathcal{J}(F_{\mathcal{G}}) = p(A_\pi)$.

Other examples of groups that are accessible via the strategy above include Grigorchuk-Erschler group and the iterated monodromy group of $z^2 + i$.

3.5. **Numerical Range.** We let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, its numerical range is defined as $W(T) = \{\langle Tu, u\rangle : u \in \mathcal{H},\ \|u\| = 1\}$. It is easy to see that $W(T)$ is a bounded subset of the complex plane $\mathbb{C}$. It is known that the classical spectrum $\sigma(T) \subset \overline{W(T)}$, and the Toeplitz-Hausdorff Theorem states that $W(T)$ is always convex. Moreover, if T is normal, i.e. the commutator $[T^*, T] = 0$, then $\overline{W(T)}$ coincides with the convex hull of $\sigma(T)$ (denoted by $\mathrm{conv}(\sigma(T))$. A natural extension of numerical range to a tuple of several operators $A = (A_1, ..., A_n)$ is defined as

$$\widetilde{W}(A) = \{(\langle A_1u, u\rangle, ..., \langle A_nu, u\rangle) \in \mathbb{C}^n :\ u \in \mathcal{H},\ \|u\| = 1\}.$$

However, it is no longer clear whether $\widetilde{W}(A)$ is always convex for any n. To avoid this issue, a modified notion of joint numerical range is defined as follows. First, we let $S(\mathcal{H})$ denote the set of positive trace-class operators ρ on $\mathcal{H}$ such that $\mathrm{Tr}\,\rho = 1$. Such operators ρ are called density operators, and they play an important role in quantum mechanics.

Definition 3.19. Given a tuple $A = (A_1, ..., A_n)$ of operators in $\mathcal{B}(\mathcal{H})$, its numerical range $W(A)$ is defined by

$$W(A) = \{(\mathrm{Tr}(A_1\rho), ..., \mathrm{Tr}(A_n\rho)) \in \mathbb{C}^n :\ \rho \in S(\mathcal{H})\}.$$

Evidently, the set $S(\mathcal{H})$ is convex, and hence the convexity of $W(A)$ is out of the question. There is an interesting connection between $W(A)$ and the projective spectrum of the pencil $A(z) = z_0I + z_1A_1 + \cdots + z_nA_n$. Suppose $\dim \mathcal{H} = k < \infty$. The joint characteristic polynomial of the tuple A is defined as $Q_A(z) = \det A(z)$ [15]. Thus, the projective spectrum $p(A)$ is the zero variety $\mathcal{Z}(Q_A)$. Note that a point λ in an irreducible variety $V = \{f(z) = 0\}$ is said to be singular if the gradient $\nabla f(\lambda) = 0$. Non-singular points are called regular points of V. Suppose V is an irreducible variety in $\mathbb{P}^n$ with the set of regular point V_{reg}. Then its dual V^* is a subset in $\mathbb{P}^n$ consisting of the normal vectors to the tangent planes of V_{reg}, i.e., $V^* = \{\nabla f(w) \in \mathbb{P}^n : w \in V_{reg}\}$. If $n = 2$, Murnaghan [13] showed that the eigenvalues of the matrix $A_1 + iA_2$ are the foci of the curve

$$T = \{y_1 + iy_2 : y_1, y_2 \in \mathbb{R}, [1 : y_1 : y_2] \in (p(A))^*\}.$$

Kippenhahn [9] observed that $W(A_1 + iA_2) = \mathrm{conv}(T)$. More generally, for self-adjoint matrices $A_1, ..., A_n$, the following theorem is shown in [14]:

$$W(A) = \{(y_1, ..., y_n) \in \mathbb{R}^n :\ [1 : y_1 : \cdots : y_n] \in \overline{\mathrm{conv}}(V_1^* \cup \cdots \cup V_m^*), \tag{6}$$

where V_j are irreducible components of $p(A)$. We use an example to illustrate this theorem.

