Physics 539 - Problem Set 3 - Due Nov. 10

(1) Let ρ be a density matrix on a Hilbert space $\mathcal{H}$ of dimension N. Show that the von Neumann entropy $S(\rho) = -\mathrm{Tr}\,\rho\log\rho$ is at most $\log N$ and find the unique ρ that achieves that value. (The method of Lagrange multipliers may be useful.)

(2) In a two-dimensional Hilbert space, a general hermitian matrix is of the form

$$M = a + \vec{b}\cdot\vec{\sigma}$$

where $\vec{\sigma}$ are the Pauli matrices, a is real, and $\vec{b}$ is a real 3-vector.

(a) What condition on $a, \vec{b}$ makes M a density matrix? What condition makes it a density matrix of rank 1?

(b) Consider a pure state $\Psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $|\alpha|^2 + |\beta|^2 = 1$. What are $a, \vec{b}$ for the rank 1 density matrix $\rho = |\Psi\rangle\langle\Psi|$? Did every a, $\vec{b}$ allowed by the answer in (a) arise in this family?

(c) Consider the density matrix

$$\rho = \begin{pmatrix}3/4 & 0\\ 0 & 1/4\end{pmatrix}.$$

One interpretation of this density matrix is that pure states $\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ 1\end{pmatrix}$ were prepared with probabilities 3/4 and 1/4. However, this interpretation is not unique. It is possible, for example, to find rank 1 density matrices ρ_1, ρ_2, ρ_3, corresponding say to pure states ψ_1, ψ_2, ψ_3, such that

$$\rho = \frac{1}{3}(\rho_1 + \rho_2 + \rho_3).$$

Can you give an example or show that this is possible? Thus another interpretation of the density matrix ρ is that pure states ψ_1, ψ_2, ψ_3 were prepared each with probability 1/3. (In answering this question, just give the ρ's, don't worry about the ψ's.)

(3) Let A, B, C be systems that consist each of a single qubit (that is, each system has a two-dimensional quantum Hilbert space). Verify the inequality of strong subadditivity of entropy, namely $S_{AB} + S_{BC} \geq S_B + S_{ABC}$, in the following two cases:

(a) The subsystem AB is in a pure state.

(b) The subsystem AC is in a pure state.

(4) Let ρ be a thermal density matrix $\rho = \frac{1}{Z}e^{-\beta H}$, where Z is such that $\mathrm{Tr}\,\rho = 1$ and $\beta = 1/T$. Define the energy by

$$E = \mathrm{Tr}\,\rho H$$

and the von Neumann entropy

$$S = -\mathrm{Tr}\,\rho \log \rho.$$

Consider an arbitrary first order deformation $\delta\rho$ of ρ and a corresponding first order deformation δE of E. Prove that

$$\delta E = T\delta S.$$

This statement holds for an arbitrary $\delta\rho$, but only to first order in $\delta\rho$. Suppose, however, that we vary ρ by varying T (or equivalently β) as a function of time. Then we always have $\rho = \frac{1}{Z(\beta)} e^{-\beta H}$, albeit with a time-dependent β, and therefore the formula just derived is true at any time. Thus in a process that is always in equilibrium but with varying temperature,

$$\mathrm{d}E = T\mathrm{d}S.$$

This should be a familiar result for a case in which temperature is assumed to be the only relevant thermodynamic variable (which is true here because we keep H fixed rather than letting it depend on volume, magnetic field, or any other variable; the derivation can be extended, of course, if H does depend on additional variables).

(5) (a) Let σ be a maximally mixed density matrix for some quantum system and let ρ be any density matrix. Compute the relative entropy $S(\rho||\sigma)$ and express it as a difference of von Neumann entropies.

(b) Consider a quantum channel under which σ is invariant. Use monotonicity of relative entropy under quantum channels to prove that such a channel can only increase the entropy of ρ.

(6) Let ψ be a given pure state of some quantum system (with N dimensional Hilbert space). Find Kraus operators of a quantum channel that maps any density matrix ρ to $|\psi\rangle\langle\psi|$. (A physical realization is to turn on a Hamiltonian for which ψ is the ground state and wait until the system relaxes to the ground state.)

(7) Find Kraus operators of a quantum channel that maps any density matrix ρ of a system with an N dimensional Hilbert space to a maximally mixed state. You can use the output of (6) as the starting point. (A physical realization is to let a system interact with a sufficiently noisy environment.)