Example 3.20. The action of the Hanoi Tower group on the first level X^1 gives rise to the 3-dim. representation

$$A_1 = \pi_1(a) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ A_2 = \pi_1(b) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ A_3 = \pi_1(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Their characteristic polynomial

$$Q_A(z) = (z_0 + z_1 + z_2 + z_3)\left(z_0^2 - (z_1^2 + z_2^2 + z_3^2) + z_1z_2 + z_2z_3 + z_1z_3\right) =: p_1(z)p_2(z).$$

Set $V_i = \{z \in \mathbb{P}^3 : p_i(z) = 0\}, i = 1, 2$. It is easy to see that V_1 has no singular points, and its dual $(V_1)^* = \{[1 : 1 : 1 : 1]\}$. For V_2, we have

$$\nabla p_2 = [2z_0 : -2z_1 + z_2 + z_3 : z_1 - 2z_2 + z_3 : z_1 + z_2 - 2z_3] =: [y_0 : y_1 : y_2 : y_3],$$

and hence it has one singular point $\{[0 : 1 : 1 : 1]\}$. Since the matrices A_i are self-adjoint, we consider the case $y_i \in \mathbb{R}$. Observe that ∇p_2 satisfies the equation $y_1 + y_2 + y_3 = 0$. Moreover, since $p_2(z) = 0$, it must also satisfy the equation

$$\frac{9}{4} y_0^2 = y_1^2 + y_2^2 + y_3^2 - (y_1 y_2 + y_1 y_3 + y_2 y_3),$$

which describes a cylinder with radius $3/2$ and the main axis in the direction of $(1, 1, 1)$. Thus the dual $(V_2)^*$ is the intersection of this cylinder with the plane $\{y_1 + y_2 + y_3 = 0\}$, which is a circle. In light of (6), the numerical range $W(A)$ is a solid cone.

The team proposed two directions of research related to self-similar groups and joint spectra.

1. As mentioned earlier, a self-similar action of G on a d-regular rooted tree naturally gives rise to d^n-dimensional Koopman representations (which we denote by π_n) for each $n \geq 0$. In the case $G = \mathcal{G}$, we let a_n, b_n, c_n, d_n be the representations of a, b, c, d under π_n. Then they are selfadjoint unitary matrices, and the characteristic polynomial $Q_n(z) = \det(z_0 I + z_1 a_n + z_2 b_n + z_3 c_n + z_4 d_n)$ can be computed through recursion.

Question 3.21. Can the numerical range for the tuple (a_n, b_n, c_n, d_n) be computed using equation (6)?

This question helps bring numerical range into the study of self-similar groups, and this could potentially offer new ideas and approaches.

2. We set $A_*(z) := z_1 A_1 + \cdots z_n A_n$ and let $u_1(z), ..., u_m(z)$ be the set of linearly independent unit eigenvectors of $A_*(z)$. The index m can vary with z, and it can be ∞ depending on the operators $A_1, ..., A_n$. The team proposed the following notion of joint spectrum for matrices, which has a clear connection with the numerical range $\widetilde{W}(A)$.

Definition 3.22. For a tuple of linear operators $A = (A_1, ..., A_n)$, define its joint spectrum

$$\Sigma(A) = \{(\langle A_1 u_j(z), u_j(z)\rangle ..., \langle A_n u_j(z), u_j(z)\rangle) \in \mathbb{C}^n : \ z \in \mathbb{C}^n, j = 1, ..., m\}.$$

Evidently, we have $\Sigma(A) \subset \widetilde{W}(A)$, and if $n = 1$ then $\Sigma(A_1) = \sigma(A_1)$. When the involved matrices are self-adjoint, it makes sense to consider the real pencil $A_*(x), x \in \mathbb{R}^n$.

Example 3.23. Consider Pauli matrices:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\Sigma(A)$ is the sphere $S^2 \subset \mathbb{R}^3$.

3.6. **Hyponormal Operators.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal if $[T^*, T] \geq 0$. Clearly, every normal operator is hyponormal, and there are plenty of hyponormal operators that are not normal. An operator T is said to be m-cyclic if there is a finite subset $\{u_1, ..., u_m\} \subset \mathcal{H}$ such that the span of $\{T^k u_j : \ j = 1, ..., m, k \geq 0\}$ is dense in $\mathcal{H}$. A celebrated theorem due to Berger and Shaw asserts that if T is a m-cyclic hyponormal operator, then $[T^*, T]$ is in the trace class, and moreover,

$$\mathrm{Tr}[T^*, T] \leq \frac{m}{\pi} \mu(\sigma(T)),$$

where μ is the Lebesgue measure on the plane. As an interesting application, if T is hyponormal, finite cyclic, and $\mu(\sigma) = 0$, then it is normal.

1. We consider the projective pencil $T(z) = z_0 + z_1 T + z_2 T^*$. In the case T is normal, it is known that the projective spectrum of the pencil $T(z)$ is a union of hyperplanes in $\mathbb{P}^2$:

$$p(T) = \bigcup_{\lambda \in \sigma(T)} \{z \in \mathbb{P}^2 : \ z_0 + z_1\lambda + z_2\overline{\lambda} = 0\}. \tag{7}$$

In other words, $\sigma(z_1 T + z_2 T^*) = \{(z_1\lambda + z_2\overline{\lambda}) : \ \lambda \in \sigma(T)\}$, and $T(z)$ is invertible if and only if

$$\inf_{\lambda \in \sigma(T)} |z_0 + z_1\lambda + z_2\overline{\lambda}| > 0.$$

Remarkably, we made the following discovery.

Theorem 3.24. *If T is hyponormal, then (7) holds.*

For hyponormal or co-hyponormal operators the following inequality is known:

$$\|(T-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}.$$

Applying this observation to the pencil one finds

$$\|(z_0 + z_1 T + z_2 T^*)^{-1}\| = \frac{1}{\inf\{|z_0 + z_1\lambda + z_2\overline{\lambda}| : \ \lambda \in \sigma(T)\}}$$

whenever $z \in p^c(T)$. The following question is intriguing.

Question 3.25. Suppose $T \in \mathcal{B}(\mathcal{H})$ and (7) holds. What can we conclude about T?

2. The particular case of pure hyponormal operators with rank-one self-commutator is notable, due to their parametrization by the *principal function*. Assuming $[T^*, T]$ is the rank-1 operator $\xi \otimes \xi := \xi\langle \cdot, \xi\rangle$, the principal function g_T is characterized by the Helton-Howe trace formula:

$$\operatorname{tr}[p(T,T^*), q(T,T^*)] = \frac{1}{\pi}\int_{\sigma(T)} (\overline{\partial}p\partial q - \overline{\partial}q\partial p) g_T d\mu.$$

Above, μ is the area measure, p, q are polynomials in the variables $z, \overline{z}, \partial = \frac{\partial}{\partial z}, \overline{\partial} = \frac{\partial}{\partial \overline{z}}$, and the functional calculus puts T, T^* in monomials in any prescribed order. The dictionary T to g_T is bijective between all pure hyponormal operators with rank-one self-commutator and classes $g_T \in L^1(\mathbb{C}, d\mu)$, $\operatorname{supp}(g_T)$ compact and $0 \leq g_T \leq 1$.

Assuming $T(z)$ invertible, we compute

$$[T(z)^{-1}, T] = T(z)^{-1}(TT(z) - T(z)T)T(z)^{-1} = -T(z)^{-1}z_2[T^*, T]T(z)^{-1},$$

therefore,

$$\operatorname{tr}[T(z)^{-1}, T] = -\operatorname{tr}\, T(z)^{-2}\xi\langle\cdot, \xi\rangle = -z_2\langle T(z)^{-2}\xi, \xi\rangle.$$

On the other hand, Helton-Howe formula yileds

$$\operatorname{tr}[T(z)^{-1}, T] = \frac{1}{\pi}\int_{\sigma(T)} \frac{-z_2 g_T(w) d\mu(w)}{(z_0 + z_1 w + z_2\overline{w})^2}.$$

We can pass to the anti-derivative with respect to z_0 and infer on the unbounded complement of the affine spectrum:

$$\langle T(z)^{-1}\xi, \xi\rangle = \frac{1}{\pi}\int_{\sigma(T)} \frac{g_T(w) d\mu(w)}{z_0 + z_1 w + z_2\overline{w}}.$$

Indeed, both expressions above are analytic in a neighborhood of $z_0 = \infty$ and vanish at that point.

3. As simple as it may look, the case of the unilateral shift $S : \ell^2(\mathbf{N}) \longrightarrow \ell^2(\mathbf{N})$ is relevant. The orthonormal basis of $\ell^2(\mathbf{N})$ is denoted $e_0, e_1, \ldots$ and S acts on it as follows:

$$Se_k = e_{k+1}, \quad k \geq 0.$$

The adjoint S^* will map e_0 to zero and e_{k+1} to e_k, $k \geq 0$. The operator S is an isometry $S^*S = I$ and hyponormal: $S^*S = I \geq SS^*$. The principal function of S is the characteristic function of the unit disk $\mathbf{D}$. In view of the preceding section:

$$\langle (I - \zeta S - \overline{\zeta} S^*)^{-1} e_0, e_0 \rangle = \int_{\mathbf{D}} \frac{d\mu(w)}{1 - \zeta w - \overline{\zeta}\overline{w}}, \quad 2|\zeta| < 1.$$

Denote by $\Sigma^{\mathrm{sym}} T^k (T^*)^n$ the symmetrized sum: over all words in T and T^* containing k-times T and n-times the adjoint T^*. A Neumann series expansion of both terms yields for all $n \geq 0$:

$$\sum_{k=0}^{n} \zeta^k \overline{\zeta}^{n-k} \langle (\Sigma^{\mathrm{sym}} S^k (S^*)^{n-k}) e_0, e_0 \rangle = \int_{\mathbf{D}} (\zeta w + \overline{\zeta}\overline{w})^n d\mu(w).$$

Since

$$\int_{\mathbf{D}} w^p \overline{w}^q d\mu(w) = \frac{\pi}{p+1} \delta_{pq},$$

we isolate the only non-vanishing expressions:

$$\langle (\Sigma^{\mathrm{sym}} S^k (S^*)^k) e_0, e_0 \rangle = \frac{(2k)!}{(k+1)(k!)^2}, \quad k \geq 0,$$

which are known as the Catalan numbers.

4. Other Topics

In addition, the team discuss a few other subjects.

(a) Functional calculus of noncommuting operators, such as the the Weyl functional calculus [1], Fantappie calculus [2].

(b) Weighted shifts on graphs [11, 12].

(c) The spectrum of the pencil $z_0 + z_1 \hat{x} + z_2 \hat{p}$, where $\hat{x}$ (position operator) and $\hat{p}$ (momentum operator) are self-adjoint unbounded operators densely defined on $L^2(\mathbb{R})$ by $\hat{x} f(x) = x f(x)$ and resp. $\hat{p} f(x) = -i \frac{df}{dx}$.

(d) The positivity problem: Given two positive operators $A, B \in \mathcal{B}(\mathcal{H})$, for x and y close to 0 we have the expansion

$$(I - (xA + yB))^{-1} = \sum_{i,j \geq 0} P_{ij}(A, B) x^i y^j, \quad x, y \in \mathbb{C},$$

where P_{ij} are polynomials in noncommuting variables. It is not hard to see that if A commutes with B, then $P_{ij}(A, B) \geq 0$ for all $i, j \geq 0$. Is the converse true?

References

[1] R. Anderson, *The Weyl functional calculus*, J. Funct. Anal 4 (1969), 240-267.

[2] L. Fantappié, *Teoría de los funcionales analíticos y sus aplicaciones*, Consejo Superior de Investigaciones Cientificas, Barcelona, 1943

[3] I. Gohberg, S. Goldberg, M. Kaashoek, *Linear operator pencil.* In: Classes of Linear Operators Vol. I. Operator Theory: Advances and Applications, vol 49 (1990), Birkhäuser, Basel. DOI: 10.1007/978-3-0348-7509-7-5.

[4] R. Grigorchuk, *Burnside's problems on periodic groups*, Funct. Anal. Appl. 14 (1980), no. 1, 41-43.

[5] R. Grigorchuk, *On the Milnor problem of group growth*, Soviet Math. Dokl. 28 (1983), no. 1, 23-26.

[6] R. Grigorchuk and Z. Šunić, *Schreier spectrum of the Hanoi Towers group on three pegs*, Proc. Sym. Pure Math. 77 (2008),183-198.

[7] R. Grigorchuk and Z. Šunić, unpublished.

[8] G. Isaev, *The linear factorization of polynomial operator pencils*, Mathematical Notes of the Academy of Sciences of the USSR 13, 333-338 (1973). DOI: 10.1007/BF01146569.

[9] R. Kippenhahn, *Über den wertevorrat einer matrix*, Mathematische Nachrichten 6 (1951), 193-228.

[10] M. Martin and M. Putinar, *Lectures on Hyponormal Operators*, Birkhäuser Verlag, Basel-Boston-Berlin, 1989.

[11] Z. Jabłoński, I.B. Jung, J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. 216 (2012), no.1017.
[12] W. Majdak, J.B. Stochel, Weighted shifts on directed semi-trees: an application to creation operators on Segal-Bargmann spaces, Complex Anal. Oper. Theory 10 (2016), 1427–1452.
[13] F. D. Murnaghan, On the field of values of a square matrix, PNAS18 (1932), 246-248.
[14] D. Plaumann, R. Sinn, and S. Weis, *Kippenhahn's theorem for joint numerical ranges and quantum states*, SIAM Journal on Applied Algebra and Geometry Vol. 5, Iss. 1 (2021), DOI: 10.1137/19M1286578
[15] R. Yang, *A spectral theory of noncommuting operators*, Springer Nature Switzerland A.G., 2024.